Vorlesung Algebraische Topologie 1

Ulrich Bunke

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1 Basic de Rham cohomology

1.1 Recap on manifolds

In this course we study topological invariants of **smooth manifolds**. We assume that the underlying topological space of a smooth manifold is Hausdorff, second countable and paracompact.

Remark 1.1. The underlying topological space of a manifold is therefore a **metrizable space**. Paracompactness is important for the existence of **smooth partitions of unity** subordiated to an open covering. Second countability implies that a manifold admits an exhaustion by a sequence of compact subsets. □

We admit manifolds with boundary, and more generally, manifolds with **corners**. A manifold with corners is locally modeled on the subspaces $[0,\infty)^n \subset \mathbb{R}^n$. A smooth map $U \to V$ between open subsets U and V of $[0,\infty)^n$ or $[0,\infty)^m$, respectively, is a continuous map which extends to a smooth map between open neighbourhoods of U or V in $\mathbb{R}^n$ or $\mathbb{R}^m$, respectively. This fixes our convention for the notion a smooth map between manifolds with corners in general, and for the coordinate transitions in particular.

By **Mf** we denote the **category of smooth manifolds** and smooth maps.

The basic examples of manifolds are $\mathbb{R}^n$ and its open subsets.

Example 1.2.

1. From the point of view of topology a simple example is the disc $D^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.
2. A topologically more interesting example is the complement of a finite set of points in $\mathbb{R}^n$.
3. Much more interesting is already a so-called **knot complement**, $\mathbb{R}^3 \setminus \phi(S^1)$, where $\phi : S^1 \to \mathbb{R}^3$ is an embedding.
4. The previous example can be generalized to complements of compact submanifolds in $\mathbb{R}^n$.

□

The basic examples of manifolds with corners of codimension k for $0 \le k \le n$ are open subsets of $[0,\infty)^k \times \mathbb{R}^{n-k}$ which contain a point $(0,x)$ for some $x \in \mathbb{R}^{n-k}$. In these cases $n \in \mathbb{N}$ is the dimension of the manifold.

A manifold with corners of codimension at most one is a manifold with **boundary** (possibly empty). For example, the unit interval $I := [0,1]$ is a manifold with boundary $\partial I = \{0,1\}$. More generally, the product $I^k \times \mathbb{R}^{n-k}$ is a manifold with corners of codimension k.

The category **Mf** has coproducts and products. In the category of manifolds certain **fibre products** exists. For example, a limit of a diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

in **Mf** of manifolds without boundary exists and and is usually denoted by $A \times_C B$, if the maps f and g are **transverse**: for every pair $(a,b) \in A \times B$ with $f(a) = g(b)$ we have

$$df(a)(T_aA) + dg(b)(T_bB) = T_cC \ ,$$

where $c := f(a)$. There are corresponding conditions for diagrams involving manifolds with corners which we will not spell out in detail.

A typical class of manifolds which are defined by fibre products are **submanifolds**. So let $f : A \to C$ be a map and $g : \{*\} \to C$ be the inclusion of an interior point c. We assume that f is transverse to g. This means in this case that for every point $a \in A$ with $f(a) = c$ the differential $df(a) : T_aA \to T_cC$ is surjective. If A has corners, then we must require that the restriction of f to all faces is transversal, too. Then we can define a submanifold $f^{-1}(c)$ as the fibre product $\{*\} \times_C A$.

Example 1.3. 1. Typical examples of manifolds naturally defined as submanifolds are the spheres

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\} \ .$$

2. Another family of examples are Lie groups

$$O(n) := \{A \in \mathtt{Mat}(n,n,\mathbb{R}) \mid AA^t = 1\} \ , \quad n \ge 1$$

Note that we must consider here AA^t as an element in the symmetric matrices in order to ensure regularity of the defining equation.

3. Let $U \subset \mathbb{R}^k$ be open and $f : U \to \mathbb{R}^m$ be a smooth map. Then the graph of f defined by

$$\mathrm{Graph}(f) := \{(u, f(u)) \in \mathbb{R}^{k+m} \mid u \in U\}$$

can be presented as a fibre product:

$$\begin{tikzcd} \mathrm{Graph}(f) \arrow[r] \arrow[d] & * \arrow[d, "0"] \\ U \times \mathbb{R}^m \arrow[r, "(u,y)\mapsto f(u)-y"] & \mathbb{R}^m \end{tikzcd}$$

□

A closed **codimension-zero submanifold** $N \subset M$ (with boundary) can be defined by an inequality $N := \{f \geq 0\}$, where $f : M \to \mathbb{R}$ is regular on the boundary $\partial N = \{f = 0\}$. We obtain N as a limit of

$$\begin{tikzcd} N \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ {[0,\infty)} \arrow[r] & \mathbb{R} \end{tikzcd} .$$

The condition on f is exactly the transversality condition for this diagram. The function f is often called a **boundary defining function**. One can find local coordinates for N near ∂N such that $f_{|N}$ is one of the coordinate functions.

More generally, a closed codimension-zero submanifold with **corners of codimension at most** k can locally be defined by a collection of inequalities $N := \bigcap_{i=1}^k \{f_i \geq 0\}$, or equivalently, as a limit of

$$\begin{tikzcd} N \arrow[r] \arrow[d] & M \arrow[d, "(f_1,\ldots,f_k)"] \\ {[0,\infty)^k} \arrow[r] & \mathbb{R}^k \end{tikzcd} .$$

We again must require an appropriate transversality condition.

In certain cases manifolds can be glued along open submanifolds. We consider a **push-out** diagram of manifolds

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \\ W & \end{tikzcd} .$$

where both maps are open embeddings. Then we can form the colimit in **Mf** denoted by $V \sqcup_U W$ if the colimit of the underlying topological spaces is Hausdorff.

Example 1.4. Consider the examples

$$\begin{tikzcd} (-1,1) \arrow[r] \arrow[d] & (-1,\infty) \\ (-\infty,1) & \end{tikzcd} \quad , \qquad \begin{tikzcd} (-1,1) \arrow[r] \arrow[d] & \mathbb{R} \\ \mathbb{R} & \end{tikzcd}$$

where all maps are the canonical inclusions. In the first case the colimit exists and is isomorphic to $\mathbb{R}$. In the second case the colimit of the underlying topological spaces is not Hausdorff. For example, the point represented by 1 in the lower left copy of $\mathbb{R}$ can not be separated from the (different) point represented by 1 in the upper right copy of $\mathbb{R}$.

Let $(U_\alpha)_{\alpha \in A}$ be a countable open covering of a manifold M. Then we can represent M as a push-out

$$\begin{tikzcd} \bigsqcup_{(\alpha,\beta)\in A^2} U_\alpha \cap U_\beta \arrow[r, "i_1"] \arrow[d, "i_2"] & \bigsqcup_{\alpha\in A} U_\alpha \arrow[d] \\ \bigsqcup_{\beta\in A} U_\beta \arrow[r] & M \end{tikzcd} ,$$

where i_1 and i_2 are induced by the two inclusions $U_\alpha \cap U_\beta \to U_\alpha$ and $U_\alpha \cap U_\beta \to U_\beta$. $\square$

Manifolds naturally appear as parametrizing objects for geometric structures.

Example 1.5. Typical examples are the **Grassmann manifolds** $Gr(k,\mathbb{R}^n)$ and $Gr(k,\mathbb{C}^n)$ of k-dimensional subspaces in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively, for $k,n \in \mathbb{N}$, $0 \leq k \leq n$. We describe the manifold structure of $Gr(k,\mathbb{R}^n)$ in greater detail. Given a decomposition $\mathbb{R}^n = V \oplus W$ with $\dim(V) = k$ we obtain a chart $\phi_{V,W} : \mathtt{Hom}(V,W) \to Gr(k,\mathbb{R}^n)$ of a neighbourhood of V

by

$$\phi_{V,W}(h) := \{v + h(v) \mid v \in V\} .$$

Of course $\mathtt{Hom}(V,W) \cong \mathbb{R}^{k(n-k)}$ after choosing bases in V and W.

The Grassmann manifolds are the receptables of the **Gauss-maps** of submanifolds. If $M \to \mathbb{R}^n$ is k-dimensional submanifold, then the Gauss map $\gamma : M \to Gr(k,\mathbb{R}^n)$

maps a point $m \in M$ to the subspace $T_m M \in Gr(k, \mathbb{R}^n)$. $\square$

Group objects in the category **Mf** are called **Lie groups**.

Example 1.6. Typical examples are the general linear groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ for $n \in \mathbb{N}$, $n \geq 1$. As manifolds they are open submanifolds of $\mathtt{Mat}(n,n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ or $\mathtt{Mat}(n,n,\mathbb{C}) \cong \mathbb{R}^{2n^2}$ defined by the condition $\det(A) \neq 0$.

Further examples of Lie groups are the orthogonal groups $O(n)$ and their connected components $SO(n)$ for $n \in \mathbb{N}$, $n \geq 1$. The group $O(n)$ consist of those elements A of $GL(n,\mathbb{R})$ which preserve the standard scalar product on $\mathbb{R}^n$. This can be expressed by the equation $A^t A = 1$. The manifold structure on $O(n)$ is defined by its presentation as a submanifold of $GL(n,\mathbb{R})$.

Similarly, the unitary group $U(n)$ is defined as the subgroup of elements A of $GL(n,\mathbb{C})$ which preserve the standard hermitean scalar product on $\mathbb{C}^n$. This can be expressed by the equation $A^* A = 1$. The special unitary groups $SU(n)$ is the subgroup of $U(n)$ defined by the additional condition that $\det(A) = 1$.

In a similar way one can define other Lie groups as subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ preserving natural geometric structures on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Examples of **abelian Lie groups** are $\mathbb{R}^n$ or the tori $T^n := (S^1)^n$. $\square$

One defines the notion of an **action** of a Lie group on a manifold using the language of the category **Mf**.

Example 1.7. Examples of actions are

1. the action of a Lie group G on itself by right or left multiplication,
2. the linear action of $O(n)$ on $\mathbb{R}^n$ and the induced action S^{n-1} or $Gr(k,\mathbb{R}^n)$,
3. the linear action of $U(n)$ on $\mathbb{C}^n$ and the induced action on $\mathbb{CP}^{n-1}$ or $Gr(k,\mathbb{C}^n)$.

$\square$

An action $a : G \times M \to M$ is **proper** if the map $(a, \mathtt{pr}_M) : G \times M \to M \times M$ is a proper map of the underlying topological spaces, i.e. it has the property that preimages of compact subsets are compact.

An action is called **free**, if for every $g \in G$ with $g \neq 1$ the subset of fixed points $M^g := \{m \in M \mid gm = m\}$ is empty.

Example 1.8. These examples illustrate the notions of proper and freeness of an action and demonstrate their independence.

1. Any action of a compact group is always proper. The action of a closed subgroup $H \subset G$ of a Lie group G on G by left- or right-multiplication is free. It is proper if and only if H is closed.
2. We consider the vector field on the torus T^2 given by $\partial_1 + \theta\partial_2$ in the natural coordinates, where $\theta \in \mathbb{R}$. Its flow is an action of $\mathbb{R}$. It is given by $(t, [x, y]) \mapsto [x + t, y + \theta t]$, where we write the points in T^2 as classes $[x, y] \in \mathbb{R}^2/\mathbb{Z}^2$. If θ is irrational, then the action is free, but not proper.
3. The trivial action of a non-trivial finite group on a manifold M is proper, but not free.
4. The action of $\mathbb{R}$ on S^1 given by $(t, u) \mapsto \exp(2\pi i t)u$ is neither free nor proper.

□

An action of a Lie group G on a manifold M induces an equivalence relation encoded in the **equalizer diagram**

$$G \times M \rightrightarrows M \ ,$$

where the two arrows are the projection and the action. If the equalizer exists, then it is called the **quotient** of M by the action of G and usually denoted by M/G.

Theorem 1.9. *If a Lie group G acts properly and freely on a manifold M, then the quotient M/G exists in* **Mf**.

In general we know that a surjective submersion $M \to X$ presents X as a quotient of M with respect to the equivalence relation $M \times_X M \rightrightarrows M$. So if we have a candidate $M \to X$ for the quotient of a proper free action, then we must only verify that this map is a surjective submersion and that the natural map

$$M \times G \to M \times_X M \ , \quad (m, g) \mapsto (m, a(g, m))$$

induces an isomorphism of equalizer diagrams.

Example 1.10. Examples of manifolds defined by forming quotients are $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ or $\mathbb{CP}^n \cong S^{2n+1}/U(1)$. $\square$

Example 1.11. Let $n \in \mathbb{N}$. If M is a manifold, then we can consider the space of ordered n-tuples of pairwise distinct points in n. It is an open submanifold

$$Conf_n^{ord}(M) \subseteq M^{\times n} := \underbrace{M \times \cdots \times M}_{n\times}$$

defined as the complement of the closed subset

$$\{(x_1, \dots x_n) \in M^{\times n} \mid (\exists i, j \in \{1, \dots, n\} \mid x_i = x_j \text{ and } i \neq j)\} \ .$$

The permutation group Σ_n acts freely on $Conf_n^{ord}(M)$ by

$$\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) \ , \quad \sigma \in \Sigma_n \ .$$

The **configuration space** of n-points in M is defined by

$$Conf_n(M) := Conf_n^{ord}(M)/\Sigma_n \ .$$

It is a interesting and in general complicated problem to understand the topology of the configuration spaces of manifolds.

We have $Conf_1(\mathbb{R}^n) \cong \mathbb{R}^n$. We now analyse the next case $Conf_2(\mathbb{R}^n)$. We have a diffeomorphism $Conf_2^{ord}(\mathbb{R}^n) \cong \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ which maps (x_1, x_2) to $(\frac{x_1+x_2}{2}, x_2 - x_1)$. The first entry is called the center of mass of the configuration. The map is Σ_2-equivariant, if we define the action of the non-trivial element in Σ_2 on the target by $(a, b) \mapsto (a, -b)$. Hence

$$Conf_2(\mathbb{R}^n) \cong \mathbb{R}^n \times (\mathbb{R}^n/(\mathbb{Z}/2\mathbb{Z})) \ ,$$

where $\mathbb{Z}/2\mathbb{Z}$ acts by reflection at the origin. This space is homotopy equivalent to $\mathbb{RP}^{n-1}$. $\square$

Example 1.12. Many interesting manifolds parametrizing geometric objects arise as **quotients of Lie groups** by the action of subgroups.

1. For example, for $k, n \in \mathbb{N}$, $1 \leq k \leq n-1$ the **Grassmann manifold** $Gr(k, \mathbb{R}^n)$ of k-dimensional subspaces in $\mathbb{R}^n$ can be presented as $O(n)/O(k) \times O(n-k)$.

To this end we observe that every k-dimensional subspace can be written in the form $A\mathbb{R}^k$ for some $A \in O(n)$. This gives a surjective map $O(n) \to Gr(k, \mathbb{R}^n)$. The block-diagonally embedded subgroup $O(k) \times O(n-k)$ is the stabiliser of the subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$. One checks that $O(n) \to Gr(k, \mathbb{R}^n)$ has the universal properties of the quotient.

2. Another example is the manifold of all **complex structures** on $\mathbb{R}^{2n}$ which can be presented as $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. To this end we fix the standard complex structure J_0 on $\mathbb{R}^{2n}$ given by the standard identification $\mathbb{R}^{2n} \cong \mathbb{C}$. Every other complex structure can be written in the form $A^{-1}J_0A$ for some $A \in GL(2n, \mathbb{R})$. The subgroup $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ is the stabilizer of J_0.

3. The manifold of **symmetric bilinear forms** of index (p, q) on $\mathbb{R}^{p+q}$ is presented as $SL(n, \mathbb{R})/SO(p, q)$, while the manifold of all orthogonal splittings of such a form into a positive and negative definite part is $O(p, q)/O(p) \times O(q)$.

4. The sphere S^{n-1} can be identified with the manifold of rays in $\mathbb{R}^n$ and presented as $SO(n)/SO(n-1)$.

$\square$

Many of these examples are related by **locally trivial fibre bundles**. In general, for a free and proper action of a Lie group G on M we have a bundle $M \to M/G$ with fibre G. The importance of recognizing manifolds as total spaces of fibre bundles is that many aspects of their topology can be understood in terms of the topology of base and fibre which have smaller dimensions and are often simpler.

In many cases we can apply the following theorem in order to detect fibre bundles.

Theorem 1.13. *A proper submersion is a locally trivial fibre bundle.*

Example 1.14. Here are examples of fibre bundles

1. $SO(n) \to S^{n-1}$ with fibre $SO(n-1)$
2. $S^{2n+1} \to \mathbb{CP}^n$ with fibre $U(1)$
3. $\mathbb{R}^n \to T^n$ with fibre $\mathbb{Z}^n$
4. $U(n+1) \to \mathbb{CP}^n$ with fibre $U(n) \times U(1)$.

□

Let M be a manifold with boundary N. Then we can choose an embedding

$$c : (-1, 0] \times N \to M$$

which identifies $\{0\} \times N$ with N. It is called a **collar**. If such a collar is chosen, then we say that N is (presented as) a right boundary. Similarly, if the collar is given by $c : [0, 1) \times N \to N$, then we say that N is a left boundary. The projection from the collar to $[0, 1)$ gives a boundary defining function which is ofter called the **normal coordinate**.

Assume that N is a right boundary of M. Let now M' be a second manifold with left boundary N' and collar $c' : [0, 1) \times N' \to M'$ and $f : N \to N'$ be a diffeomorphism. Then we can form a new manifold $M \cup_f M'$ (often denoted by $M \cup_N M'$) called the **connected sum of M and M' along the boundary N**. Its underlying topological space is the quotient of $M \sqcup M'$ by the relation $c(0, n) \sim c'(0, f(n))$. The smooth structure is defined such that the union of the two collars is diffeomorphic to $(-1, 1) \times N$ via $c(t, n) \mapsto (t, n)$ for $t \leq 0$ and $c'(t, f(n)) \mapsto (t, n)$ otherwise.

Let M be a manifold with boundary N. Then we present N as a right boundary by choosing a collar c. It induces a presentation of N as a left boundary by setting $c'(t, n) := c(-t, n)$. We can now form a manifold without boundary $M \cup_{\mathtt{id}_N} M$ called the **double of M along N**.

The construction of the connected sum along the boundary involves the choice of the collars. By the following theorem these choices do not influence the resulting diffeomorphism type.

Theorem 1.15. *In the constructions above the isomorphism classes of $M \cup_f M'$ or the double $M \cup_{\mathtt{id}_N} M$ do not depend on the choice of the collars*

Example 1.16.
1. We can present the sphere S^n as a double of a disc D^n so that the two copies of the disc correspond to the lower and upper hemispheres, and the boundary gives rise to the equator.
2. We can present torus T^n as a double of $I \times T^{n-1}$.

□

Assume that we are given an embedding $i : S^k \times D^{n-k} \to M$ as a codimension zero submanifold with boundary. Then $M \setminus i(S^k \times \mathrm{int}(D^{n-k}))$ is a manifold with boundary diffeomorphic to $S^k \times S^{n-k-1}$. We can form

$$M' := M \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1} .$$

We say that M' is obtained from M by a **surgery in codimension** $n-k$ **along** i.

For example, assume that M has two connected components M_0 and M_1. Then, using charts, we can find embedded discs $D^n \hookrightarrow M_i$ for $i = 0, 1$. We consider this data as an embedding $S^0 \times D^n \to M$. If we do surgery on this datum, then we get a connected manifold M' usually called the **connected sum** $M_0 \sharp M_1$. One can check that up to diffeomorphism it does not depend on the choices.

Example 1.17. For $g \in \mathbb{N}$, $g \geq 1$ we can form a connected sum Σ^g of g copies of T^2. We set $\Sigma^0 := S^2$. We have the following classification of surfaces.

Theorem 1.18. *Let Σ be a compact connected surface. Then there exists a unique $g \in \mathbb{N}$ called the **genus of** Σ such that Σ is isomorphic to:*

1. *Σ^g if Σ is orientable,*
2. *$\Sigma^g \sharp \mathbb{RP}^2$ if is Σ is not orientable.*

□

The following theorem is one starting point for the classification of closed manifolds.

Theorem 1.19. *Every closed manifold of dimension n can be obtained from S^n by a sequence of surgeries.*

1.2 Recap of the basic definitions of de Rham cohomology

Let M be a smooth manifold. By

$$\Omega(M) \, : \, 0 \to \Omega^0(M) \stackrel{d}{\to} \Omega^1(M) \stackrel{d}{\to} \Omega^2(M) \stackrel{d}{\to} \dots$$

we denote the **de Rham complex** of M consisting of real-valued smooth differential forms. A form $\omega \in \Omega^k(M)$ is called **closed**, if $d\omega = 0$, and it is called **exact**, if there exists $\alpha \in \Omega^{k-1}(M)$ such that $\omega = d\alpha$.

Example 1.20. 1. The 1-form $\omega := xdy + ydx$ on $\mathbb{R}^2$ is closed. Indeed,

$$d\omega = d(xdy + ydx) = dx \wedge dy + dy \wedge dx = 0 \ .$$

It is in fact exact: $d(xy) = \omega$.

2. The form $\omega := dt$ on S^1 (we parametrize S^1 by $t \mapsto e^{2\pi i t}$) is closed, but not exact. If it would be, say $\omega = df$, then

$$1 = \int_{S^1} dt \overset{Stokes}{=} \int_{\partial S^1} f = 0 \ .$$

Note that the parameter t does not give a smooth function on all of S^1 since it jumps at 1.

3. If f is a complex valued function defined on some open subset of $\mathbb{C}$, then fdz is a (complex valued) form. The function f is holomorphic exactly if fdz is closed. Indeed, we have

$$d(fdz) = (\partial_z f dz + \partial_{\bar z} f d\bar z) = \partial_{\bar z} f \ \partial \bar z \wedge \partial z \ .$$

□

Since $d \circ d = 0$, an exact form is closed. The converse is not true in general. In order to formalize this effect one introduces the $\mathbb{R}$-vector spaces

$$H^*_{dR}(M) := H^*(\Omega(M))$$

called the **de Rham cohomology** of M. In detail, for $k \in \mathbb{Z}$ de k'th de Rham cohomology is the $\mathbb{R}$-vector space

$$H^k_{dR}(M) := \frac{\mathtt{ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathtt{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} \ .$$

A closed k-form ω is exact iff its class $[\omega] \in H^k_{dR}(M)$ vanishes.

The number

$$b^k(M) := \dim_{\mathbb{R}} H^k_{dR}(M) \in \mathbb{N} \cup \{\infty\}$$

is called the **k'th Betti number** of M. The Betti numbers are the most basic invariants of M defined through de Rham cohomology.

Example 1.21. We have

$$H^0_{dR}(M) \cong \{f \in C^\infty(M, \mathbb{R}) \mid f \text{ is locally constant}\} \ .$$

Let $\pi_0(M)$ denote the set of connected components of M. Then

$$H^0_{dR}(M) \cong \mathbb{R}^{\pi_0(M)} \ .$$

The zeroth Betti number $b^0(M)$ is equal to the number of connected components of M. □

Example 1.22. In this example we list calculations of Betti numbers of various manifolds. One goal of this course is to develop the methods to do these calculations.

1. For $n \geq 1$ we have
$$b^i(S^n) = \begin{cases} 1 & i = 0, n \\ 0 & else \end{cases}$$

2. For $n \geq 1$, we have
$$b^i(\mathbb{RP}^{2n+1}) = \begin{cases} 1 & i = 0, 2n+1 \\ 0 & else \end{cases}$$

3. For $n \geq 1$, we have
$$b^i(\mathbb{RP}^{2n}) = \begin{cases} 1 & i = 0 \\ 0 & else \end{cases}$$

4. For $n \geq 1$ and $i \in \mathbb{Z}$ we have
$$b^i(T^n) = \begin{pmatrix} n \\ i \end{pmatrix} \ .$$

5. For $n \geq 1$ we have
$$b^i(\mathbb{CP}^n) = \begin{cases} 1 & i = 0, 2, \ldots, 2n \\ 0 & else \end{cases}$$

6. For a connected orientable surface Σ of genus g we have
$$b^i(\Sigma) = \begin{cases} 1 & i = 0, 2 \\ 2g & i = 1 \\ 0 & else \end{cases}$$

7. For a connected non-orientable surface Σ of genus g we have
$$b^i(\Sigma) = \begin{cases} 1 & i = 0 \\ 2g & i = 1 \\ 0 & else \end{cases}$$

$\square$

A smooth map $f : M \to N$ induces a map of complexes $f^* : \Omega(N) \to \Omega(M)$ and therefore a map between cohomology groups $H_{dR}(f) : H^*_{dR}(N) \to H^*_{dR}(M)$. It is given by application of f^* to representatives
$$H_{dR}(f)([\omega]) := [f^*\omega] \ .$$
From now one we write f^* instead of $H_{dR}(f)$. These constructions are functorial, i.e. we have the rule $(g \circ f)^* = f^* \circ g^*$ for composeable smooth maps f and g. The Rham cohomology group thus constitutes a functor H^*_{dR} from $\mathbf{Mf}^{op}$ to the category of $\mathbb{Z}$-graded real vector spaces.

The $\wedge$-product turns $\Omega(M)$ into a **commutative differential graded algebra**. Consequently, the de Rham cohomology $H^*_{dR}(M)$ is a graded commutative algebra whose product will be denoted by $\cup$. The product in cohomology is given in terms of representatives by
$$[\alpha] \cup [\omega] = [\alpha \wedge \omega] \ .$$
The pull-back operations f^* on the level forms and cohomology are compatible with the products. So the functor H^*_{dR} actually takes values in graded commutative $\mathbb{R}$-algebras

Example 1.23. In the following we present the structure of de Rham cohomology as a ring in a number of examples.

1. $H^*_{dR}(S^n) \cong \mathbb{R}[z]/(z^2)$, where z is a generator in degree n.
2. $H^*_{dR}(\mathbb{RP}^{2n+1}) \cong \mathbb{R}[z]/(z^2)$, where z is a generator in degree n.
3. $H^*_{dR}(\mathbb{RP}^{2n}) \cong \mathbb{R}$.
4. $H^*_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[z]/(z^{n+1})$, where z is a generator in degree 2.
5. The product $S^2 \times S^4$ has the same Betti numbers as $\mathbb{CP}^3$. So the de Rham cohomology groups are isomorphic. But the ring structures are different: We have $H^*_{dR}(S^2 \times S^4) \cong \mathbb{R}[x, y]/(x^2, y^2)$ with $|x| = 2$ and $|y| = 4$.

6. $H^*_{dR}(T^n) \cong \Lambda^* H^1_{dR}(T^n)$.

□

Let $f : I \times M \to N$ be a smooth map and $f_i : M \to N$, $i = 0, 1$ be the restrictions of f to the boundary faces of the interval. The map f is called a **homotopy** from f_0 to f_1. If we are given f_0 and f_1, then we say that these maps are homotopic if such a map f as above exists. We have the **homotopy formula**

$$f_1^* - f_0^* = dh + hd : \Omega(N) \to \Omega(M) \ , \tag{1}$$

where $h : \Omega(N) \to \Omega(M)$ is given by the degree -1-map

$$h(\omega) := \int_{[0,1]} \iota_{\partial_t}(f^*\omega)_{|\{t\times M\}} dt \ .$$

The homotopy formula implies that

$$f_1^* = f_0^* : H^*_{dR}(N) \to H^*_{dR}(M) \ ,$$

i.e. the de Rham cohomology functor is **homotopy invariant**. Indeed, if $d\omega = 0$, then

$$f_1^*\omega - f_0^*\omega = dh\omega \ .$$

A map $f : M \to N$ is called a **homotopy equivalence** if there exists a map $g : N \to M$ called **inverse up to homoptopy** such that $f \circ g$ is homotopic to $\mathtt{id}_M$ and $g \circ f$ is homotopic to $\mathtt{id}_N$. In this case $g^* : H^*_{dR}(M) \to H^*_{dR}(N)$ is inverse to $f^* : H^*_{dR}(N) \to H^*_{dR}(M)$. In particular, f^* is an isomorphism.

Example 1.24. The inclusion $f : * \to \mathbb{R}^n$ of the origin is a homotopy equivalence. In fact, the inverse up to homoptopy $g : \mathbb{R}^n \to *$ is the unique map. We have $g \circ f = \mathtt{id}_*$ and $h(t, x) := tx$ is a homotopy from $f \circ g$ to $\mathtt{id}_{\mathbb{R}^n}$. Consequently

$$\mathbb{R} = H^*_{dR}(*) \cong H^*_{dR}(\mathbb{R}^n) \ ,$$

where we consider $\mathbb{R}$ as a graded commutative algebra in the natural way. □

Example 1.25. The inclusion $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence. An inverse up to homoptopy is given by $g : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$, $g(x) := \frac{x}{\|x\|}$. We have $g \circ f = \mathtt{id}_{S^n}$ and $h(t, x) := tx + (1-t)\frac{x}{\|x\|}$ is a homotopy from $f \circ g$ to $\mathtt{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ □

Example 1.26. One can use de Rham cohomology in order to show fixed point theorems, for example the **Brouwer fixed point theorem**.

Theorem 1.27. *If $f : D^n \to D^n$ is a smooth map, then it has a fixed point.*

Proof. We argue by contraction. Assume that f has no fixed point. Then we construct $F : D^n \to S^{n-1}$ such that $F(x)$ is the intersection of the ray with S^{n-1} starting in $f(x)$ and going through x.

In order to see that F is smooth we argue as follows. We consider the pull-back

$$\begin{tikzcd} U \arrow[r] \arrow[d] & * \arrow[d, "1"] \\ D^n \times [0,\infty) \arrow[r] & \mathbb{R} \end{tikzcd} ,$$

where the lower horizontal map is $(x,t) \mapsto \|f(x)+t(x-f(x)\|^2$. We check transversality: The derivative of the lower horizontal map with respect to t is

$$2\langle f(x), x-f(x)\rangle + 2t\|x-f(x)\|^2 \ . \tag{2}$$

If $\|f(x)+t(x-f(x)\|^2 = 1$, then this is non-zero. If it would be zero, then (for the first transition we multiply by t and add and substract $\|f(x)\|^2$)

$$\begin{aligned} 0 &= 2t\langle f(x), x-f(x)\rangle + 2t^2\|x-f(x)\|^2 \\ &= \|f(x)+(t(x-f(x))\|^2 - \|f(x)\|^2 + t^2\|x-f(x)\|^2 \\ &= \big(1-\|f(x)\|^2\big) + t^2\|x-f(x)\|^2 \ . \end{aligned}$$

Since $\|f(x)\|^2 \le 1$ we conclude that both summand must vanish separately, and therefore that $t=0$ and $\|f(x)\| = 1$. But then from (2) we would have

$$\langle f(x), x\rangle = \|f(x)\|^2 = 1 \ ,$$

and since x and $f(x)$ belong to D^n, also $f(x) = x$, a contradiction.

The map $U \to S^{n-1}$ given by the projection to $D^n \times [0,\infty)$ and application of $f(x)+t(x-f(x))$ is the required map F.

The composition $S^{n-1} \xrightarrow{i} D^n \xrightarrow{F} S^{n-1}$ is the identity. Hence

$$F^*i^* = \mathtt{id} : H^{n-1}_{dR}(S^{n-1}) \to H^{n-1}_{dR}(S^{n-1})$$

is an isomorphism of a non-trivial $\mathbb{R}$-vector space. But $H^{n-1}_{dR}(D^n) = 0$ and thus $i^* = 0$. □

Remark 1.28. In the theorem it suffices to assume that f is continuous. But then the proof must be modified.

□

Example 1.29. We have seen in Example 1.11 that

$$Conf_2(\mathbb{R}^n) \cong \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/\mathbb{Z}/2\mathbb{Z} \ .$$

This space is homotopy equivalent to $\mathbb{RP}^{n-1}$. To see this we represent $\mathbb{RP}^{n-1}$ as quotient $S^{n-1}/\mathbb{Z}/2\mathbb{Z}$. Then we define the map $f : Conf_2(\mathbb{R}^n) \to \mathbb{RP}^{n-1}$ by $f(a, [b]) := [b/\|b\|]$. An inverse up to homotopy is given by $g([u]) := (0, [u])$. Note that $f \circ g = \mathtt{id}_{\mathbb{RP}^{n-1}}$. A homotopy from $g \circ f$ to $\mathtt{id}_{Conf_2(\mathbb{R}^n)}$ is given by

$$h(t, a, [b]) := (ta, [tb + (1-t)b/\|b\|]) \ .$$

We conclude that (using the still unproven calculation of the cohomology of $\mathbb{RP}^{n-1}$)

$$H^*_{dR}(Conf_2(\mathbb{R}^n)) \cong H^*_{dR}(\mathbb{RP}^{n-1}) \cong \left\{ \begin{array}{cc} \mathbb{R}[z]/(z^2) \ , |z| = n-1 & n \text{ even} \\ \mathbb{R} & n \text{ odd} \end{array} \right. .$$

□

If M is an oriented manifold and $A \subseteq M$ is a precompact Borel measurable subset, then we have an **integration**

$$\int_A : \Omega(M) \to \mathbb{R} \ .$$

In particular, if $A \subseteq M$ is a compact codimension zero submanifold with boundary ∂A, then we have **Stokes' theorem**

$$\int_A d\omega = \int_{\partial A} \omega \ .$$

Here ∂A has the induced orientation. We represent orientations by nowhere vanishing forms of maximal degree. If $\nu \in \Omega^{\dim(M)}$ represents the orientation of M, then

$$(\iota_n \nu)_{|\partial A} \in \Omega^{\dim(M)-1}(\partial A)$$

represents the orientation of ∂A, where n is some outward pointing normal vector field on ∂A.

Example 1.30. In this example we illustrate the induced orientation on a boundary. We consider the sphere $S^{n-1} \subset \mathbb{R}^n$. We equip $\mathbb{R}^n$ with the standard orientation represented by $\mathrm{vol}_{\mathbb{R}^n} = dx^1 \wedge \cdots \wedge dx^n$. We consider the chart of the upper hemisphere $\{x^n > 0\} \cap S^{n-1} \to \mathbb{R}^{n-1}$ given by the projection along the last coordinate. We ask when this chart is **compatible with the orientation**.

The outward pointing unit normal vector at the north pole $N := (0, \ldots, 0, 1) \in S^{n-1}$ is ∂_n. Therefore $(i_{\partial_n} \mathrm{vol}_{\mathbb{R}^n})(N) \in \Lambda^{n-1}(T^*_N S^{n-1})$ represents the orientation in this point. In the chart it is sent to $(-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1}$. Hence this chart is compatible with the orientation exactly if n is odd.

$\square$

If M is a closed oriented manifold, then the integral induces a homomorphism

$$\int_{[M]} : H^{\dim(M)}_{dR}(M) \to \mathbb{R} \ , \quad [\omega] \mapsto \int_M \omega \ .$$

Example 1.31. Let $\nu \in \Omega^{\dim(M)}(M)$ represent the orientation of a closed manifold M. Then we have $[\nu] \neq 0$ in $H^{\dim(M)}_{dR}(M)$. In fact, $\int_{[M]} [\nu] > 0$. For example, we have $H^n_{dR}(S^n) \not\cong 0$. Using the module structure of $H^n_{dR}(M)$ over $H^0_{dR}(M)$ one can show that for a compact oriented M we have $b^n(M) \geq b^0(M)$.

For example, the volume form of the sphere vol_{S^n} represents a non-trivial class in $H^n_{dR}(S^n)$. $\square$

Example 1.32. Consider $r, n \in \mathbb{N}$ such that $0 \leq r \leq n$. We consider the index set

$$J_r := \{(i_1, \ldots, i_r) \in \{1, \ldots, n\}^r \mid 1 \leq i_1 < \cdots < i_r \leq n\} \ .$$

For $i := (i_1, \ldots, i_r) \in J_r$ we define $dt^i := dt^{i_1} \wedge \cdots \wedge dt^{i_r}$. We show that the set of forms

$$\{dt^i \in \Omega^r(T^n) \,|i \in J_r\}$$

represents a linearly independent subset of $H^r_{dR}(T^n)$ (in fact a basis, but this can not be shown here).

To this end for $j \in J_r$ we consider the map $f_j : T^r \to T^n$ given by sending $(s_1, \ldots, s_r)$ to $(1, \ldots, s_1, \ldots, s_r, \ldots, 1)$, where s_i is put in place j_i.

Then $f_j^* dt^i = \mathrm{vol}_{T^r}$ if the indices match, and zero else. We get for all $i, j \in J_r$ that

$$\int_{[T^n]} f_j^*[dt^i] = \delta^i_j \ .$$

□

1.3 Recap on basic homological algebra

Let C^* be a **cohomological chain complex** of abelian groups. In detail

$$C : \cdots \xrightarrow{d} C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} \dots \ .$$

It can equivalently be considered as a homological chain complex

$$\cdots \xrightarrow{d} C_{1-n} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots$$

by setting $C_{-n} := C^n$ for all $n \in \mathbb{N}$.

Example 1.33. 1. The de Rham complex $\Omega(M)$ of a smooth manifold M is an example.

2. For $n \in \mathbb{Z}$ and an abelian group A we can form the chain complex $S^n(A)$ whose only non-trivial entry is A in degree n.

□

A **morphism of chain complexes** $C \to D$ is a commutative diagram

$$\begin{tikzcd} \dots \arrow[r, "d"] & C^n \arrow[r, "d"] \arrow[d] & C^{n+1} \arrow[r, "d"] \arrow[d] & \dots \\ \dots \arrow[r, "d"] & D^n \arrow[r, "d"] & D^{n+1} \arrow[r, "d"] & \dots \end{tikzcd} \ .$$

If f is a symbol for the morphism, then we let f_n be the symbol for its component in degree n.

We get a category of chain complexes **Ch** and morphisms of chain complexes.

For $n \in \mathbb{Z}$ the degree n-**cohomology** of a chain complex C is the abelian group defined by
$$H^n(C) := \frac{\mathtt{ker}(d : C^n \to C^{n+1})}{\mathtt{im}(d : C^{n-1} \to C^n)} \ .$$
A standard notation for cohomology classes is $[x] \in H^n(C)$, where $x \in C^n$ is a cycle, i.e. $dx = 0$ and $[-]$ denotes the class in the quotient by the image of d. If $f : C \to D$ is a morphism of chain complexes, then we define a map
$$H(f) : H^n(C) \to H^n(F) \ , \quad H(f)[x] := [f_n(x)] \ .$$
It is well-defined and functorial. Hence we can consider H^n as a functor $\mathbf{Ch} \to \mathbf{Ab}$. We often write $f_* := H(f)$.

A morphism between chain complexes is called a **quasi isomorphism** if it induces an isomorphism in cohomology groups.

Example 1.34. 1. We have a quasi-isomorphism
$$f : S^0(\mathbb{Z}) \xrightarrow{\simeq} (\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$$
such that $f_0 : \mathbb{Z} \to \mathbb{Q}$ is the inclusion.

2. We have a quasi-isomorphism
$$0 \simeq (\mathbb{Z} \xrightarrow{\mathtt{id}} \mathbb{Z}) \ .$$

Example 1.35. If A is an abelian group and (C, d) is a chain complex, then we can form new chain complexes
$$C \otimes A \ , \quad \mathtt{Hom}(C, A) \ , \quad \mathtt{Hom}(A, C) \ .$$
In the first case for $c \otimes a \in (C \otimes A)^i := C^i \otimes A$ we set $d(c \otimes a) := dc \otimes a$. In the second case, $\mathtt{Hom}(C, A)^i := \mathtt{Hom}(C^{-i}, A)$ and $d\phi := (-1)^i \phi \circ d$. Finally, for $\phi \in \mathtt{Hom}(A, C)^i := \mathtt{Hom}(A, C^i)$ we define $d\phi := d \circ \phi$.

The starting point of homological algebra is the observation that in general these **operations do not preserve quasi-isomorphisms**.

We consider $A := \mathbb{Z}/3\mathbb{Z}$ and the quasi-isomorphism $f : S^0(\mathbb{Z}) \to (\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}))$. Then
$$f \otimes A \cong S^0(\mathbb{Z}/3\mathbb{Z}) \to 0$$

which is not a quasi-isomorphism. The morphism

$$\mathtt{Hom}(\mathbb{Z}/3\mathbb{Z}, f) \cong 0 \to (0 \to \mathbb{Z}/3\mathbb{Z}) \ ,$$

again not a quasi-isomorphism.

The idea of derived functors is to improve this situation and to define versions of these operations which preserve quasi-isomorphisms.

□

A **homotopy between morphisms** $f_0, f_1 : C \to D$ of chain complexes as a map $h : C \to D$ of degree -1 such that

$$hd + dh = f_1 - f_0 \ .$$

In this case

$$f_{0,*} = f_{1,*} : H^*(C) \to H^*(D) \ .$$

In particular, if $f = dh + hd$, then $f_* = 0$, and if $f - \mathtt{id} = dh + hd$, then f is a quasi-isomorphism.

Remark 1.36. Note that the operations discussed in 1.35 preserve homotopy equivalences. This shows that homotopy equivalence is a strictly stronger notion of equivalence than quasi-isomorphism. □

Example 1.37. If $f : M \to N$ is a morphism of manifolds, then $f^* : \Omega(N) \to \Omega(M)$ is a morphism of chain complexes. If f is a homotopy equivalence, then f^* is a quasi isomorphism, in fact a homotopy equivalence. A homotopy $f : I \times M \to M$ between f_0 and f_1 induces a homotopy between f_0^* and f_1^*. See (1). □

An **exact sequence** of chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a sequence of morphisms of chain complexes such that for every $n \in \mathbb{Z}$ the sequence of abelian groups

$$0 \to A^n \xrightarrow{f_n} B^n \xrightarrow{g_n} C^n \to 0$$

is exact. In this case we have the **long exact sequence** in cohomology

$$\dots \to H^n(A) \xrightarrow{f_*} H^n(B) \xrightarrow{g_*} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \to \dots \ .$$

The map ∂ is called the **boundary operator**. Explicitly, it is given by

$$\partial[c] = [db] \ , \tag{3}$$

where $b \in B^n$ is a lift of c and we observe that $db \in A^n$.

The boundary operator depends naturally on the exact sequence. A morphism between short exact sequences is a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd} \ .$$

Let ∂ and ∂' be the associated boundary operators. Then we have the relation

$$\partial' \circ h_* = f_* \circ \partial \ .$$

A map of short exact sequences induces a morphism between long exact sequence.

Example 1.38. We consider the chain complex

$$A : 0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0 \ .$$

We form the exact sequence of chain complexes

$$0 \to \mathtt{Hom}(A, \mathbb{Z}) \to \mathtt{Hom}(A, \mathbb{R}) \to \mathtt{Hom}(A, \mathbb{R}/\mathbb{Z}) \to 0 \ .$$

The boundary operator induces an isomorphism

$$H^{-1}(\mathtt{Hom}(A, \mathbb{R}/\mathbb{Z})) \xrightarrow{\partial, \cong} H^0(\mathtt{Hom}(A, \mathbb{Z})) \ .$$

This can be verified by an explicit calculation. Let $[\phi] \in H^{-1}(\mathtt{Hom}(A, \mathbb{R}/\mathbb{Z}))$ be represented by $\phi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. The condition $d\phi = 0$ says that ϕ has values in $\frac{1}{5}\mathbb{Z}/\mathbb{Z}$. We choose a lift $\tilde{\phi} : \mathbb{Z} \to \mathbb{R}$ of ϕ. Such a lift is fixed by the choice of $\tilde{\phi}(1) \in \mathbb{R}$. Then $\partial[\phi] = [5\tilde{\phi}]$. The homomorphism $5\tilde{\phi}$ has values in $\mathbb{Z}$. Its class is zero if $5\tilde{\phi}(1) \in 5\mathbb{Z}$. This is exactly the case when $\phi = 0$. In the other direction, given a class $[\kappa] \in H^0(\mathtt{Hom}(A, \mathbb{Z}))$ we can take $\phi := [\frac{1}{5}\kappa]$ as a preimage. This shows surjectivity of ∂. $\square$

If C is a chain complex, then we define the shift $C[n]$ by

$$C[n]^k := C^{n+k} \ , \quad \text{with differential } (-1)^n d \tag{4}$$

Example 1.39. We have $S^n(A) \cong S^0(A)[-n]$. □

A basic tool of homological algebra is the **Five Lemma**:

Lemma 1.40. *Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a morphism between exact sequences of abelian groups. If a, b, d, e are isomorphisms, then c is an isomorphism, too.

Example 1.41. If $f : A \to B$ is a morphism of chain complexes, then we can form a new chain complex $\mathtt{Cone}(f) := A[1] \oplus B$ called the **cone** of f. Its differential is given by

$$d(a, b) = (-da, -f(a) + db) \ .$$

We define $H^*(f) := H^*(\mathtt{Cone}(f))$. We have an obvious exact sequence

$$0 \to B \to \mathtt{Cone}(f) \to A[1] \to 0$$

and a long exact sequence

$$\cdots \to H^n(B) \to H^n(f) \to H^{n+1}(A) \stackrel{f_*}{\to} H^{n+1}(B) \to \dots \ .$$

If f is injective, then we have a quasi isomorphism $\operatorname{coker}(f) \simeq \mathtt{Cone}(f)$ given by

$$\mathtt{Cone}(f) \to B/A \ , \quad (a, b) \to [b] \ .$$

This follows from the Five Lemma 1.40. A similar statement holds for surjective maps. See Example 3.1. □

Further aspects of homological algebra will be developed in Section 2.3.

1.4 The Mayer-Vietoris sequence

Let $U \cup V = M$ be a decomposition of a manifold into two open submanifolds.

Lemma 1.42. *The following sequence of complexes is exact:*

$$0 \to \Omega(M) \xrightarrow{\gamma \mapsto (\gamma_{|U}, \gamma_{|V})} \Omega(U) \oplus \Omega(V) \xrightarrow{(\alpha,\beta) \mapsto \alpha_{|U\cap V} - \beta_{|U \cap V}} \Omega(U \cap V) \to 0 \ . \qquad (5)$$

Proof. The only non-trivial place is the surjectivity of the second map. To see this we choose a partition of unity $\{\chi_U, \chi_V\}$ subordinated to the covering $\{U, V\}$ of M. If $\omega \in \Omega(U \cap V)$, then $\chi_U \omega$ is defined on $U \cap V$ and vanishes identically near $\partial \bar{U} \cap V$. Therefore it can be extended by zero to V. We will denote the extension still by $\chi_U \omega$. Similarly, $\chi_V \omega$ can be extended by zero to U. Therefore the second map in the sequence maps $(\chi_V \omega, -\chi_U \omega)$ to ω. □

The long exact sequence associated to the short exact sequence (5) of de Rham complexes is called the **Mayer-Vietoris sequence**:

$$\dots \to H^{n-1}_{dR}(U \cap V) \stackrel{\partial}{\to} H^n_{dR}(M) \to H^n_{dR}(U) \oplus H^n_{dR}(V) \to H^n_{dR}(U \cap V) \stackrel{\partial}{\to} H^{n+1}_{dR}(M) \to \dots$$

Here is a explicit formula for the **boundary operator**. It uses the explicit formula for the preimage of ω obtained in the proof of Lemma 1.42 and (3). We let β be the form given by

$$\beta_{|U} := d\chi_V \wedge \omega \ , \quad \beta_{|V} = -d\chi_U \wedge \omega \ . \qquad (6)$$

Then

$$\partial[\omega] = [\beta] \ .$$

Example 1.43. In this example we calculate $H^*_{dR}(S^n)$. We can write $S^n \cong \mathbb{R}^n \cup \mathbb{R}^n$, where the two copies of $\mathbb{R}^n$ are the complements of the north- and the southpoles which intersect in a manifold diffeomorphic to $\mathbb{R}^n \setminus \{0\}$. We can calculate $H^*_{dR}(S^n)$ using the Mayer-Vietoris sequence and induction. We claim that for $n \geq 1$ we have

$$H^k_{dR}(S^n) \cong \left\{ \begin{array}{cc} \mathbb{R} & k = 0, n \\ 0 & else \end{array} \right. \ .$$

Note that we have a homotopy equivalence $\mathbb{R}^n \setminus \{0\} \sim S^{n-1}$ (Example 1.25). We start with the case $n = 1$. The non-trivial segment of the MV-sequence for $n = 1$ is

$$0 \to H^0_{dR}(S^1) \to H^0_{dR}(\mathbb{R} \sqcup \mathbb{R}) \to H^0_{dR}(S^0) \to H^1_{dR}(S^1) \to 0 \ .$$

The middle map is, after natural identifications, given by $(x,y)\mapsto(x-y,x-y)$ and has rank one. This implies the claim for $n=1$. The beginning of the MV-sequence for $n\geq 2$ has the form

$$0\to H^0_{dR}(S^n)\to H^0_{dR}(\mathbb{R}^n)\oplus H^0_{dR}(\mathbb{R}^n)\to H^0_{dR}(\mathbb{R}^n)\to H^1_{dR}(S^n)\to 0\ .$$

The second map is $(x,y)\mapsto x-y$, hence surjective. We conclude $H^0_{dR}(S^n)\cong\mathbb{R}$ as expected since S^n is connected, and $H^1_{dR}(S^n)=0$. In higher degree the only non-trivial segment of the MV-sequence is

$$0\to H^{n-1}_{dR}(S^{n-1})\to H^n_{dR}(S^n)\to 0\ .$$

This gives $H^n_{dR}(S^n)\cong\mathbb{R}$ by induction.

Example 1.44. In this example we illustrate the explicit formula for the boundary operator (6). We consider the circle which we cover as before by the two hemispheres U,V. The intersection $U\cap V$ is the disjoint union of two intervals,

$$U\cap V\cong I^+\sqcup I^-=(-\pi,\pi)\sqcup(-\pi,\pi)+\pi$$

in the natural parametrization. The boundary operator maps the class $[(1,0)]\in H^0_{dR}(I^+\sqcup I^-)$ to $[\beta]\in H^1_{dR}(S^1)$, where $\beta\in\Omega^1(S^1)$ is given by

$$\beta_{|U\cap I^+}=d\chi_V\ ,\quad \beta_{|U\cap I^-}=0\ ,\quad \beta_{|V\cap I^+}=-d\chi_U\ ,\quad \beta_{|V\cap I^-}=0\ .$$

We have

$$\int_{S^1}\beta=\int_{I^+}d\chi_V=-1\ .$$

Therefore, $[\beta]=\partial[(0,1)]$ indeed represents the generator of $H^1_{dR}(S^1)$

Example 1.45. This is a higher-dimensional generalization of Example 1.44. We consider a codimension-one submanifold $N\subset M$ such that $M\setminus N$ has two connected components $M_\pm$. We can extend the embedding of N to an embedding of a collar $(-1,1)\times N\to M$ such that $M_-\cap(-1,1)\times N=(-1,0)\times N$ and define the open subsets $\tilde{M}_\pm:=M_\pm\cup(-1,1)\times N\subseteq M$. The inclusion $M_\pm\to\tilde{M}_\pm$ is a homotopy equivalence. Moreover, the inclusion $N\to\tilde{M}_+\cap\tilde{M}_-\cong(-1,1)\times N$ is a homotopy equivalence. We consider the covering $\{\tilde{M}_+,\tilde{M}_-\}$ of M. The Mayer-Vietoris sequence reads after the obvious identifications

$$\dots\to H^{k-1}_{dR}(N)\xrightarrow{\partial}H^k_{dR}(M)\to H^k_{dR}(M_+)\oplus H^k_{dR}(M_-)\to H^k_{dR}(N)\to\dots\ .$$

If $[\omega] \in H^{k-1}_{dR}(N)$, then $\partial[\omega] \in H^k_{dR}(M)$ is represented by the form α which is supported in $(-1,1) \times N$ and characterized by

$$\alpha_{|(-1,1)\times N} = d\chi_{\tilde{M}_+} \wedge \mathtt{pr}_N^* \omega \ .$$

Let us now assume that M is closed and oriented, and that N is closed and has the orientation induced as the boundary of $[0,1) \times N$. Assume that $k = \dim(M)$. Then we have

$$\int_{[M]} \partial[\omega] = \int_M \alpha = \int_{(-1,1)\times N} d\chi_{\tilde{M}_+} \wedge \mathtt{pr}_N^* \omega = (-1)^k \int_N \omega = (-1)^k \int_{[N]} [\omega] \ . \quad (7)$$

Consequently, the boundary operator $\partial : H^{\dim(N)}_{dR}(N) \to H^{\dim(M)}_{dR}(M)$ has at least rank one.

Actually, it has rank one since $b_{\dim(M)}(M) = 1$, but we have not shown this at this moment. $\square$

Example 1.46. Let $n \geq 2$. We consider an embedding $rD^n := \bigsqcup_r D^n \to S^n$ of r pairwise disjoint discs into the sphere and let S^n_r be the manifold with boundary $\partial S^n_r \cong \bigsqcup_{i=1}^r S^{n-1} =: rS^{n-1}$ obtained by removing the interior of the image of this embedding. We calculate the cohomology of S^n_r using the Mayer-Vietoris sequence. Its beginning is

$$0 \to H^0_{dR}(S^n) \to H^0_{dR}(S^n_r) \oplus \bigoplus_{i=1}^r H^0_{dR}(D^n) \to \bigoplus_{i=1}^r H^0_{dR}(S^{n-1}) \xrightarrow{\partial} \ .$$

By counting connected components we see that $\partial = 0$.

For $k \notin \{0, n-1, n\}$ we get

$$0 \cong H^k_{dR}(S^n) \cong H^k_{dR}(S^n_r) \ .$$

The remaining piece of the sequence is

$$0 \to H^{n-1}_{dR}(S^n_r) \xrightarrow{i} \bigoplus_{i=1}^r H^{n-1}_{dR}(S^{n-1}) \xrightarrow{\partial} H^n_{dR}(S^n) \to H^n_{dR}(S^n_r) \to 0 \ .$$

Since $b_n(S^n) = 1$ we see from Example 1.45 that ∂ has rank one. This gives $H^n_{dR}(S^n_r) = 0$ and $b_{n-1}(S^n_r) = r - 1$. For later calculations we must understand the kernel of ∂, i.e. the image of i. We equip all boundary components with the induced orientation from S^n. The choices of orientations induces via integration an isomorphism

$$\bigoplus_{i=1}^{r} H^{n-1}_{dR}(S^{n-1}) \cong \mathbb{R}^r \ .$$

In view of (7) we have $\partial x = 0$ iff $\sum_{i=1}^r x_i = 0$.

$\square$

Example 1.47. We now calculate the cohomology of the manifold M_r obtained from S^n by attaching r handles of dimension 0 (r surgeries of codimension n).

We can write M_r as a boundary sum along $2rS^{n-1}$ of S^n_{2r} with $r(D^1 \times S^{n-1})$. We get the Mayer-Vietoris sequence

$$\cdots \to \bigoplus_{i=1}^{r} H^{k-1}_{dR}(S^{n-1}) \xrightarrow{\partial} H^k(M_r) \to H^k_{dR}(S^n_{2r}) \oplus \bigoplus_{i=1}^{r} H^k_{dR}(S^{n-1}) \xrightarrow{!} \bigoplus_{i=1}^{2r} H^k_{dR}(S^{n-1}) \xrightarrow{\partial} \dots \ .$$

Note that we identify the degree $n-1$-cohomology of every component of ∂S^n_{2r} with $\mathbb{R}$ using some orientation orientation. For $k = n-1$ the second component of the map ! maps $(x_1, x_2, \dots) \in \mathbb{R}^r$ to $(\pm x_1, \mp x_1, \pm x_2, \mp x_2, \dots) \in \mathbb{R}^{2r}$. All these elements belong to the image of the first component $H^{n-1}_{dR}(S^n_{2r}) \to \mathbb{R}^{2r}$ of !. In order to see this note that the canonical identification of S^{n-1} with the boundary components of $[0,1] \times S^{n-1}$ induces orientations on these boundary components. One of them is compatible with the orientation induced from viewing the S^{n-1} as a boundary component of S^2_{2r}, and the other is not. We conclude that for $k = n-1$ the marked map has rank $2r - 1$.

We can now evaluate the Betti numbers. Assume first that $n \geq 3$.

$$H^*_{dR}(M_r) \cong \begin{cases} \mathbb{R} & * = 0 \\ \mathbb{R}^r & * = 1, n-1 \\ \mathbb{R} & * = n \\ 0 & else \end{cases} \ .$$

Similarly we get in the case $n = 2$ with $M_r = \Sigma_r$ the surface of genus r:

$$H^*_{dR}(\Sigma_r) \cong \begin{cases} \mathbb{R} & * = 0 \\ \mathbb{R}^{2r} & * = 1 \\ \mathbb{R} & * = 2 \\ 0 & else \end{cases} \ .$$

□

1.5 Coverings by a finite group

Let G be a finite group acting freely on a manifold M. Then we consider the quotient M/G. In this subsection we want to calculate the de Rham cohomology of M/G using the knowledge of the de Rham cohomology of M. Here is a list of examples:

1. For $n \geq 1$ we can represent the real projective spaces as $\mathbb{RP}^n \cong S^n/\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on the sphere by the antipodal map.
2. For coprime integers p, q we define the action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{C}^2$ by
$$[n](z_0, z_1) \mapsto (e^{2\pi i \frac{n}{p}} z_0, e^{2\pi i \frac{qn}{p}} z_1) \ .$$
This action preserves the unit sphere $S^3 \subset \mathbb{C}^2$ and has no fixed points there. The quotient
$$L(p, q) := S^3/\mathbb{Z}/p\mathbb{Z} \tag{8}$$
is called a **lense space** of type (p, q).
3. The configuration space of k pairwise distinct points in a manifold M can be written as a quotient
$$Conf_k(M) := Conf_k^{ord}(M)/\Sigma_k \ .$$

As a preparation consider an action $(g, v) \mapsto gv$ of a finite group G on a real vector space V by linear transformations. We define the subspace of G-invariant vectors
$$V^G := \{v \in V \mid (\forall g \in G \mid gv = v)\} \ .$$
We further define the linear endomorphism
$$P : V \to V \ , \quad P(v) := \frac{1}{|G|} \sum_{g \in G} gv \ .$$

Lemma 1.48. *The endomorphism P of V is a projection onto V^G.*

Proof. We calculate for $v \in V$ and $h \in G$ that

$$hP(v) = h\frac{1}{|G|}\sum_{g\in G} gv = \frac{1}{|G|}\sum_{g\in G} hgv = \frac{1}{|G|}\sum_{g\in G} gv = Pv \ ,$$

in particular we get $P(v) \in V^G$. If $v \in V^G$, then

$$P(v) = \frac{1}{|G|}\sum_{g\in G} gv = \frac{1}{|G|}\sum_{g\in G} v = v \ .$$

This implies $P(P(v)) = v$. Therefore P is a projection whose image is exactly V^G. □

We now come back to $\pi : M \to M/G$. The group G acts on the complex $\Omega(M)$.

Lemma 1.49. *The pull-back by π^* induces an isomorphism of complexes $\Omega(M/G) \cong \Omega(M)^G$.*

Proof. Since π is a surjective submersion the pull-back $\pi^* : \Omega(M) \to \Omega(M/G)$ is injective.

If $\omega \in \Omega(M/G)$ and $g \in G$, then the equality $\pi \circ g = \pi$ implies that $g^*\pi^*\omega = \pi^*\omega$, hence $\pi^*\omega \in \Omega^*(M)^G$.

Vice-versa, if $\beta \in \Omega^*(M)^G$, then there exists $\omega \in \Omega(M/G)$ such that $\pi^*\omega = \beta$. In order to define β near a point $x \in M/G$ we choose a neighbourhood U of x such that $\pi^{-1}(U) \cong U \times G$. Fixing a point $g \in G$ we get a section $s : U \to U \times G \to M$. We set $\omega_{|U} := s^*\beta$. The result is independent of the choice of g or the trivialization. Indeed, if s' is defined with a second choice then in a neighbourhood of x we have $s' = hs$ for some $h \in G$. Then $s'^*\beta = s^*h^*\beta = s^*\beta$. □

By functoriality of the de Rham cohomology the group G acts on the real vector space $H^*_{dR}(M)$.

Proposition 1.50. *The pull-back $\pi^* : H_{dR}(M/G) \to H^*_{dR}(M)$ induces an isomorphism $H^*_{dR}(M/G) \cong H^*_{dR}(M)^G$.*

Proof. By Lemma 1.49 it is clear that the image of π^* is contained in $H^*_{dR}(M)^G$.

We first show injectivity of π^*. Note that $g^* : \Omega(M) \to \Omega(M)$ and hence P preserve the differential. Let $[\omega] \in H^k_{dR}(M/G)$ be such that $\pi^*[\omega] = 0$. Then there exists

$\alpha \in \Omega^{k-1}(M)$ such that $d\alpha = \pi^*\omega$. Then $dP\alpha = Pd\alpha = P\pi^*\omega = \pi^*\omega$. We now use Lemma 1.49. Let $\beta \in \Omega(G/M)$ be such that $\pi^*\beta = P\alpha$. Then $d\beta = \omega$ and therefore $[\omega] = 0$.

We now show surjectivity of π^*. Let $[\gamma] \in H^k_{dR}(M)^G$. For every $g \in G$ there exists $\alpha_g \in \Omega^{k-1}(M)$ such that $g^*\omega - \omega = d\alpha_g$. This gives

$$\omega = P\omega - d\frac{1}{|G|}\sum_{g\in G}\alpha_g \ .$$

Hence $[\omega] = [P\omega]$. By Lemma 1.49 there exists $\beta \in \Omega^k(M/G)$ with $\pi^*\beta = P\omega$. Moreover, $d\beta = 0$. Hence $[\omega] = [P\omega] = \pi^*[\beta]$. □

The upshot of this proof is that if a finite group G acts on a chain complex of rational vector spaces C, then $H^*(C^G) \cong H^*(C)^G$. The assumptions are needed in order to be able to divide by the order $|G|$ of G.

Example 1.51. In this example we calculate the cohomology of $\mathbb{RP}^n \cong S^n/\mathbb{Z}/2\mathbb{Z}$. We first observe that the antipodal map on $\mathbb{R}^{n+1}$ preserves the orientation if and only if n is odd. Since it also preserves the outer normal field on $S^n \cong \partial D^{n+1}$ we see that it preserves the orientation of S^n iff n is odd.

The non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $H^0(S^n)$. This implies

$$H^0_{dR}(\mathbb{RP}^n) = H^0_{dR}(S^n)^{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{R} \ .$$

For odd n it acts trivially on $H^n_{dR}(S^n)$ so that

$$H^n_{dR}(\mathbb{RP}^n) = H^n_{dR}(S^n)^{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{R} \ .$$

For even n it acts by -1 so that

$$H^n_{dR}(\mathbb{RP}^n) = H^n_{dR}(S^n)^{\mathbb{Z}/2\mathbb{Z}} \cong 0 \ .$$

The de Rham cohomology of $\mathbb{RP}^n$ in all other degrees vanishes since the cohomology of the sphere vanishes. The result of our calculation is:

1. For $n \geq 1$, we have

$$b^i(\mathbb{RP}^{2n+1}) = \left\{ \begin{array}{ll} 1 & i = 0, 2n+1 \\ 0 & else \end{array} \right.$$

2. For $n \geq 1$, we have

$$b^i(\mathbb{RP}^{2n}) = \begin{cases} 1 & i = 0 \\ 0 & else \end{cases}$$

□

Example 1.52. In this example we demonstrate that the assumption of finiteness of G in Proposition 1.50 is essential. We consider the usual action of $\mathbb{Z}^n$ (which is not finite) on $\mathbb{R}^n$ with $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$.

Then

$$H^i_{dR}(\mathbb{R}^n)^{\mathbb{Z}^n} \cong \begin{cases} \mathbb{R} & i = 0 \\ 0 & else \end{cases}$$

In the other hand

$$H^i_{dR}(T^n) \cong \mathbb{R}^{\binom{n}{i}} \ , \quad i \in \mathbb{N} \ .$$

□

2 Spectral sequence for filtered complexes and first applications

2.1 Spectral sequence of a filtered chain complex

A **decreasing filtration** $\mathcal{F}$ of an abelian group A is a decreasing family of subgroups

$$\dots \subseteq \mathcal{F}^{p+1}A \subseteq \mathcal{F}^p A \subseteq \dots \subseteq A$$

indexed by $p \in \mathbb{Z}$. In the following we introduce some properties which a filtration can have.

1. We say that $\mathcal{F}$ is **separated** if $\bigcap_{p\in\mathbb{Z}} \mathcal{F}^p A = 0$.
2. We say that it is **exhaustive** if $\bigcup_{p\in\mathbb{Z}} \mathcal{F}^p A = A$.
3. We say that the filtration is **bounded below**, if $\mathcal{F}^p A = \mathcal{F}^{p-1}A$ for all sufficiently small p and **bounded above** if $\mathcal{F}^p A = \mathcal{F}^{p+1}A$ for sufficiently large p.

4. We say that the filtration is **finite** if it is bounded below and above.

For a filtered abelian group $(A, \mathcal{F})$ and $p \in \mathbb{Z}$ we define the pth **graded component** by

$$\mathrm{Gr}^p A := \mathcal{F}^p A / \mathcal{F}^{p+1} A \ .$$

A **morphism of filtered abelian groups** $(A, \mathcal{F}) \to (B, \mathcal{F})$ is a morphism $f : A \to B$ of abelian groups such that $f(\mathcal{F}^p A) \subseteq \mathcal{F}^p B$ for all $p \in \mathbb{Z}$.

Lemma 2.1. *Let $f : (A, \mathcal{F}_A) \to (B, \mathcal{F}_B)$ be a morphism of filtered abelian groups such that $\mathrm{Gr}^p(f) : \mathrm{Gr}^p A \to \mathrm{Gr}^p B$ is an isomorphism for all $p \in \mathbb{Z}$. If both filtrations are exhaustive, separating and bounded above, then $f : A \to B$ is an isomorphism of abelian groups.*

Proof. We consider the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^{p+1} A \arrow[r] \arrow[d] & \mathcal{F}^p A \arrow[r] \arrow[d] & \mathrm{Gr}^p A \arrow[r] \arrow[d] & 0 \ . \\
0 \arrow[r] & \mathcal{F}^{p+1} B \arrow[r] & \mathcal{F}^p B \arrow[r] & \mathrm{Gr}^p B \arrow[r] & 0
\end{tikzcd}$$

Both filtrations as separating and bounded above. Consequently there exists some $p_0 \in \mathbb{Z}$ such that $\mathcal{F}^{p_0+1} A = 0$ and $\mathcal{F}^{p_0+1} B = 0$. We can start a downward induction at $p = p_0$ and use the Five Lemma in order to conclude that $\mathcal{F}^p A \to \mathcal{F}^p B$ is an isomorphism for all $p \in \mathbb{Z}$. Since both filtrations are exhaustive we can conclude that $f : A \to B$ is an isomorphism. $\square$

A **filtration of a chain complex** of abelian groups is a chain complex $C = (C^q, d)$ together with filtrations $(\mathcal{F}^p C^q)_{p \in \mathbb{Z}}$ for all $q \in \mathbb{Z}$ such that the differential is a morphism of filtered groups. A filtered chain complex induces a sequence of chain complexes $(\mathcal{F}^p C)_{q \in \mathbb{Z}}$. We have natural morphisms of chain complexes

$$\ldots \hookrightarrow \mathcal{F}^{p+1} C \hookrightarrow \mathcal{F}^p C \hookrightarrow \ldots \hookrightarrow C \ .$$

We define an **induced filtration on the cohomology** $H^*(X)$ by

$$\mathcal{F}^p H^q(C) := \mathtt{im}(H^q(\mathcal{F}^p C) \to H^q(C)) \ .$$

Note that we can not commute the operations of taking the graded components and cohomology.

Example 2.2. We consider the filtered chain complex C given in the form

$$\mathcal{F}^2 C \subseteq \mathcal{F}^1 C \subseteq \mathcal{F}^0 C$$

by

$$(0 \to 0) \subseteq (0 \to 0 \oplus \mathbb{Z}) \subseteq (\mathbb{Z} \xrightarrow{d} \mathbb{Z} \oplus \mathbb{Z}) \ , \quad dx := 0 \oplus x \ .$$

The filtration of the cohomology

$$\left(\begin{array}{c} H^0(C) \\ H^1(C) \end{array} \right)$$

is given by

$$\left(\begin{array}{c} 0 \\ 0 \end{array} \right) \subseteq \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \subseteq \left(\begin{array}{c} 0 \\ \mathbb{Z} \end{array} \right) \ .$$

The associated graded groups given in the form $\mathrm{Gr}^1 \oplus \mathrm{Gr}^0$ are

$$\left(\begin{array}{c} \mathrm{Gr}^* H^0(C) \\ \mathrm{Gr}^* H^1(C) \end{array} \right) \cong \left(\begin{array}{c} 0 \oplus 0 \\ 0 \oplus \mathbb{Z} \end{array} \right) \ .$$

The graded chain complex is given by

$$\mathrm{Gr}^*(C) = (0 \oplus \mathbb{Z}) \xrightarrow{0} (\mathbb{Z} \oplus \mathbb{Z}) \ .$$

Its cohomology is

$$\left(\begin{array}{c} H^0(\mathrm{Gr}^*(C) \\ H^1(\mathrm{Gr}^*(C)) \end{array} \right) \cong \left(\begin{array}{c} 0 \oplus \mathbb{Z} \\ \mathbb{Z} \oplus \mathbb{Z} \end{array} \right) \ .$$

$\square$

Our goal in the following is to calculate the groups $\mathrm{Gr}^p H^q(C)$ starting from the groups $H^q(\mathrm{Gr}^p C)$. The tool is called a **spectral sequence**.

For every $p \in \mathbb{Z}$ we have an exact sequence of chain complexes

$$0 \to \mathcal{F}^{p+1} C \to \mathcal{F}^p C \to \mathrm{Gr}^p C \to 0 \ .$$

It gives rise to a long exact sequence

$$\dots \to H^q(\mathcal{F}^{p+1} C) \xrightarrow{i} H^q(\mathcal{F}^p C) \xrightarrow{\mathtt{pr}} H^q(\mathrm{Gr}^p C) \xrightarrow{\partial} H^{q+1}(\mathcal{F}^{p+1} C) \to \dots$$

We form the following triangle (called **exact couple**)

$$\begin{tikzcd} \bigoplus_{p,q} H^q(\mathcal{F}^pC) \arrow[rr, "i"] & & \bigoplus_{p,q} H^q(\mathcal{F}^pC) \arrow[dl, "\mathtt{pr}"] \\ & \bigoplus_{p,q} H^q(\mathrm{Gr}^pC) \arrow[ul, "\partial"] & \end{tikzcd} \tag{9}$$

Spectral sequences are derived from exact couples. In the following we explain this construction in general. We consider two abelian groups E and W which are connected by a triangle of homomorphisms

$$\begin{tikzcd} W \arrow[rr, "i"] & & W \arrow[dl, "\mathtt{pr}"] \\ & E \arrow[ul, "\partial"] & \end{tikzcd}$$

such that the sequence

$$\dots \to W \xrightarrow{i} W \xrightarrow{\mathtt{pr}} E \xrightarrow{\partial} W \to \dots$$

is exact. This datum is called **an exact couple**. Given an exact couple we define the **derived exact couple**

$$\begin{tikzcd} W' \arrow[rr, "i'"] & & W' \arrow[dl, "\mathtt{pr}'"] \\ & E' \arrow[ul, "\partial'"] & \end{tikzcd}$$

as follows:

1. $E' := \mathtt{ker}(d)/\mathtt{im}(d)$, where $d := \mathtt{pr} \circ \partial : E \to E$
2. $W' := \mathtt{im}(i)$
3. $i' := i_{|W'}$
4. $\mathtt{pr}' : W' \to E'$ is given by $w \mapsto [\mathtt{pr}(\tilde{w})]$, where $\tilde{w} \in W$ is chosen such that $i(\tilde{w}) = w$.
5. $\partial'[e] = \partial e$.

Lemma 2.3. *The derived exact couple is well-defined.*

Proof. We first must show that the maps are well-defined. Then we must verify exactness of the derived couple.

1. A priory the map i' maps W' to W. But it is clear from the construction that it takes values in the subspace $W' \subseteq W$.

2. Given $w \in W'$ we can find $\tilde{w} \in W$ such that $i(\tilde{w}) = w$. We can therefore try to define $\mathtt{pr}'(w) := [\mathtt{pr}(\tilde{w})]$, where $[e]$ denotes the equivalence class modulo $\mathtt{im}(d)$ of an element $e \in E$ with $de = 0$. It is clear that $\partial'[\mathtt{pr}(\tilde{w})] = [\partial \mathtt{pr}(\tilde{w})] = 0$ and hence $[\mathtt{pr}(\tilde{w})] \in E'$. We must check that $[\mathtt{pr}(\tilde{w})]$ is independent of the choice of $\tilde{w}$. A different choice can be written in the form $\tilde{w} + \partial e$ for some $e \in E$. But then $[\mathtt{pr}(\tilde{w} + \partial e)] = [\mathtt{pr}(\tilde{w})] + [\mathtt{pr}(\partial(\tilde{e}))] = [\mathtt{pr}(\tilde{w})] + [de] = [\mathtt{pr}(\tilde{w})]$ in view of the definition of E'.

3. If $de = 0$, then $\mathtt{pr}(\partial e) = 0$, hence $\partial e \in \mathtt{im}(i) = W'$. Hence the definition of $\mathtt{pr}'$ produces elements in the correct target. We must show that ∂' is well-defined. A different representative of $[e]$ can be written in the form $e + d\tilde{e} = e + \mathtt{pr}(\partial(\tilde{e}))$. But then $\partial(e + \mathtt{pr}(\partial(\tilde{e}))) = \partial e$.

4. We show exactness for $i' \circ \partial'$. Let $w \in W'$. Assume that $i'(w) = 0$. Then there exists $\tilde{w} \in W$ such that $i(\tilde{w}) = w$ and $i(w) = 0$. We have $w = \partial e$ for some $e \in E$. But then $de = \mathtt{pr}(\partial e) = \mathtt{pr}(w) = \mathtt{pr}(i(\tilde{w})) = 0$. Hence $w = \partial'[e]$.

   On the other hand if $[e] \in E'$, the $i'(\partial'([e])) = i(\partial(e)) = 0$.

5. We show exactness for $\mathtt{pr}' \circ i'$. Let $w \in W'$ be such that $\mathtt{pr}'(w) = 0$. Let us write $w = i(\tilde{w})$. Then $\mathtt{pr}(\tilde{w}) = de = \mathtt{pr}(\partial e)$ and hence $\mathtt{pr}(\tilde{w} - \partial e) = 0$. We can replace $\tilde{w}$ by $\tilde{w} - \partial e$ and thus assume that $\tilde{w} = i(\hat{w})$. We conclude that $\tilde{w} \in W'$ and therefore $w \in \mathtt{im}(i')$.

   On the other hand, if $w \in W'$, then we can write $w = i(\tilde{w})$. We have $i'(w) = i(i(\tilde{w}))$ and hence $\mathtt{pr}'(i'(w)) = \mathtt{pr}(i(\tilde{w})) = 0$.

6. Finally we show exactness at E. Let $[e] \in E'$ be such that $\partial'[e] = 0$. Then we have $\partial e = 0$ and hence $e = \mathtt{pr}(w)$ for some $w \in W$. We thus have $e = \mathtt{pr}'(i(w))$.

   On the other hand, if $[e] = \mathtt{pr}'(w)$, then we find $\tilde{w} \in W$ such that $i(\tilde{w}) = w$ and $e = \mathtt{pr}(\tilde{w}) + d(\tilde{e})$ for some $\tilde{e} \in E$. Then $\partial'[e] = \partial e = 0$.

□

We can interate the formation of the derived couple. The $r - 1$'th derivation will be denoted by

$$\begin{tikzcd} W_r \arrow[rr, "i_r"] & & W_r \arrow[dl, "\mathtt{pr}_r"] \\ & E_r \arrow[ul, "\partial_r"] & \end{tikzcd} .$$

In particular we get a sequence of groups $(E_r)_{r\geq 1}$ and maps $d_r : E_r \to E_r$ such that $d_r \circ d_r = 0$ and $E_{r+1} \cong \frac{\mathtt{ker}(d_r)}{\mathtt{im}(d_r}$. This sequence $(E_r, d_r)_{r\geq 1}$ is called the **spectral sequence** associated to the exact couple.

We say that the spectral sequence **degenerates at the rth page** if $d_{r'} = 0$ for all $r' \geq r$. This is e.g. the case if the rth derived couple is stable under further derivation. In this case i_r is surjective and we have an exact sequence

$$0 \to E_r \xrightarrow{\partial} W_r \xrightarrow{i} W_r \to 0 \ .$$

We now come back to the exact couple defined by a filtered chain complex. In this case the groups E and W are bigraded as follows:

$$W^{p,q} := H^q(\mathcal{F}^p C) \ , \quad E^{p,q} := H^{q+p}(\mathrm{Gr}^p C) \ .$$

We further set

$$W := \bigoplus_{p,q\in\mathbb{Z}} W^{p,q} \ , \quad E := \bigoplus_{p,q\in\mathbb{Z}} E^{p,q} \ .$$

The arrows have the following bidegrees:

1. $i : W^{p+1,q} \to W^{p,q}$
2. $\partial : E^{p,q} \to W^{p+1,q+p+1}$
3. $\mathtt{pr} : W^{p,q} \to E^{p,q-p}$

We now analyse the derivation of (9). We first calculate the bidegrees of the morphisms by induction.

1. $i_r : W_r^{p+1,q} \to W_r^{p,q}$.
2. $\partial_r : E_r^{p,q} \to W_r^{p+1,q+p+1}$
3. $\mathtt{pr}_r : W_r^{p,q} \to E_r^{p+r-1,q-p-r+1}$ - This is shown by induction on r.
4. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$

The case $r = 1$ is discussed above. Assume now that we have shown the case $r - 1$. Then:

1. $i_r : W_r^{p+1,q} \to W_r^{p,q}$ is the restriction of $i_{r-1} : W_{r-1}^{p+1,q} \to W_{r-1}^{p,q}$ into the image of $i_{r-1} : W_{r-1}^{p+2,q} \to W_{r-1}^{p+1,q}$. Therefore i_r has the same bidgree as i_{r-1}.

2. $\partial_r : E_r^{p,q} \to W_r^{p+1,q+p+1}$ is induced by the application of $\partial_{r-1} : E_{r-1}^{p,q} \to W_{r-1}^{p+1,q+p+1}$ to representatives. It has the same bidegree as ∂_{r-1}.

3. The value of $\mathtt{pr}_r : W_r^{p,q} \to E_r^{p+r-1,q-p-r+1}$ on $w \in W_r^{p,q}$ is given by $\mathtt{pr}_{r-1}(\tilde{w})$, where $\tilde{w} \in W_{r-1}^{p+1,q}$ is such that $i_{r-1}(\tilde{w}) = w$. Then by induction assumption $\mathtt{pr}_{r-1}(\tilde{w}) \in E_{r-1}^{p+r-1,q-p-r+1}$.

Here is a picture of a part of an E_4-term, where we have indicated the only differentials which act between groups in this piece.

$$
\begin{tikzcd}
4 & E_4^{0,4} \arrow[dddrrrr] & E_4^{1,4} \arrow[dddrrrr] & E_4^{2,4} & E_4^{3,4} & E_4^{4,4} & E_4^{5,4} \\
3 & E_4^{0,3} \arrow[dddrrrr] & E_4^{1,3} \arrow[dddrrrr] & E_4^{2,3} & E_4^{3,3} & E_4^{4,3} & E_4^{5,3} \\
2 & E_4^{0,2} & E_4^{1,2} & E_4^{2,2} & E_4^{3,2} & E_4^{4,2} & E_4^{5,2} \\
1 & E_4^{0,1} & E_4^{1,1} & E_4^{2,1} & E_4^{3,1} & E_4^{4,1} & E_4^{5,1} \\
0 & E_4^{0,0} & E_4^{1,0} & E_4^{2,0} & E_4^{3,0} & E_4^{4,0} & E_4^{5,0} \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}
$$

Consider $p, q \in \mathbb{Z}$. We have a sequence of subquotients $(E_r^{p,q})_{r\geq 1}$ of $E_1^{p,q}$. We can define increasing (resp. decreasing) sequences of subgroups

$$\cdots \subseteq B_r^{p,q} \subset B_{r+1}^{p,q} \subseteq \ldots Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \ldots$$

such that $E_r^{p,q} := Z_r^{p,q}/B_r^{p,q}$ for all $r \geq 1$. We define

$$B_\infty^{p,q} := \bigcup_{r\geq 1} B_r^{p,q} \ , \quad Z_\infty^{p,q} := \bigcap_{r\geq 1} Z_r^{p,q} \ , \quad E_\infty^{p,q} := \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} \ .$$

Lemma 2.4. *We assume that for each q the filtration of $(\mathcal{F}^pC^q)_{p\in\mathbb{Z}}$ is finite, exhaustive and separating. Then for every $p,q\in\mathbb{Z}$ there exists r_0 (possibly depending on p,q) such that for all $r\geq r_0$ the differentials starting or ending at $E_r^{p,q}$ are trivial. Furthermore, for $r\geq r_0$ we have*

$$E_r^{p,q}\cong E_\infty^{p,q}$$

and an isomorphism

$$\mathrm{Gr}^pH^{p+q}(C)\cong E_\infty^{p,q}\ .$$

Proof. We get by inspection

$$W_r^{p,q}=\mathtt{im}(H^q(\mathcal{F}^{p+r-1}C)\to H^q(\mathcal{F}^pC))\ . \tag{10}$$

We set $p:=p'-r+1$ and get

$$W_r^{p'-r+1,q}\cong\mathtt{im}(H^q(\mathcal{F}^{p'}C)\to H^q(\mathcal{F}^{p'-r+1}C))\ .$$

The assumption implies that for every $q\in\mathbb{Z}$ there are $p^\pm(q)\in\mathbb{Z}$ such that we have $\mathcal{F}^pC^q=C^q$ for $p\leq p_-(q)$ and $\mathcal{F}^pC^q=0$ for $p\geq p^+(q)$. Then for $r\geq p'+2-p^-(q)$ we have

$$W_r^{p'-r+1,q}\cong\mathtt{im}(H^q(\mathcal{F}^{p'}C)\to H^q(C))=\mathcal{F}^{p'}H^q(C)\ .$$

Moreover, $i_r:W_r^{p'-r+2,q}\to W_r^{p'-r+1,q}$ is the injective and we have an exact sequence

$$0\to\mathcal{F}^{p'+1}H^q(C)\xrightarrow{i_r}\mathcal{F}^{p'}H^q(C)\to E_r^{p',q-p}\to W_r^{p'+1,q+1}\ .$$

If $p'+r\geq p^+(q+1)$, then in view of (10) we have $W_r^{p'+1,q+1}=0$, and we get the exact sequence

$$0\to\mathcal{F}^{p'+1}H^q(C)\to\mathcal{F}^{p'}H^q(C)\to E_r^{p',q-p'}\to 0\ ,$$

i.e.

$$\mathrm{Gr}^{p'}H^q(C)\cong E_r^{p',q-p'}\ ,$$

if $r\geq\max\{p'+2-p^-(q),p^+(q+1)-p'\}$.

The assumption 2.4 thus implies that for every $p',q\in\mathbb{Z}$ there exists $r_0\in\mathbb{N}$ such that for $r\geq r_0$ we have

$$E_r^{p',q}=E_\infty^{p',q}\ ,\quad \mathrm{Gr}^{p'}H^q(C)\cong E_\infty^{p',q-p}\ .$$

$\square$

A **morphism between exact couples**

$$(e,w):(E_1,W_1,i_1,\mathtt{pr}_1,\partial_1)\to(E_2,W_2,i_2,\mathtt{pr}_2,\partial_2)$$

is a pair of morphisms of abelian groups $e:E_1\to E_2$ and $w:W_1\to W_2$ which are compatible with the structure maps, i.e. the following relations hold:

$$w\circ i_1=i_2\circ w\ ,\quad e\circ\mathtt{pr}_1=\mathtt{pr}_2\circ w\ ,\quad w\circ\partial=\partial\circ e\ .$$

The construction of the derived exact couple is functorial, i.e. we get an induced morphism

$$(e',w'):=(E_1',W_1',i_1',\mathtt{pr}_1',\partial_1')\to(E_2',W_2',i_2',\mathtt{pr}_2',\partial_2')\ .$$

We refrain from writing out the details.

The construction of the spectral sequence associated to an exact couple is therefore also functorial. We get an induced morphism of spectral sequences $(E_{1,r},d_{1,r})_{r\ge1}\to(E_{2,r},d_{2,r})_{r\ge1}$. In detail this morphism is given by the collection $(e_r)_{r\ge1}$ of derivations of e. Note that

$$e_r\circ d_{1,r}=d_{2,r}\circ e_r\ .$$

If the couples are bigraded as above, then these morphisms are compatible with the induced bigradings.

If $(E_{1,r},d_{1,r})_{r\ge1}\to(E_{2,r},d_{2,r})_{r\ge1}$ is a morphism of spectral sequences such that for some $r_0\in\mathbb{N}$ the map $e_{r_0}:E_{1,r_0}\to E_{2,r_0}$ is an isomorphism, then by induction we see that $e_r:E_{1,r}\to E_{2,r}$ is an isomorphism for all $r\ge r_0$.

If

$$f:(C,\mathcal{F}_C)\to(D,\mathcal{F}_D)$$

is a morphism of filtered chain complexes, then we get a morphism of exact couples and therefore a morphism of spectral sequences

$$E_r(f):(E_r(C),d_r)\to(E_r(D),d_r)\ .$$

Lemma 2.5. *Assume that the filtrations of C^q and D^q are exhaustive, separating and finite for all q. If for some $r\in\mathbb{N}$ the induced morphism $E_r(C)\to E_r(D)$ is an isomorphism, then $H^*(f):H^*(C)\to H^*(D)$ is an isomorphism.*

Proof. By Lemma 2.4 the map $\mathrm{Gr}^*(H^*(f)) : \mathrm{Gr}^* H^*(C) \to \mathrm{Gr}^* H^*(C)$ is an isomorphism. We check by inspection that the filtrations on $H^q(C)$ and $H^q(D)$ are finite, exhaustive and separating. We now use Lemma 2.1 in order to transfer the statement about the isomorphism from the graded groups to the cohomology groups themselfes. □

The case $r = 1$ is of particular interest.

Corollary 2.6. *If*

$$f : (C, \mathcal{F}_C) \to (D, \mathcal{F}_D)$$

is a morphism of filtered chain complexes such that

$$\mathrm{Gr}(f) : \mathrm{Gr}(C) \to \mathrm{Gr}(D)$$

is a quasi-isomorphism and assume that the filtrations of C^q and D^q are exhaustive, separating and finite for all q. Then $f : C \to D$ is a quasi-isomorphism.

Proof. Note that $E_1(f) : E_1(C) \to E_1(D)$ is the map $\mathrm{Gr}(f) : \mathrm{Gr}(C) \to \mathrm{Gr}(D)$. Apply Lemma 2.5. □

Example 2.7. A **double complex** is a given by a family of abelian groups $(C^{p,q})_{p,q\in\mathbb{Z}}$ together with homomorphisms $d_1 : C^{p,q} \to C^{p+1,q}$ and $d_2 : C^{p,q} \to C^{p,q+1}$ for all $p, q \in \mathbb{Z}$ which turn $(C^{*,q}, d_1)$ and $(C^{p,*}, d_2)$ into chain complexes for all $p \in \mathbb{Z}$ or $q \in \mathbb{Z}$, and which satisfy $d_1 d_2 + d_2 d_1 = 0$.

Given a double complex $C := ((C^{p,q})_{p,q}, d_1, d_2)$ we can form its **total complex** $\mathtt{tot}(C)$. Its is given by

$$\mathtt{tot}(C)^n := \bigoplus_{p+q=n} C^{p,q} \ , \quad d := d_1 + d_2 \ .$$

Indeed,

$$d : \mathtt{tot}(C)^n \to \mathtt{tot}(C)^{n+1} \ , \quad dd = (d_1+d_2)(d_1+d_2) = d_1d_1+(d_1d_2+d_2d_1)+d_2d_2 = 0 \ .$$

The total complex of a double complex has two natural filtrations ${}^I\mathcal{F}^*$ and ${}^{II}\mathcal{F}^*$ given by

$${}^I\mathcal{F}^k \,\mathtt{tot}(C)^n := \bigoplus_{p+q=n, p\ge k} C^{p,q} \ , \quad {}^{II}\mathcal{F}^k \,\mathtt{tot}(C)^n := \bigoplus_{p+q=n, q\ge k} C^{p,q} \ .$$

We thus get two spectral sequences $({}^{I}E_r, {}^{I}d_r)$ and $({}^{II}E_r, {}^{II}d_r)$. We have

$$ {}^{I}\mathrm{Gr}^p(\mathtt{tot}(C), d) \cong (C^{p,*}, d_2) \ , \quad {}^{II}\mathrm{Gr}^p(\mathtt{tot}(C), d) \cong (C^{*,q}, d_1) $$

and hence

$$ {}^{I}E_1^{p,q} \cong H^q(C^{p,*}, d_2) \ , \quad {}^{II}E_1^{p,q} \cong H^q(C^{*,p}, d_1) $$

For given $n \in \mathbb{Z}$ the filtrations ${}^{\cdots}\mathcal{F}^* \, \mathtt{tot}(C)^n$ are exhaustive and separating. They are finite if $C^{n-q,q} = 0$ for sufficiently large or small q.

If this is the case, then we can apply Lemma 2.4 and get

$$ {}^{II}E_\infty^{p,q} = {}^{II}\mathrm{Gr}^p H^{p+q}(\mathtt{tot}(C)) \ . $$

Observe that the finiteness assumption is satisfied if the double complex is supported in the **right upper quadrant**, i.e. if $C^{p,q} = 0$ if one of the indices p, q is negative.

A typical application goes as follows. Let $C^{*,*} \to D^{*,*}$ be a morphism of double complexes such that $C^{*,p} \to D^{*,p}$ is a quasi-isomorphism for all p. Assume that for all $n \in \mathbb{Z}$ we have $C^{n-q,q} = 0$ and $D^{n-q,q} = 0$ for sufficiently large or small q. Then $\mathtt{tot}\, C \to \mathtt{tot}\, D$ is a quasi isomorphism. Indeed, for all $p, q \in \mathbb{Z}$ we get an isomorphism

$$ {}^{II}E_1^{p,q}(C) \stackrel{\cong}{\to} {}^{II}E_1^{p,q}(D) \ . $$

We now apply Lemma 2.5.

$\square$

Example 2.8. If (A, d_A) and (B, d_B) are chain complexes, then we define the double complex C by

$$ C^{p,q} := A^p \otimes B^q \ , \quad d_1 := d_A \otimes \mathtt{id} : C^{p,q} \to C^{p+1,q} \ , \quad d_2 := (-1)^p \mathtt{id} \otimes d_B : C^{p,q} \to C^{p,q+1} \ . $$

The total complex of C is called the **tensor product of the chain complexes** A **and** B

$$ A \otimes B := \mathtt{tot}(C) \ . $$

This construction defines a symmetric monoidal structure on the category of chain complexes.

We have a canonical map $H^*(A) \otimes H^*(B) \to H^*(A \otimes B)$ which sends $[a] \otimes [b]$ to $[a \otimes b]$. Under certain assumptions it is an isomorphism. See Lemma 2.30. $\square$

2.2 Good coverings, Čech complex and finiteness of de Rham cohomology

In this section we construct and analyse the Čech complex of sections of a vector bundle $V \to M$ associated to an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of M. We apply this to the bundle of alternating forms. Our main result in this subsection shows that the de Rham cohomology of a compact manifold is finite dimensional.

The natural home for these constructions is **sheaf theory**. The construction of the Čech complex works for arbitrary sheaves of abelian groups. The proof of Lemma 2.11 apples equally well if one replaces the sheaf of sections of the vector bundle by an arbitrary sheaf which admits the multiplication by a partition of unity, i.e. a **fine sheaf**. We refrain from developing the sheaf language at this point.

For $n \in \mathbb{N}$ and $\alpha \in (\alpha_0, \dots, \alpha_n) \in A^{n+1}$ we define the open subset

$$U_\alpha := U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$$

of M. We further define the vector space

$$\check{C}^n(\mathcal{U}, V) := \prod_{\alpha \in A^{n+1}} \Gamma(U_\alpha, V) \ .$$

Our notation for elements in this vector space is

$$\phi = (\phi_\alpha)_{\alpha \in A^{n+1}} \ , \quad \phi_\alpha \in \Gamma(U_\alpha, V) \ ,$$

and we call ϕ_α a component of ϕ. For every $i \in \{0, \dots, n+1\}$ and $\alpha \in A^{n+2}$ we have $U_\alpha \subseteq U_{(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1})}$, where $(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1}) \in A^{n+1}$ is the tuple derived from α by omission of the i'th entry. We define homomorphisms

$$d_i : \check{C}^n(\mathcal{U}, V) \to \check{C}^{n+1}(\mathcal{U}, V)$$

given on the components by

$$(d_i\phi)_\alpha := (\phi_{(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1})})_{|U_\alpha} \ .$$

We further define the **Čech differential** by

$$d : \check{C}^n(\mathcal{U}, V) \to \check{C}^{n+1}(\mathcal{U}, V) \ , \quad d := \sum_{i=0}^{n+1} (-1)^i d_i \ . \qquad (11)$$

Lemma 2.9. *For every $n \in \mathbb{N}$ the composition*

$$\check{C}^{n-1}(\mathcal{U}, V) \stackrel{d}{\to} \check{C}^{n}(\mathcal{U}, V) \stackrel{d}{\to} \check{C}^{n+1}(\mathcal{U}, V)$$

is trivial.

Proof. We use the relations (called **cosimplicial relations**)

$$d_i \circ d_j = \left\{ \begin{array}{ll} d_j \circ d_{i-1} & j < i \\ d_{j+1} \circ d_i & j \geq i \end{array} \right\} .$$

We calculate

$$\begin{aligned}
d(df) &= d \sum_{j=0}^{n} (-1)^j d_j \\
&= \sum_{i=0}^{n+1} (-1)^i d_i \sum_{j=0}^{n} (-1)^j d_j = \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{i=0}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} d_i d_j + \sum_{i=0}^{n+1} \sum_{j=i}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{j=0}^{n} \sum_{i=j+1}^{n+1} (-1)^{i+j} d_i d_j + \sum_{i=0}^{n+1} \sum_{j=i}^{n} (-1)^{i+j} d_{j+1} d_i \\
&= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j} d_j d_i + \sum_{i=0}^{n+1} \sum_{j=i+1}^{n+1} (-1)^{i+j-1} d_j d_i \\
&= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j} d_j d_i + \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j-1} d_j d_i \\
&= 0 .
\end{aligned}$$

□

Definition 2.10. *The complex $(\check{C}^*(\mathcal{U}, V), d)$ is called the* ***Čech complex*** *of sections of V.*

We have a natural map of complexes

$$i : \Gamma(M, V) \to \check{C}(\mathcal{U}, V) \ , \quad \Gamma(M, V) \ni \phi \mapsto (\phi_{|U_\alpha})_{\alpha \in A} \in \check{C}^0(\mathcal{U}, F) \ ,$$

where we view $\Gamma(M,V)$ as a chain complex concentrated in degree zero. If $\omega \in \Gamma(M,V)$, then $d\omega = 0$ in the complex $\Gamma(M,V)$ and one must check that $di(\omega) = 0$. Indeed, we have

$$(d(i\omega))_{\alpha_0,\alpha_1} = (\omega_{|U_{\alpha_1}})_{|U_{\alpha_0,\alpha_1}} - (\omega_{|U_{\alpha_0}})_{|U_{\alpha_0,\alpha_1}} = 0 \ .$$

Lemma 2.11. *The map $i : \Gamma(M,V) \to \check{C}(\mathcal{U},V)$ is a quasi-isomorphism.*

Proof. Let (χ_α) be a partition of unity for $\mathcal{U}$. We define a map of complexes

$$r : \check{C}(\mathcal{U},V) \to \Gamma(M,V) \ , \quad r(\omega) := \begin{cases} \sum_\alpha \chi_\alpha \omega_\alpha & \omega \in \check{C}^0(\mathcal{U},V) \\ 0 & \omega \in \check{C}^q(\mathcal{U},V) \ , q \geq 1 \end{cases} \ .$$

The compatibility with differentials is trivially satisfied. Note that $r \circ i = \mathtt{id}_{\Gamma(M,V)}$. It therefore suffices to show that $i \circ r \sim \mathtt{id}_{\check{C}(\mathcal{U},V)}$by construction a homotopy h. For $q \geq 1$ we define the component $h : \check{C}^q(\mathcal{U},V) \to \check{C}^{q-1}(\mathcal{U},V)$ of the homotopy by

$$h(\omega)_{\alpha_0,\dots,\alpha_{q-1}} := (-1)^q \sum_\alpha \chi_\alpha \omega_{\alpha_0,\dots,\alpha_{q-1},\alpha} \ .$$

Note that $\mathtt{supp}(\chi_\alpha \omega_{\alpha_0,\dots,\alpha_{q-1},\alpha})$ is closed in $U_{\alpha_0,\dots,\alpha_{q-1}}$. Therefore we can interpret the terms on the right-hand side by extension by zero as an element in $\Gamma(U_{\alpha_0,\dots,\alpha_{q-1}},V)$. Indeed, if $x \in U_{\alpha_0,\dots,\alpha_{q-1}}$, then we have two cases. In the first we have $x \in U_{\alpha_0,\dots,\alpha_{q-1},\alpha}$ and $\chi_\alpha \omega_{\alpha_0,\dots,\alpha_{q-1},\alpha}$ is smooth near x. In the second case $x \in U_{\alpha_0,\dots,\alpha_{q-1}} \setminus U_{\alpha_0,\dots,\alpha_{q-1},\alpha}$. But then a neighbourhood of x does not intersect $\mathtt{supp}(\chi_\alpha \omega_{\alpha_0,\dots,\alpha_{q-1},\alpha})$.

For $q \geq 1$ we have

$$\begin{aligned} (dh+hd)(\omega)_{\alpha_0,\dots,\alpha_q} &= (-1)^q \sum_{i=0}^q (-1)^i \sum_\alpha \chi_\alpha \omega^q_{\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_q,\alpha} \\ &\quad +(-1)^{q+1} \sum_{i=0}^q (-1)^i \sum_\alpha \chi_\alpha \omega_{\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_q,\alpha} + \sum_\alpha \chi_\alpha \omega_{\alpha_0,\dots,\alpha_q} \\ &= \omega_{\alpha_0,\dots,\alpha_q} \\ &= ((\mathtt{id} - (i \circ r))(\omega))_{\alpha_0,\dots,\alpha_q} \end{aligned}$$

and for $q = 0$

$$\begin{aligned} (dh+hd)(\omega)_\alpha &= (hd\omega^0)_\alpha \\ &= h(((\omega_\beta)_{|U_{(\alpha,\beta)}} - (\omega_\alpha)_{|U_{(\alpha,\beta)}})_{(\alpha,\beta)}) \\ &= \sum_\beta \chi_\beta \omega_\alpha - \sum_\beta \chi_\beta \omega_\beta \\ &= ((\mathtt{id} - (i \circ r))(\omega))_\alpha \ . \end{aligned}$$

□

We now consider the Čech complex of the de Rham complex. In this case we write $\check{d}$ for the Čech differential. For every $q \in \mathbb{N}$ we have a chain complex $(\check{C}(\mathcal{U}, \Lambda^q T^*M), \check{d})$. For every $p \in \mathbb{N}$ the de Rham differential d_{dR} induces homomorphism

$$d_{p,dR} : \check{C}^p(\mathcal{U}, \Lambda^q T^*M) \to \check{C}^p(\mathcal{U}, \Lambda^{q+1} T^*M) \ .$$

Since $\check{d}$ is defined using restriction along smooth maps it commutes with the de Rham differential. If we set $d_1 := \check{d}$ and $d_2 := (-1)^p d_{p,dR}$, then we get a double complex $(\check{C}^p(\mathcal{U}, \Lambda^q T^*M, d_1, d_2)$.

We can consider the de Rham complex of M as a double complex with

$$\Omega(M)^{p,q} = \begin{cases} \Omega^q(M) & p = 0 \\ 0 & else \end{cases}$$

Note that $\mathtt{tot}(\Omega(M)^{*,*}) \cong \Omega(M)$. We have a natural map of double complexes

$$i : \Omega(M)^{*,*} \to \check{C}(\mathcal{U}, \Lambda T^*M) \ .$$

Lemma 2.12. *The induced map of total complexes*

$$\Omega(M) \to \mathtt{tot}(\check{C}(\mathcal{U}, \Lambda T^*M))$$

is a quasi-isomorphism.

Proof. We consider the induced map of spectral sequences

$${}^{II}E_r(\Omega(M)) \to {}^{II}E_r(\mathtt{tot}(\check{C}(\mathcal{U}, \Lambda T^*M))) \ .$$

The first page is given by the cohomology of d_1. We clearly have

$${}^{II}E_1^{p,*}(\Omega(M)) \cong \begin{cases} \Omega^*(M) & p = 0 \\ 0 & else \end{cases} \ .$$

By Lemma 2.11 we also have

$${}^{II}E_1^{p,*}(\mathtt{tot}(\check{C}(\mathcal{U}, \Lambda T^*M))) \cong \begin{cases} \Omega^*(M) & p = 0 \\ 0 & else \end{cases} \ .$$

Under these identifications the map induced by i on the first page is the identity, hence an isomorphism.

We can now apply Lemma 2.5. Note that the spectral sequence is supported in the right upper quadrant. Therefore the assumptions of the Lemma are satisfied. □

We now study the spectral sequence $({}^I E_r(\mathtt{tot}(\check{C}(\mathcal{U},\Lambda T^*M))))_{r\geq 1}$. It is called the **Čech-de Rham** spectral sequence. Note that

$$ {}^I E_1^{p,q}(\mathtt{tot}(\check{C}(\mathcal{U},\Lambda T^*M))) \cong \prod_{\alpha\in A^{p+1}} H^q_{dR}(U_\alpha) \ . $$

The Čech-de Rham spectral is a useful tool if the cohomology appearing on the right-hand side of this isomorphism is simpler then the cohomology of M itself. In the best situation the manifolds U_α are contractible.

Definition 2.13. *A covering $\mathcal{U}$ is called **good**, if for every $q\in\mathbb{N}$ and $\alpha\in A^{q+1}$ the open subset U_α is either empty or contractible.*

We define

$$A^n_{\mathcal{U}} := \{\alpha\in A^{n+1}\mid U_\alpha\neq\emptyset\}$$

If $\mathcal{U}$ is good, then we have

$$ {}^I E_1^{p,q}(\mathtt{tot}(\check{C}(\mathcal{U},\Lambda T^*M))_r) = \begin{cases} \prod_{\alpha\in A^p_{\mathcal{U}}}\mathbb{R} & q=0 \\ 0 & else \end{cases} \ . $$

If $\mathcal{U}$ is finite and good, then ${}^I E_\infty^{p,q}$ is finite-dimensional for every p and q.

Corollary 2.14. *If M admits a finite good coverings, then $H^n_{dR}(M)$ is finite-dimensional for every $n\in\mathbb{N}$.*

Proof. For every p,q the $\mathbb{R}$-vector space ${}^I E_1^{p,q}$ is finite-dimensional. Hence the $\mathbb{R}$-vector space $\bigoplus_{p+q=n,p,q\geq 0}{}^I E_\infty^{p,q}$ is finite-dimensional for every $n\in\mathbb{N}$. Therefore $\mathrm{Gr}H^n_{dR}(M)$ is finite-dimensional, and so is $H^n_{dR}(M)$. □

Proposition 2.15. *Every manifold admits a good covering. If M is compact, then M admits a finite good covering.*

Proof. The second assertion follows easily from the first. A proof of the first using some basic Riemann geometry goes as follows: We choose a Riemannain metric on M. If M has a boundary we take care that the metric has a product structure near the boundary.

Recall that a subset $U \subset M$ is called geodesically convex if for every two points $x, y \in U$ there exists a unique minimizing geodesic from x to y in U. The intersection of a finite number of geodecsically convex subsets is again geodesically convex. A geodesically convex subset is star shaped with respect to each of its points and is thus contractible.

We now use the fact that small balls in Riemannian manifold are geodesically convex. We therefore can fine a good covering of M by sufficiently small balls. $\square$

Corollary 2.16. *If M is compact, then $H^n_{dR}(M)$ is finite-dimensional for every $n \in \mathbb{N}$.*

Example 2.17. The de Rham cohomology $H^*_{dR}(Gr(k, \mathbb{R}^n))$ of the Grassmann manifolds is finite-dimensional. To see this we observe that $Gr(k, \mathbb{R}^n)$ is compact. To this end we use the presentation

$$Gr(k, \mathbb{R}^n) \cong \frac{O(n)}{O(k) \times O(n-k)}$$

and the fact that the orthogonal groups are compact. $\square$

Example 2.18. Let M be a compact manifold and $p : V \to M$ be a vector bundle. We identify M with the zero section in V. We claim that $H^*_{dR}(V)$ and $H^*_{dR}(V \setminus M)$ are finite-dimensional. For the first we argue that p is a homotopy equivalence, hence $p^* : H^*_{dR}(M) \to H^*_{dR}(V)$ is an isomorphism, and $H^*_{dR}(M)$ is finite-dimensional since M is compact.

For the second we choose a metric $\|-\|$ on V and form the sphere bundle

$$S(V) := \{v \in V \mid \|v\| = 1\} .$$

This is a closed submanifold of $V \setminus M$. The inclusion $S(V) \to V \setminus M$ is a homotopy equivalence with inverse given by $v \mapsto \|v\|^{-1}v$ and homotopy $(t, v) \mapsto tv + (1 - t)v\|v\|^{-1}$. The fibre of $p_{|S(V)} : S(V) \to M$ is diffeomorphic to the sphere $S^{\dim(V)-1}$ and hence compact. Therefore $S(V)$ is compact and $H^*_{dR}(S(V)) \cong H^*_{dR}(V \setminus M)$ is finite-dimensional.

$\square$

Definition 2.19. *For a compact manifold M we define its* ***Euler characteristic*** *by*

$$\chi(M) := \sum_{i=0}^{\dim(M)} (-1)^i b^i(M) \ .$$

The Euler characteristic is a numerical invariant of a manifold which is often easy to calculate.

Example 2.20.

1. $\chi(S^{2n+1}) = 0$, $n \in \mathbb{N}$.
2. $\chi(S^{2n}) = 2$, $n \in \mathbb{N}$.
3. $\chi(\Sigma_g) = 2 - 2g$, *where Σ_g is a compact oriented surface of genus g.*

2.3 Filtered colimits, cohomology and tensor products

Let I be some small category. For an auxiliary category $\mathcal{C}$ we can consider the functor category $\mathcal{C}^I$. We define the functor

$$\mathtt{const} : \mathcal{C} \to \mathcal{C}^I$$

which maps an object $C \in \mathcal{C}$ to the constant functor $\mathtt{const}(C) \in \mathcal{C}^I$ with value C. The **limit and colimit** are functors $\mathcal{C}^I \to \mathcal{C}$ defined as right- or left-adjoints of $\mathtt{const}$ (if they exist):

$$\mathtt{colim}_I : \mathcal{C}^I \leftrightarrows \mathcal{C} : \mathtt{const} \ , \quad \mathtt{const} : \mathcal{C} \leftrightarrows \mathcal{C}^I : \mathtt{lim}_I \ .$$

We call $\mathcal{C}$ **complete (cocomplete)** if the limit (colimit) exists for every small index category I.

Example 2.21. Pull-backs, fibre products or equalizers are examples of limits. Push-outs, quotients and coequalizers are examples of colimits.

□

By an evaluation of the definition of limits and colimits we have the following natural isomorphisms for $X, Y \in \mathcal{C}$ and $\mathcal{X}, \mathcal{Y} \in \mathcal{C}^I$:

$$\mathtt{Hom}(\mathtt{colim}_I \mathcal{X}, Y) \cong \mathtt{lim}_{I^{op}} \mathtt{Hom}(\mathcal{X}, Y) \ , \quad \mathtt{Hom}(X, \mathtt{lim}_I \mathcal{Y}) \cong \mathtt{lim}_I \mathtt{Hom}(X, \mathcal{Y}) \ ,$$

where the limits or colimits on the right-hand sides are taken in the category **Set**.

Let k be a commutative ring and consider the category $\mathbf{Mod}(k)$ of k-modules.

Fact 2.22. *The category of k-modules is complete and cocomplete.*

In the following we analyze the compatibility of colimits in $\mathbf{Mod}(k)$ with tensor products. The basic ingredient from algebra is the natural isomorphism

$$\mathtt{Hom}(A \otimes B, C) \cong \mathtt{Hom}(A, \hom(B, C)) \ ,$$

where $\hom(B, C) \in \mathbf{Mod}(k)$ denotes the k-module of homomorphisms from B to C.

Let W be a k-module and $\mathcal{V} \in \mathbf{Mod}(k)^I$ be a diagram. Then we can define a diagram $\mathcal{V} \otimes W \in \mathbf{Mod}(k)^I$ in the natural way.

Lemma 2.23. *There is a natural isomorphism*

$$\mathtt{colim}_I(\mathcal{V} \otimes W) \cong (\mathtt{colim}_I \mathcal{V}) \otimes W \ .$$

Proof. It suffices to define for every k-module T an isomorphism

$$\mathtt{Hom}(\mathtt{colim}_I(\mathcal{V} \otimes W), T) \cong \mathtt{Hom}((\mathtt{colim}_I \mathcal{V}) \otimes W, T)$$

which is natural in T. Indeed, such an isomorphism is given by

$$\begin{aligned} \mathtt{Hom}(\mathtt{colim}_I(\mathcal{V} \otimes W), T) &\cong \mathtt{lim}_{I^{op}} \mathtt{Hom}(\mathcal{V} \otimes W, T) \\ &\cong \mathtt{lim}_{I^{op}} \mathtt{Hom}(\mathcal{V}, \hom(W, T)) \\ &\cong \mathtt{Hom}(\mathtt{colim}_I \mathcal{V}, \hom(W, T)) \\ &\cong \mathtt{Hom}(\mathtt{colim}_I \mathcal{V} \otimes W, T) \ . \end{aligned}$$

□

Next we study the compatibility of cohomology with filtered colimits. We simplify the discussion and restrict our attention to a special class of filtered index categories. Let P be a **filtered partially order set**, i.e. a partially ordered set such that for all $p, q \in P$ there exists $r \in P$ with $p \leq r$ and $q \leq r$. We consider P as a category such that

$$\mathtt{Hom}_P(p, q) = \begin{cases} * & q \geq p \\ \emptyset & q < p \end{cases}$$

Example 2.24. Consider a module V over some ring k. Let P be the partially ordered set of finitely generated submodules of V. For $p \in P$ let $V_p \subseteq V$ be the corresponding submodule. Then the natural morphism is an isomorphism

$$\mathtt{colim}_{p\in P} V_p \stackrel{\cong}{\to} V \ .$$

In order to see this we argue as follows: Let T be some k-module. Then we have a natural map

$$\mathtt{Hom}(V,T) \to \mathtt{lim}_{p\in P^{op}} \mathtt{Hom}(V_p, T)$$

given by $\phi \mapsto (\phi_{|V_p})_{p\in P^{op}}$. We check that it is an isomorphism.

1. Injectivity: Let $\phi, \phi' \in \mathtt{Hom}(V,T)$ be mapped to the same element in the limit. Then for every $v \in V$ they coincide on the k-module generated by v. Hence $\phi(v) = \phi'(v)$.
2. Surjectivity: Let $(\phi_p)_{p\in P^{op}} \in \mathtt{lim}_{p\in P^{op}} \mathtt{Hom}(V_p, T)$. For $v \in V$ choose $p \in P$ such that $v \in V$. Then we define $\phi(v) := \phi_p(v)$. One easily checks that ϕ is well-defined. Furthermore it is the required preimage of $(\phi_p)_{p\in P^{op}}$.

$\square$

In the following we make the structure of objects in $\mathcal{C}^P$ more explicit. For brevity we take the example $\mathcal{C} := \mathbf{Ch}$, the category of chain complexes.

An object of $\mathbf{Ch}^P$ is called a P**-indexed family** of chain complexes. In detail this datum associates to every $p \in P$ a chain complex C_p. Furthermore, for every $q \in P$ with $q \geq p$ we are given a map $i_p^q : C_p \to C_q$ such that for every additional $r \in P$ with $r \geq q$ we have the relation $i_p^r = i_q^r i_p^q$.

In a similar manner we define P-indexed families of objects in any category, e.g. abelian groups or chain complexes. Let us continue with chain complexes.

We have a very simple description of the chain complex $\mathtt{colim}_{p\in P} C_p$ as a quotient

$$\begin{tikzcd} \bigoplus_{(p,q)\in P, q\geq p} C_p \arrow[r, shift left, "i_p^q"] \arrow[r, shift right, "\mathtt{id}"'] & \bigoplus_{p\in P} C_p \arrow[r] & \mathtt{colim}_{p\in P} C_p \end{tikzcd} \ .$$

For every $p \in P$ we have the canonical map $i_p : C_p \to \mathtt{colim}_{p\in P} C_p$. For $q \in P$ with $q \geq p$ we have the relation $i_q i_p^q = i_p$.

Every element in $\mathtt{colim}_{p\in P} C_p$ is of the form $i_p(c_p)$ for some $p \in P$ and $c_p \in P$. Furthermore, $i_p(c_p) = 0$ if and only if there exists $q \in P$ with $q \geq p$ such that $i_p^q(c_p) = 0$.

We now show that **cohomology commutes with filtered colimits**.

Lemma 2.25. *Let P be a filtered partially ordered set and C be a P-indexed family of chain complexes. Then we have a natural isomorphism*

$$\mathtt{colim}_{p\in P} H^*(C_p) \cong H^*(\mathtt{colim}_p C_p) \ .$$

Proof. The family of maps of complexes $i_p : C_p \to \mathtt{colim}_p C_p$ for $p \in P$ induces a map $\mathtt{colim}_{p\in P} H^*(C_p) \to H^*(\mathtt{colim}_p C_p)$. We must show that it is an isomorphism.

We first show surjectivity. Let $[c] \in H^*(\mathtt{colim}_p C_p)$. Then there exists $p \in P$ and $c_p \in C_p$ such that $i_p(c_p) = c$. Since $dc = 0$ there exists $q \in P$ with $q \geq p$ such that $d i_p^q c_p = 0$. Then $i_q[i_p^q(c_p)] = [c]$.

Next we show injectivity. Let $[c_p] \in H^*(C_p)$ be such that $[i_p c_p] = 0$. Then there exists $q \in P$ with $q \geq p$ and $e \in C_q$ such that $i_p^q i_p(c_p) = de$. But then $i_p^q[c_p] = [i_p^q c_p] = [de] = 0$ so that $i_p[c_p] = 0$ in $\mathtt{colim}_{p\in P} H^*(C_p)$. $\square$

Example 2.26. The following example shows that in Lemma 2.25 one can not omit the condition that the index category is filtered. Recall that a coequalizer is a special case of a colimit. Its index category is not filtered. We consider the exact chain complex

$$C: \quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

Then we form the diagram

$$\mathcal{D}: \quad \begin{tikzcd} C \arrow[r, bend left, "2"] \arrow[r, bend right, "0"'] & C \end{tikzcd}$$

where the first copy of $\mathbb{Z}$ is in degree 0. Its colimit is given by

$$\mathtt{colim}\ \mathcal{D}: \quad 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathtt{id}} \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

We have

$$\mathtt{colim}\ H^*(\mathcal{D}) \cong 0 \ , \quad H^k(\mathtt{colim}\ \mathcal{D}) \cong \left\{ \begin{array}{cc} \mathbb{Z}/2\mathbb{Z} & k = 0 \\ 0 & else \end{array} \right.$$

$\square$

We consider a field k and chain complexes of k-vector spaces. In this case **cohomology commutes with tensor products**. We start with a first result in this direction.

Lemma 2.27. *If C is a complex of k-vector spaces and V a k-vector space, then we have a canonical isomorphism*

$$H^*(C) \otimes_k V \xrightarrow{\cong} H^*(C \otimes_k V) \ , \quad [c] \otimes v \mapsto [c \otimes v] \ .$$

Proof. For every $n \in \mathbb{Z}$ we choose a basis of $H^n(C)$ and cycles in C^n representing the basis elements. Mapping the basis elements to the cycles we define a chain map $c : H(C) \to C$, where we consider $H(C)$ as a chain complex with trivial differentials. This map is in fact a quasi-isomorphism. We now define

$$H^*(C) \otimes_k V \to H^*(C \otimes_k V) \ , \quad [x] \otimes v \mapsto [c(x) \otimes v] \ . \tag{12}$$

This map is independent of the choices made in the construction of c. Indeed, if c' is a different choice, then for $[x] \in H^n(C)$ we have $c(x) - c'(x) = dy$ for some $y \in C^{n-1}$. Hence

$$c(x) \otimes v - c'(x) \otimes v = d(y \otimes v) \ .$$

We must show that (12) is an isomorphism. This is obvious if $\dim(V) = 1$. The map is functorial in V and therefore compatible with direct sums. It follows that it is an isomorphism for finite-dimensional V. Since cohomology and tensor products are compatible with filtered colimits and every vector space is a filtered colimit of finite-dimensional ones, the map is an isomorphism in general. □

Remark 2.28. Note that the argument would work for abelian groups instead of k-vector spaces if we assume that $H^*(C)$ and V are free.

If k is a ring and we consider chain complexes of k-modules, then Lemma 2.27 holds (with a different argument) if one assumes that V is flat, i.e. if the functor $(-) \otimes V$ preserves short exact sequences. □

Example 2.29. The assertion of Lemma 2.27 does not hold in general if one replaces vector spaces over a field by modules over a ring k. Here is an example for $\mathbb{Z}$-modules. We consider the exact complex

$$C : \quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and $V := \mathbb{Z}/2\mathbb{Z}$. Then we have

$$C \otimes V : \quad 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathtt{id}} \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

We have

$$H^*(C) \otimes \mathbb{Z}/2\mathbb{Z} \cong 0 \ , \quad H^k(C \otimes \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 0 \\ 0 & else \end{cases}$$

This example can also be considered as a counter example to a corresponding generalization of Lemma 2.30. □

Lemma 2.30. *We assume that C and D are complexes of k-vector spaces. Then we have a canonical isomorphism $H^*(C \otimes D) \cong H^*(C) \otimes H^*(D)$.*

Proof. We construct quasi-isomorphisms $c : H(C) \to C$ and $d : H(D) \to D$ as in Lemma 2.27. They induce a map of double complexes

$$c \otimes d : H(C) \otimes H(D) \to C \otimes D \ .$$

We now study the induced map of spectral sequences ${}^I E(H(C) \otimes H(D)) \to {}^I E(C \otimes D)$. On the level of $E_1^{p,q}$-terms it is given by

$$c \otimes \mathtt{id}_{H(D)} : H^p(C) \otimes H^q(D) \to H^q(C^p \otimes D^*) \stackrel{Lemma 2.27}{\cong} C^p \otimes H^q(D) \ .$$

The differential on the target is induced by the differential of C so that the induced map on E_2-terms is an isomorphism, again by Lemma 2.27.

Let us first assume that the complexes are lower bounded. Then we can apply Lemma 2.5 in order to conclude that $c \otimes d$ is a quasi-isomorphism. The assumption that the complexes are lower bounded ensures the finiteness of the relevant filtrations.

In order to treat the general case note that every chain complex can be written as a filtered colimit of lower bounded chain complexes

$$C \cong \mathtt{colim}(\dots \subseteq C^{\geq p+1} \subseteq C^{\geq p} \subseteq C^{\geq p-1} \subseteq \dots) \ ,$$

where

$$C^{\geq p} : \dots 0 \to C^p \to C^{p+1} \to \dots$$

is the subcomplex of C of chains of degree $\geq p$. We now use that cohomology and tensor products commute with filtered colimits. □

Remark 2.31. The argument for Lemma 2.30 generalizes to the case of chain complexes of modules over an arbitrary ring k if we assume that the k-modules C^* and $H^*(C)$, $H^*(D)$ are free. $\square$

Lemma 2.32. *We consider chain complexes of k-vector spaces C, D, D' over a field k. If $g : D \to D'$ is a quasi isomorphism, then $\mathtt{id}_{\mathbb{C}} \otimes g : C \otimes D \to C \otimes D'$ is a quasi-isomorphism.*

Proof. The map ${}^{II}E_1^{p,*}(C \otimes D) \to {}^{II}E_1^{p,*}(C \otimes D')$ is the map $\mathtt{id}_{H^p(C)} \otimes g : H^p(C) \otimes D \to H^p(C) \otimes D'$. In cohomology it induces the map

$$H^q(H^p(C) \otimes D) \overset{2.27}{\cong} H^p(C) \otimes H^q(D) \overset{\cong}{\to} H^p(C) \otimes H^q(D') \overset{2.27}{\cong} H^q(H^p(C) \otimes D') ,$$

hence an isomorphism. We now apply Lemma 2.5 in order to conclude that $\mathtt{id}_{\mathbb{C}} \otimes g$ is a quasi-isomorphism. Here again we first consider the case of lower bounded chain complexes and then extend the result to all as at the end of the proof of Lemma 2.30. $\square$

Remark 2.33. The argument of Lemma 2.32 generalizes to the case of k-modules over a ring k if we assume that $H^*(C)$ and D, D' are flat. $\square$

2.4 The Künneth formula

We now consider two manifolds M, N. We have a natural morphism of complexes

$$\times : \Omega(M) \otimes \Omega(N) \to \Omega(M \times N) , \quad \omega \otimes \alpha \mapsto \mathtt{pr}_M^* \omega \wedge \mathtt{pr}_N^* \alpha .$$

If both manifolds are not zero dimensional, then this map is far from being an isomorphism.

Proposition 2.34 (Künneth formula)**.** *If N admits a finite good covering, then the map $\times$ induces an isomorphism*

$$\times : H^*_{dR}(M) \otimes H^*_{dR}(N) \overset{\cong}{\to} H^*_{dR}(M \times N) . \tag{13}$$

Proof. We fix a finite good covering $\mathcal{U}$ of N. Then we have a diagram

$$\begin{tikzcd}
\Omega(M)\otimes\Omega(N) \arrow[r] \arrow[d, "\simeq"] & \Omega(M\times N) \arrow[d, "\simeq"] \\
\Omega(M)\otimes \mathtt{tot}(\check{C}(\mathcal{U},\Lambda^*T^*M)) \arrow[r, "v"] & \mathtt{tot}(\check{C}(M\times\mathcal{U},\Lambda^*T^*M))
\end{tikzcd}$$

The right vertical map is a quasi-isomorphism by Lemma 2.12. The left vertical map is the product of $\mathtt{id}_{\Omega(M)}$ and a quasi-isomorphism (again by Lemma 2.12), hence itself a quasi-isomorphism by Lemma 2.32.

We now consider the lower horizontal map. We filter both complexes such that $\mathcal{F}^p$ is the subcomplex of Čech degree $\geq p$. For fixed p the map of $E_1^{p,*}(v)$ is a finite product of maps

$$\Omega(M)\otimes\Omega(U_\alpha)\to\Omega(M\times U_\alpha)$$

for $\alpha\in A^{p+1}$. This map fits into the square

$$\begin{tikzcd}
\Omega(M)\otimes\Omega(*) \arrow[r, "\cong"] \arrow[d, "\mathtt{id}_{\Omega(M)}\otimes\mathtt{pr}_*^*"] & \Omega(M) \arrow[d, "\mathtt{pr}_M^*"] \\
\Omega(M)\otimes\Omega(U_\alpha) \arrow[r] & \Omega(M\times U_\alpha)
\end{tikzcd} ,$$

where the vertical maps are quasi-isomorphisms since U_α is contractible (and for the left map we also use Lemma 2.32). We conclude that $E_1^{p,*}(v)$ is a quasi-isomorphism. We apply Lemma 2.5 in order to conclude that (13) is an isomorphism. □

Note that the Künneth isomorphism is induced by a morphism of rings.

Example 2.35. We can calculate the de Rham cohomology of tori using the Künneth isomorphism and the presentation $T^n := S^1\times\cdots\times S^1$ with n factors. We have an isomorphism of rings $H^*_{dR}(S^1)\cong\mathbb{R}[x]$ with $|x|=1$. It follows that

$$H^*_{dR}(T^n)\cong\mathbb{R}[x_1]\otimes\cdots\otimes\mathbb{R}[x_n]\cong\mathbb{R}[x_1,\dots,x_n] ,$$

where all generators are of degree 1. For the Betti numbers we get

$$b^i(T^n)=\binom{n}{i} . \tag{14}$$

□

Example 2.36. Let M be any manifold and $n \in \mathbb{N}$. Then for $k \in \mathbb{Z}$ we have an isomorphism

$$H^{k-n}_{dR}(M) \oplus H^k_{dR}(M) \stackrel{\cong}{\to} H^k_{dR}(S^n \times M)$$

given by

$$(\alpha, \omega) \mapsto 1 \times \alpha + [\mathrm{vol}_{S^n}] \times \omega \ .$$

□

Lemma 2.37. *If M and N are compact manifolds, then we have*

$$\chi(M \times N) = \chi(M)\chi(N) \ .$$

Proof. We have

$$\begin{aligned}
\chi(M \times N) &= \sum_n (-1)^n \dim(H^n_{dR}(M \times N)) \\
&= \sum_n (-1)^n \sum_{i+j=n} \dim(H^i_{dR}(M) \otimes H^j_{dR}(N)) \\
&= \sum_n \sum_{i+j=n} (-1)^{i+j} \dim(H^i_{dR}(M)) \ \dim(H^j_{dR}(N)) \\
&= \sum_i (-1)^i \dim(H^i_{dR}(M)) \sum_j (-1)^j \dim(H^j_{dR}(N)) \\
&= \chi(M)\chi(N)
\end{aligned}$$

□

Example 2.38. For a compact manifold M we have $\chi(S^1 \times M) = 0$ and $\chi(S^2 \times M) = 2\chi(M)$. Observe that $\chi(T^n) = 0$. Using the formula (14) this gives the identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0 \ .$$

□

3 Poincaré duality

3.1 Relative de Rham cohomology

The cone construction (Example 1.41) provides a functorial extension of a map between chain complexes to a short exact sequence of chain complexes. This allows us to define functorally the relative cohomology of a morphism which fits into a long exact sequence. In detail, let

$$f : (C, d_C) \to (D, d_D)$$

be a map of chain complexes. Then we form the short exact sequence of chain complexes

$$0 \to D \to \mathtt{Cone}(f) \to C[1] \to 0 \ ,$$

where

$$\mathtt{Cone}(f) := C[1] \oplus D \ , \quad d_{\mathtt{Cone}(f)}(c, d) := (-d_C c, -f(c) + d_D d)$$

and the maps are the obvious inclusion and projection. We define the **relative cohomology** of f by

$$H^*(f) := H^*(\mathtt{Cone}(f)[-1]) \ .$$

It fits into the long exact sequence in cohomology

$$\cdots \to H^n(f) \to H^n(C) \to H^n(D) \to H^{n+1}(f) \to \ldots \ . \tag{15}$$

Note that

$$\mathtt{Cone}(f)[-1]^n = C^n \oplus D^{n-1} \ , \quad d_{\mathtt{Cone}(f)[-1]}(c, d) = (d_C c, f(c) - d_D d) \ .$$

Example 3.1. If f is a surjection, then we have a smaller model for the relative cohomology. In this case we consider the map of chain complexes

$$\mathtt{ker}(f) \to \mathtt{Cone}(f)[-1] \ , \quad c \mapsto (c, 0) \ .$$

The following diagram commutes

$$\begin{tikzcd}
\ldots \arrow[r] & H^{n-1}(D) \arrow[r] \arrow[d, equal] & H^n(\mathtt{ker}(f)) \arrow[r] \arrow[d] & H^n(C) \arrow[r] \arrow[d, equal] & \ldots \ , \\
\ldots \arrow[r] & H^{n-1}(D) \arrow[r] & H^n(f) \arrow[r] & H^n(C) \arrow[r] & \ldots
\end{tikzcd}$$

and consequenty, by the Five Lemma, we have an isomorphism

$$H^n(\mathtt{ker}(f)) \stackrel{\cong}{\to} H^n(f) \ .$$

$\square$

Let $f : N \to M$ be a smooth map between manifolds. Then we apply the above construction to the morphism of chain complexes $f^* : \Omega(M) \to \Omega(N)$ and get the **relative de Rham cohomology** $H^*_{dR}(f)$ of f.

Let $N \to M$ is the inclusion of a closed submanifold. If M has boundaries, then this assumption includes additional conditions as follows. We assume that there is a face $M' \subseteq M$ such that the embedding factorizes as $N \to M' \to M$, and that $N \to M'$ is transversal to all boundary faces of M'. In fact, all what we need is that N admits an open neighbourhood in M which has the structure of a bundle over N.

In this case we write

$$H_{dR}(M, N) := H^*_{dR}(f) \ .$$

Example 3.2. Let $M := [-1, 1]^2 \subseteq \mathbb{R}^2$. The following examples are admitted.

1. $S^1 \to M$, $f(\exp^{2\pi it}) = (\frac{1}{2}\sin(2\pi t), \frac{1}{2}\cos(2\pi t))$.
2. $\{*\} \to M$ the inclusion of the point $(0, 1)$. It is contained in the face $[-1, 1] \times \{1\}$ and transversal to the boundary faces of that face.
3. $[-1, 1] \to \{-1\} \times [-1, 1]$, the inclusion of the boundary face.

The following examples are not admitted:

1. $S^1 \to M$, $f(\exp^{2\pi it}) = (\sin(2\pi t), \cos(t))$. This is not transversal to the boundary faces.
2. $(0, 1) \to \{-1\} \times (0, 1)$, the inclusion of the open boundary face. This is not closed.
3. $[-1, 1] \to M$, $t \mapsto (t, t)$, a diagonal. This is not considered to be transversal to the boundary.

$\square$

We must understand the topology of M near N. In the discussion below we assume for simplicity that N is a closed submanifold of the interior of M. But if N intersects boundary faces, then the constructions are similar. For example, if it is the embedding of a boundary face, then we must consider the half sided normal bundle pointing into interior of M. The conclusions hold in general.

Since f is an immersion we have an inclusion of vector bundles

$$df : TN \to f^*TM \ .$$

The quotient $\mathcal{N} := f^*TM/df(TN)$ is called **the normal bundle** of f. Let $z : N \to \mathcal{N}$ denote the zero section. The following differential geometric fact generalizes the existence of a collar for a codimension one submanifold:

Fact 3.3. *There exists a smooth map $F : \mathcal{N} \to M$ which is a diffeomorphism onto an open neighbourhood of N such that $f = F \circ z$.*

Proof. We sketch the idea. We choose a Riemannian metric on M and identify $\mathcal{N}$ with the orthogonal complement of $df(TN) \subseteq f^*TM$. The exponential map of M provides a diffeomorphism of a neighborhood of the zero section of $\mathcal{N}$ with a neighborhood of N in M. We now precompose with a scaling diffeomorphism which maps $\mathcal{N}$ into a suitable neighborhood of its zero section. □

From now on we identify $\mathcal{N}$ with its image under F.

Lemma 3.4. *If $N \to M$ is the inclusion of a closed submanifold, then $f^* : \Omega(M) \to \Omega(N)$ is surjective.*

Proof. We consider the open covering $\{U, \mathcal{N}\}$ of M with $U := M \backslash N$ and let $\{\chi_U, \chi_{\mathcal{N}}\}$ be an associated partition of unity. For $\omega \in \Omega(N)$ we define $\tilde{\omega} \in \Omega(M)$ as the extension by zero of $\chi_{\mathcal{N}} \mathtt{pr}^*\omega$, where $\mathtt{pr} : \mathcal{N} \to N$ is the bundle projection. Then $\tilde{\omega}_{|N} = \omega$. □

Since f^* is surjective, by Example 3.1 we have $H_{dR}(M, N) \cong H^*(\Omega(M, N))$, where $\Omega(M, N) := \mathtt{ker}(f^*)$. The sequence

$$\cdots \to H^{n-1}(N) \stackrel{\partial}{\to} H^n(M, N) \to H^n(M) \to H^n(N) \to \dots$$

is called the long exact sequence of the pair. Using the notation introduced in the proof of Lemma 3.4 the boundary operator can be calculated as follows. If

$[\omega] \in H^{n-1}(N)$, then we have

$$\partial[\omega] = [d\chi_{\mathcal{N}} \wedge \mathtt{pr}^*\omega] \ . \tag{16}$$

Assume that M is compact and oriented of dimension n and N is the boundary of M. Then the integration

$$\int_M : \Omega^n(M) \to \mathbb{R}$$

induces a homomorphism

$$\int_M : H^n_{dR}(M,N) \to \mathbb{R} \ .$$

Indeed, for $\omega \in \Omega^{n-1}(M,N)$ we have by Stoke's theorem

$$\int_M d\omega = \int_N \omega = 0 \ .$$

Example 3.5. We calculate $H^*_{dR}(D^n, S^{n-1})$ using the long exact sequence. The beginning is

$$0 \to H^0_{dR}(D^n, S^{n-1}) \to H^0_{dR}(D^n) \to H^0_{dR}(S^{n-1}) \to H^1_{dR}(D^n, S^{n-1}) \to 0 \ .$$

The second map is injective. We conclude that $H^0_{dR}(D^n, S^n) = 0$ and

$$H^1_{dR}(D^n, S^n) \cong \left\{ \begin{array}{cc} \mathbb{R} & n=1 \\ 0 & else \end{array} \right. \ .$$

We further have the segment

$$0 \to H^{n-1}_{dR}(S^{n-1}) \to H^n_{dR}(D^n, S^{n-1}) \to 0$$

showing that

$$H^n_{dR}(D^n, S^{n-1}) \cong \mathbb{R} \ .$$

We get

$$H^k_{dR}(D^n, S^{n-1}) \cong \left\{ \begin{array}{cc} \mathbb{R} & n=k \\ 0 & else \end{array} \right. \ . \tag{17}$$

Let $\chi \in C^\infty_c(D^n \setminus S^{n-1})$ be such that $\int_{D^n} \chi(x)dx = 1$. Then $\chi \mathrm{vol}_{\mathbb{R}^n} \in \Omega^n(D^n, S^{n-1})$ is a closed from. Its cohomology class generates $H^n_{dR}(D^n, S^{n-1})$ since

$$\int_{D^n} [\chi \mathrm{vol}_{\mathbb{R}^n}] = 1 \ .$$

□

Let $f : M \to N$ be a morphism between manifolds. If $H^*_{dR}(M)$ and $H^*_{dR}(N)$ are finite-dimensional, then so is $H^*_{dR}(f)$ by the long exact sequence (3.1). In this case we define

$$\chi(f) := \sum_{i\in\mathbb{Z}} (-1)^i \dim\ H^i(f)\ .$$

We have the relation

$$\chi(M) = \chi(N) + \chi(f)$$

which immediately follows from the long exact sequence (3.1). If f is the inclusion of a closed submanifold $N \hookrightarrow M$, then we write $\chi(M, N) := \chi(f)$.

Example 3.6. For $n \in \mathbb{N}$ we consider $S^{n-1} \to D^n$. We have the relation $\chi(D^n) = \chi(S^{n-1}) + \chi(D^n, S^{n-1})$. We get

$$\chi(D^n, S^{n-1}) = (-1)^n$$

as expected by (17). □

3.2 Compactly supported cohomology

For a manifold M let $\Omega_c(M)$ denote the complex of forms with compact support. The **de Rham cohomology of M with compact support** is defined by

$$H^*_{dR,c}(M) := H^*(\Omega_c(M))\ .$$

If M is oriented of dimension n and without boundary, then the integration over M induces a homomorphism

$$\int_M : H^n_{dR,c}(M) \to \mathbb{R}\ .$$

In the following we discuss the functoriality of the cohomology with compact support. If $M \to N$ is a proper map, then $f^* : \Omega_c(N) \to \Omega_c(M)$. Therefore $H^*_{dR,c}$ is a contravariant functor defined on the category of manifolds and proper maps.

If $i^U : U \to M$ is the inclusion of an open submanifold, then we have a map

$$i^U_! : \Omega_c(U) \to \Omega_c(M)$$

given by extension by zero. We get an induced map

$$i^U_! : H_{dR,c}(U) \to H_{dR,c}(M)\ .$$

Lemma 3.7. *Let $M = U \cup V$ be a covering of M by open subsets. Then we have the following Mayer-Vietoris sequence*

$$\dots \to H^{n-1}_{dR,c}(M) \stackrel{\partial}{\to} H^n_{dR,c}(U \cap V) \to H^n_{dR,c}(U) \oplus H^n_{dR,c}(V) \to H^n_{dR,c}(M) \to \dots \quad (18)$$

Proof. The Mayer-Vietoris sequence is the long exact sequence associated to the short exact sequence of complexes

$$0 \to \Omega_c(U \cap V) \xrightarrow{(i^{U\cap V}_{U,!}, -i^{U\cap V}_{V,!})} \Omega_c(U) \oplus \Omega_c(V) \xrightarrow{(\alpha,\omega)\mapsto i^U_!(\alpha)+i^V_!(\omega)} \Omega_c(M) \to 0 \ .$$

In order verify the surjectivity of the second map let $\omega \in \Omega_c(M)$. Using a partition of unity $\{\chi_U, \chi_V\}$ we get a preimage $(\chi_U\omega, \chi_V\omega)$. In order to show exactness in the middle assume that $i^U_!(\alpha)+i^V_!(\omega) = 0$. Then $\omega_{|U\setminus V} = 0$ and $\alpha_{|V\setminus U} = 0$. In particular, both forms are supported in $U \cap V$ and therefore coincide up to sign. The injectivity of the first map is clear. □

Consider an inclusion $f : N \to M$ of a closed submanifold into a compact manifold. Then we have an inclusion

$$i : \Omega_c(M \setminus N) \to \Omega(M, N) \ .$$

Proposition 3.8. *Let $f : N \to M$ the inclusion of a closed submanifold into a compact manifold. Then the map i induces an isomorphism in cohomology*

$$H^*_{dR,c}(M \setminus N) \to H^*_{dR}(M, N) \ .$$

Proof. We again assume for simplicity that N is a submanifold in the interior of M, but the general case is similar using a modified notion of a normal bundle (e.g. half sided for the inclusion of a boundary face).

We use embedding of the normal bundle $\mathcal{N}$ into M as a neighborhood of N. We choose a metric on $\mathcal{N}$. For $r \in (0, r)$ we let

$$\mathcal{N}_r := \{v \in \mathcal{V} \mid \|v\| \le r\} \subset \mathcal{N}$$

be the subbundle of discs of radius r. We define the complex

$$\Omega_r(M) := \{\omega \in \Omega(M) \mid \omega_{|\mathcal{N}_r} = 0\} \ .$$

Note that for $r' < r$ we have $\mathcal{N}_{r^{-1}} \subseteq \mathcal{N}_{(r')^{-1}}$ and hence $\Omega_{(r')^{-1}}(M) \subseteq \Omega_{r^{-1}}(M)$. We thus get an incresing family of chain complexes $(\Omega_{r^{-1}}(M))_{r\in(0,\infty)}$. Note that for all $r \in (0,\infty)$ we have

$$\Omega_r(M) \subseteq \Omega_c(M \setminus N) \ ,$$

and

$$\Omega_c(M \setminus N) = \bigcup_{r\in(0,\infty)} \Omega_r(M) \ . \tag{19}$$

Lemma 2.25 implies in view of (19) that

$$H^*(\Omega_c(M \setminus N)) \cong \mathtt{colim}_{r\in(0,\infty)} H^*(\Omega_r(M)) \ . \tag{20}$$

The projection $p : \mathcal{N}_r \to N$ is a homotopy equivalence with inverse the zero section z. Indeed, $p\circ z = \mathtt{id}_N$. The homotopy $z\circ p \sim \mathtt{id}_{\mathcal{N}_r}$ is given by $I\times\mathcal{N}_r \ni (t,v) \mapsto tv \in \mathcal{N}_r$. It follows that $z^* : H^*_{dR}(\mathcal{N}_r) \to H^*_{dR}(N)$ is an isomorphism.

We have an exact sequence

$$0 \to \Omega_r(M) \to \Omega(M) \to \Omega(\mathcal{N}_r) \to 0 \ .$$

We argue that the second map is surjective. Let $\omega \in \Omega(\mathcal{N}_r)$. Then there exists an open neighbourhood U of $\mathcal{N}_r$ and $\tilde\omega \in \Omega(U)$ such that $\tilde\omega_{|\mathcal{N}_r} = \omega$. We choose a cut-off function $\chi \in C^\infty_c(M)$ such that $\chi_{|\mathcal{N}_r} = 1$ (note that $\{\chi, 1-\chi\}$ is a partition of unity for the covering $(N \setminus \mathcal{N}_r, U)$ of M). Then $\chi\tilde\omega$ extends to all of M and is a preimage of ω.

For all $r \in (0,\infty)$ we get a map of long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^{n-1}_{dR}(\mathcal{N}_{r^{-1}}) \arrow[r, "\partial'"] \arrow[d, "z^*", "\cong"'] & H^n(\Omega_{r^{-1}}(M)) \arrow[r] \arrow[d, "i_r"] & H^n_{dR}(M) \arrow[r] \arrow[d, equal] & H^n_{dR}(\mathcal{N}_{r^{-1}}) \arrow[r, "\partial"] \arrow[d, "z^*", "\cong"'] & {} \\
{} \arrow[r] & H^{n-1}_{dR}(N) \arrow[r, "\partial"] & H^n_{dR}(M,N) \arrow[r] & H^n_{dR}(M) \arrow[r] & H^n_{dR}(N) \arrow[r] & {}
\end{tikzcd} \ . \tag{21}$$

The two right squares obviously commute. In order to show that the left square commutes we start with $[\omega] \in H^{n-1}_{dR}(N)$ and show that $i_r\partial' p^*[\omega] = \partial\omega$. Indeed, on the one hand, using the notation from above and $\tilde\omega := \mathtt{pr}^*\omega$, we have $i_r\partial' p^*[\omega] = [d(\chi\tilde\omega)] = [d\chi \wedge \mathtt{pr}^*\omega]$, where $\chi \in C^\infty(M)$ is such that $\chi_{|\mathcal{N}_{r^{-1}}} \equiv 1$ and $\chi_{|M\setminus\mathcal{N}_{2r^{-1}}} \equiv 0$, and $\mathtt{pr} : \mathcal{N} \to N$ is the bundle projection. On the other hand, $\partial[\omega] = [d\chi \wedge \mathtt{pr}^*\omega]$ by (16)

Applying the Five Lemma to (21) we see that

$$i_r : H^n(\Omega_{r^{-1}}(M)) \to H^n_{dR}(M, N)$$

is an isomorphism for all $r \in (0, \infty)$. We conclude that

$$H^n_{dR,c}(M \setminus N) \overset{(20)}{\cong} \mathtt{colim}_{r\in(0,\infty)} H^n(\Omega_{r^{-1}}(M)) \cong H^n_{dR}(M, N)$$

□

Example 3.9. Since $D^n \setminus S^{n-1}$ is diffeomorphic to $\mathbb{R}^n$ we have

$$H^k_{dR,c}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & n = k \\ 0 & else \end{cases}$$

We can use the integral

$$\int_{\mathbb{R}^n} : H^n_{dR,c}(\mathbb{R}^n) \to \mathbb{R}$$

in order to detect the cohomology. □

3.3 Poincaré duality

If C is a chain complex over some ring k, then we define the dual chain complex $C^* = \hom(C, n)$ by

$$(C^*)^n := \hom(C^{-n}, k) \ , \quad d : C^{*,n} \to C^{*,n+1} \ , \quad d(\phi) := (-1)^n \phi \circ d \ .$$

Let M be an oriented manifold M of dimension n. We define maps

$$\phi_k : \Omega^k(M) \to \Omega^{n-k}_c(M)^* \ , \quad \omega \mapsto (\alpha \mapsto (-1)^{(n+1)k} \int_M \omega \wedge \alpha)$$

for all $k \in \mathbb{Z}$. Note that $\Omega^{n-k}_c(M)^* = (\Omega_c(M)[n]^*)^k$.

Lemma 3.10. *If M has no boundary, then the collection of homomorphims $\phi = (\phi_k)_k$ induces a morphism of complexes*

$$\phi : \Omega(M) \to \Omega_c(M)[n]^* \ .$$

Proof. The proof is a calculation using Stoke's theorem. For this reason we must exclude the presence of a boundary. We are careful and distinguish the de Rham differential d_{dR} from the differential $(-1)^n d_{dR} = d$ of the shifted complex $\Omega_c(M)[n]$. For $\alpha \in \Omega_c(M)[n]^{-k-1} = \Omega_c^{n-k-1}(M)$ and $\omega \in \Omega^k(M)$ we have

$$\begin{aligned}
(d\phi_k(\omega))(\alpha) &= (-1)^k \phi_k(\omega)(d\alpha) \\
&= (-1)^{k+n} \phi_k(\omega)(d_{dR}\alpha) \\
&= (-1)^{k+n+(n+1)k} \int_M \omega \wedge d_{dR}\alpha \\
&= (-1)^{n+(n+1)k} \int_M d_{dR}(\omega \wedge \alpha) - (-1)^{n+(n+1)k} \int_M d_{dR}\omega \wedge \alpha \\
&= (-1)^{n-1+(n+1)k} \int_M d_{dR}\omega \wedge \alpha \\
&= (-1)^{n-1+(n+1)k-(n+1)(k+1)} \phi_{k+1}(d\omega)(\alpha) \\
&= \phi_{k+1}(d\omega)(\alpha) \ .
\end{aligned}$$

□

In this section we analyze conditions under which the morphism ϕ in Lemma 3.10 is a quasi-isomorphism.

Let C again be a complex of k-modules over some ring. We have an evalution pairing

$$C \otimes C^* \to k$$

which is a morphism of complexes if we consider the target as a complex concentrated in degree zero. It induces the second morphism in the composition

$$H(C) \otimes H(C^*) \to H(C \otimes C^*) \to k$$

and hence a morphism

$$H(C^*) \to H(C)^* \ .$$

Applying this to $C = \Omega_c(M)[n]$ we get the second map in the composition

$$\mathcal{P}_M : H_{dR}(M) \xrightarrow{\phi} H(\Omega_c(M)[n]^*) \to H_{dR,c}(M)[n]^* \ .$$

This map is natural in the sense that for an inclusion $i^U : U \to M$ we have the equality

$$\mathcal{P}_M(\omega)(i^U_!(\alpha)) = \mathcal{P}_U(i^{U,*}\omega)(\alpha) \ . \qquad (22)$$

Example 3.11. If $M = \mathbb{R}^n$, then

$$\mathcal{P}_{\mathbb{R}^n} : H^k_{dR}(\mathbb{R}^n) \to H^{n-k}_{dR,c}(\mathbb{R}^n)^*$$

is an isomorphism for all $k \in \mathbb{Z}$. Indeed we must consider the case $k = 0$, since for $k \neq 0$ the target and domain both vanish. Let $[1] \in H^0_{dR}(\mathbb{R}^n)$ be the generator. A generator of $H^n_{dR,c}(\mathbb{R}^n)$ is given by $[\chi \mathrm{vol}_{\mathbb{R}^n}]$, where $\chi \in C^\infty_c(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Note that

$$\mathcal{P}_{\mathbb{R}^n}([1])([\chi \mathrm{vol}_{\mathbb{R}^n}]) = (-1)^{(n+1)n} \int_{\mathbb{R}^n} 1 \wedge \chi \mathrm{vol}_{\mathbb{R}^n} = 1 \ .$$

□

We generalize the Example 3.11 from $\mathbb{R}^n$ to star shaped open subsets $G \subset \mathbb{R}^n$ and geodesically convex subsets of a Riemannian manifold.

Lemma 3.12. *If $G \subseteq \mathbb{R}^n$ is open and star shaped with respect to $0 \in \mathbb{R}^n$, then the Poincaré duality map*

$$\mathcal{P}_G : H^k_{dR}(G) \to H^{n-k}_{dR,c}(G)^*$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof. Note that by the Poincaré Lemma

$$H^k_{dR}(G) \cong \left\{ \begin{array}{cc} 0 & k \neq 0 \\ \mathbb{R} & k = 0 \end{array} \right\} \ .$$

It is clear by a similar argument as in Lemma 3.11 that $\mathcal{P}_G$ is injective. It suffices to show that

$$H^k_{dR,c}(G) \cong \left\{ \begin{array}{cc} 0 & k \neq n \\ \mathbb{R} & k = n \end{array} \right\} \ .$$

For simplicity we assume that G is bounded. In the following, our notation for an inclusion of an open subset is $i^X_Y : X \to Y$. Let B be an open ball at zero such that $\bar{B} \subseteq G$. It suffices to show that

$$i^B_{G,!} : H_{dR,c}(B) \to H_{dR,c}(G)$$

is an isomorphism. Since G is bounded there exists $r \in (0,1)$ such that $\overline{rG} \subset B$.

Let $f_r : B \to B$ be the multiplication with r. We have an obvious homotopy $H : [r,1] \times B \to B$, $H(s,x) := sx$, from f_r to $\mathtt{id}_B$. We get a chain homotopy

$$h : \Omega(B) \to \Omega(B)$$

such that $dh + hd = \mathtt{id} - f_r^*$. Recall that

$$h(\omega) := \int_r^1 (\iota_{\partial_s} H^*\omega)_{|\{s\}\times M} ds \ .$$

From the formula we deduce that if $\omega \in i^B_{rB,!}(\Omega_c(rB))$, then $h(\omega) \in \Omega_c(B)$.

If we precompose the identity $dh + hd = \mathtt{id} - f_r^*$ with $i^{rB}_{B,!}$ we get a chain homotopy

$$h \circ i^{rB}_{B,!} : \Omega_c(rB) \to \Omega_c(B)$$

from $f_r^* \circ i^{rB}_{B,!}$ to $i^{rB}_{B,!}$. The first map is an isomorphism since it is just the pull-back with the diffeomorphism $f_r : B \to rB$. Hence $i^{rB}_{B,!} : H_{dR,c}(rB) \to H_{dR,c}(B)$ is an isomorphism.

In a similar manner we show that $i^{rG}_{G,!} : H_{dR,c}(rG) \to H_{dR,c}(G)$ is an isomorphism. We now have factorizations

$$i^{rB}_{B,!} = i^{rG}_{B,!} \circ i^{rB}_{rG,!} \ , \quad i^{rG}_{G,!} = i^{B}_{G,!} \circ i^{rG}_{B,!} \ .$$

The first shows that $i^{rB}_{rG,!}$ is injective. Consequently, the equivalent map

$$i^{B}_{G,!} : H_{dR,c}(B) \to H_{dR,c}(G)$$

is injective, too. The second implies that

$$i^{B}_{G,!} : H_{dR,c}(B) \to H_{dR,c}(G)$$

is surjective.

If G is not bounded, then we must modify the rescaling map f_r appropriately and argue similarly. □

Corollary 3.13. *If M is an n-dimensional Riemannian manifold without boundary and $G \subseteq M$ is a geodesically convex subset, then*

$$\mathcal{P}_G : H^k_{dR}(G) \to H^{n-k}_{dR,c}(G)^*$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof. We use the exponential map in order to reduce to the case considered in Lemma 3.12. □

Proposition 3.14. *If M is a manifold without boundary which admits a Riemannian metric and a finite good covering by geodesically convex subsets, then*

$$\mathcal{P}_M : H_{dR}(M) \to H_{dR,c}(M)[n]^*$$

is an isomorphism.

Proof. We argue by induction on the number of elements of the covering. The case of one element is done by Corollary 3.13.

Let us now assume that the case of at most $n-1$ element is settled. Let $\mathcal{U} := \{U_1, \ldots, U_n\}$ be a good covering of M by geodesically convex subsets for some metric. We define $U := \bigcup_{i=1}^{n-1} U_n$ and consider the following map of long exact sequences. The upper sequence is the Mayer-Vietoris sequence associated to the covering $\{U, U_n\}$ of M.

$$\begin{tikzcd}
H^{k-1}_{dR}(U\cap U_n) \arrow[r, "\partial"] \arrow[d, "\mathcal{P}_{U\cap U_n}"'] & H^k_{dR}(M) \arrow[r] \arrow[d, "\mathcal{P}_M"] & H^k_{dR}(U)\oplus H^k_{dR}(U_n) \arrow[r] \arrow[d, "\mathcal{P}_U\oplus\mathcal{P}_{U_n}"] & H^k_{dR}(U\cap U_n) \arrow[d, "\mathcal{P}_{U\cap U_n}"'] \\
H^{n-k+1}_{dR,c}(U\cap U_n)^* \arrow[r, "\partial^*"] & H^{n-k}_{dR,c}(M)^* \arrow[r] & H^{n-k}_{dR,c}(U)^*\oplus H^{n-k}_{dR,c}(U_n)^* \arrow[r] & H^{n-k}_{dR,c}(U\cap U_n)^*
\end{tikzcd}$$

The lower sequence is the dual of the sequence (18). The right two squares commute by the naturality (22) of $\mathcal{P}_-$. We must verify that the square involving the boundary operators commutes. For our purpose it suffices that it commutes up to sign. Let $\omega \in \Omega^{k-1}(U \cap U_n)$ and $\alpha \in \Omega_c^{n-k+1}(M)$ be closed. Then we have $\partial[\omega] = d\chi_{U_n} \wedge \omega$, where (χ_U, χ_{U_n}) is a partition of unity. We get

$$\begin{aligned}
\mathcal{P}_M(\partial[\omega])([\alpha] &= \pm \int_M (d\chi_{U_n} \wedge \omega) \wedge \alpha \\
&= \pm \int_M \omega \wedge (d\chi_{U_n} \wedge \alpha) \\
&= \pm \mathcal{P}_{U\cap U_n}(\omega)(\partial[\alpha]) \\
&= \pm \mathcal{P}_{U\cap U_n}(\partial^*\omega)([\alpha]) \ ,
\end{aligned}$$

where the signs only depend on k and n. Now note that $\{U_1 \cap U_n, \ldots, U_{n-1} \cap U_n\}$ and $\{U_1, \ldots, U_{n-1}\}$ are a good coverings of $U \cap U_n$ and $\{U_1, \ldots, U_{n-1}\}$ by geodesically convex subsets, respectively, by $n-1$ open subsets. Consequently, $\mathcal{P}_U$, $\mathcal{P}_{U_n}$ and $\mathcal{P}_{U_n \cap U}$ are isomorphisms. We use the Five Lemma in order to conclude that $\mathcal{P}_M$ is an isomorphism. □

Corollary 3.15 (Poincaré duality)**.** *If M is a closed oriented n-dimensional manifold, then for all $k \in \mathbb{Z}$ we have an isomorphism*

$$\mathcal{P}_M : H^k_{dR}(M) \xrightarrow{\cong} H^{n-k}_{dR}(M)^* \ .$$

In particular, we have the identity of Betti numbers

$$b^k(M) = b^{n-k}(M) \ .$$

Proof. We choose a Riemannian metric g on M. If M is compact, then the proof of Proposition 2.15 produces a finite good covering of M by geodesically convex subsets. We also use compactness of M for the isomorphism $H_{dR,c}(M) \cong H_{dR}(M)$. □

Corollary 3.16. *If M is a closed oriented and connected n-dimensional manifold, then we have $b^n(M) = 1$.*

Proof. We have $b^n(M) = b^0(M) = 1$ since M is connected. □

Corollary 3.17 (Alexander duality)**.** *If M is a compact oriented n-dimensional manifold and $N \hookrightarrow M$ is a closed submanifold such that $M \setminus N$ has no boundary, then for all $k \in \mathbb{Z}$ we have an isomorphism*

$$\mathcal{P}_M : H^k_{dR}(M \setminus N) \xrightarrow{\cong} H^{n-k}_{dR}(M, N)^* \ .$$

In particular, we have the identity of Betti numbers

$$b^k(M \setminus N) = b^{n-k}(M, N) \ .$$

Proof. We use the Poincaré duality isomorphism

$$\mathcal{P}_{M \setminus N} : H^k_{dR}(M \setminus N) \xrightarrow{\cong} H^{n-k}_{dR,c}(M \setminus N)^*$$

and Proposition 3.8 which states that $H_{dR,c}(M \setminus N) \cong H_{dR}(M, N)$. □

Note that the condition on the pair (M, N) in 3.17 says that either M is closed and N is a closed submanifold in M or $N \hookrightarrow M$ is the inclusion of the boundary of M.

Corollary 3.18 (Intersection form)**.** *Let M is a closed oriented $2m$-dimensional manifold. Then we have a non-degenerated bilinear form*

$$\langle -, - \rangle : H^m(M) \otimes H^m(M) \to \mathbb{R} \ , \quad \langle [\omega], [\alpha] \rangle := \int_M \omega \wedge \alpha \ .$$

*It is symmetric for even m and antisymmetric for odd m. This form is called the **intersection form** of M.*

Proof. The form is nondegenerated since it induces the Poincaré duality isomorphism $\mathcal{P}_M : H^m_{dR}(M) \xrightarrow{\sim} H^m_{dR}(M)^*$ up to sign. □

Corollary 3.19. *Let M be a closed oriented $4k+2$-dimensional manifold. Then $b^{2k+1}(M)$ is even.*

Proof. The antisymmetric intersection form is non-degenerated. Hence it lives on an even-dimensional vector space. □

We consider a closed oriented manifold M of dimension $4k$. Its intersection form is a non-degenerated symmetric bilinear form on $H^{2k}_{dR}(M)$.

Definition 3.20. *The signature of the intersection form of a $4m$-dimensional manifold is called the **signature** of M and denoted by $\mathtt{sign}(M)$.*

If M^{op} denotes M with the opposite orientation, then $\mathtt{sign}(M^{op}) = -\mathtt{sign}(M)$.

Example 3.21. We consider the manifold $S^2 \times S^2$. Then by the Künneth formula $H^2_{dR}(S^2 \times S^2) \cong H^2_{dR}(S^2) \otimes H^0_{dR}(S^2) \oplus H^0_{dR}(S^2) \otimes H^2_{dR}(S^2)$ is spanned by $\mathtt{pr}_1^* \mathrm{vol}_{S^2}$ and $\mathtt{pr}_2^* \mathrm{vol}_{S^2}$. In this basis the intersection form is given by the matrix

$$\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) .$$

In particular, its signature is 0. □

Corollary 3.22. *If M is a closed oriented manifold of dimension n, then*

$$\chi(M) \in \left\{ \begin{array}{cc} 2\mathbb{Z} + \mathtt{sign}(M) & n \equiv 0(4) \\ 2\mathbb{Z} & n \equiv 2(4) \\ \{0\} & n \equiv 1(2) \end{array} \right. .$$

Proof. If $n = 2k$, then

$$\chi(M) = \sum_{i=0}^{2k}(-1)^i b^i(M) = b^k(M) + \sum_{i=0}^{k-1}(-1)^i(b^i(M)+b^{2k-i}(M)) = b^k(M) + 2\sum_{i=0}^{k}(-1)^i b^i(M)$$

and note that $b^k(M)$ is even for odd k by Corollary 3.19 and $b^k(M) \equiv \mathtt{sign}(M)$ mod 2 for even k. If $n = 2k+1$, then

$$\chi(M) = \sum_{i=0}^{2k}(-1)^i b^i(M) = \sum_{i=0}^{k}(-1)^i(b^i(M) - b^{2k+1-i}(M)) = 0$$

□

Let $\langle ., . \rangle$ be a bilinear form on a vector space V. A subspace $L \subseteq V$ is called **isotropic** if the restriction of $\langle ., . \rangle$ vanishes. Is is called **maximally isotropic** if it is isotropic and maximal with this property. If the form is non-degenerated, then L is called **Lagrangian** if it is isotropic of dimension $\dim(L) = \dim(V)/2$. A non-degenerated symmetric bilinear form on a real vector space admits a lagrangian subspace if and only if its signature vanishes.

Example 3.23. Let M be closed oriented of dimension n and consider the non-degenerated form

$$\langle ., . \rangle : H_{dR}(M) \otimes H_{dR}(M) \to \mathbb{R} , \quad \langle [\omega], [\alpha] \rangle = \int_M \omega \wedge \alpha \ .$$

For example, the subspace

$$\bigoplus_{k < \frac{n}{2}} H^k_{dR}(M) \subseteq H_{dR}(M)$$

is isotropic. It is Lagrangian if n is odd.

Assume now that the closed manifold M is the boundary of a compact oriented manifold W. We consider the restriction $r : H_{dR}(W) \to H_{dR}(M)$.

Lemma 3.24. *The subspace* $\mathtt{im}(r) \subseteq H_{dR}(M)$ *is Lagrangian.*

Proof. We first show that $\mathtt{im}(r)$ is isotropic. We have by Stoke's theorem for $[\omega], [\alpha] \in H_{dR}(W)$

$$\langle r([\omega]), r([\alpha]) \rangle = \int_M (\omega \wedge \alpha)_{|M} = \int_W d(\omega \wedge \alpha) = 0 \ .$$

The pairing induces a map $\kappa : \mathtt{im}(r) \to (H_{dR}(M)/\mathtt{im}(r))^*$. We show that this map is an isomorphism.

We first show surjectivity of κ. Let $[\omega] \in H_{dR}(W)$, $[\beta] \in H_{dR}(M)$ and $\tilde{\beta} \in \Omega(W)$ be some extension of β. Then we have

$$\langle r([\omega]), [\beta] \rangle = \int_M \omega_{|M} \wedge \beta = \int_W d(\omega \wedge \tilde{\beta}) = \pm \int_W \omega \wedge d\tilde{\beta} = \pm \mathcal{P}_W([\omega])(\partial[\beta]) \ .$$

We now fix $[\beta] \in H(M)$ and assume that $\langle r([\omega]), [\beta] \rangle = 0$ for all $[\omega] \in H(W)$. Since $\mathcal{P}_W : H_{dR}(W) \to H_{dR}(W, N)[n]^*$ is an isomorphism we conclude that $\partial[\beta] = 0$ and hence $[\beta] \in r(H_{dR}(W))$ so that $[\beta]$ represents zero in the quotient $(H_{dR}(M)/\mathtt{im}(r))^*$.

In order to show injectivity of κ we now fix $[\omega] \in H(W)$ and assume that $\langle r([\omega]), [\beta] \rangle = 0$ for all $[\beta] \in H_{dR}(M)$. Then we conclude that $r([\omega]) = 0$ since the pairing is non-degenerated.

The fact that κ is an isomorphism implies that $\mathtt{im}(r)$ is maximally isotropic of dimension $\dim H(M)/2$. $\square$

Assume now that $n = 4m$. Then the intersection $\mathtt{im}(r) \cap H^{2m}_{dR}(M)$ is Lagragian for the intersection form.

Corollary 3.25. *If a closed oriented manifold M of dimension $4m$ is a boundary of a compact oriented manifold, then $\mathtt{sign}(M) = 0$. The signature is an* ***obstruction against being an oriented boundary****.*

At the moment we do not have an example of non-trivial signature but we will see some later in the course.

$\square$

4 De Rham cohomology with coefficients in a flat bundle

4.1 Connections, curvature, flatness, cohomology

In this section we consider some elements of the local geometry of vector bundles. We discuss connections and its curvature. We are in particular interested in flat connections.

Definition 4.1. *A* ***connection*** ∇ *on a vector bundle* $V \to M$ *is a* $\mathbb{R}$*-linear map*

$$\nabla : \Gamma(M, V) \to \Gamma(M, T^*M \otimes V)$$

which satisfies the ***Leibniz rule***

$$\nabla(f\phi) = f\nabla\phi + df \otimes \phi \ , \quad f \in C^\infty(X) \ , \phi \in \Gamma(M, V) \ .$$

Usually one writes for a vector field $X \in \Gamma(M, TM)$ and a section $\phi \in \Gamma(M, V)$

$$\nabla_X \phi := \iota_X \nabla\phi \ .$$

In this notation, for $f \in C^\infty(M)$, we have the relations

$$\nabla_{fX}\phi = f\nabla_X\phi \ , \quad \nabla_X f\phi = f\nabla_X\phi + df \otimes \phi \ .$$

Lemma 4.2. *A vector bundle* $V \to M$ *admits connections.*

Proof. Assume that $V \cong M \times \mathbb{R}^k$. Such an isomorphism of vector bundles is called a **trivialization** and induces an isomorphism of section spaces

$$\Gamma(M, V) \cong C^\infty(M) \otimes \mathbb{R}^k \ .$$

We define the associated **trivial connection** on V by $\nabla^{triv} := d \otimes \mathtt{id}_{\mathbb{R}^k}$.

In the general case we fix an open covering $(U_\alpha)_{\alpha \in A}$ of M by domains of trivializations $V_{|U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ and a subordinated partition of unity $(\chi_\alpha)_{\alpha \in A}$. The trivializations induce trivial connections ∇^{triv}_α on $V_{|U_\alpha}$ for all $\alpha \in A$. For $\phi \in \Gamma(M, V)$ we can consider $\nabla^{triv}_\alpha(\chi_\alpha \phi_{|U_\alpha})$ as a section in $\Gamma(M, V)$ be extension by zero. We define

$$\nabla\phi := \sum_\alpha \nabla^{triv}_\alpha(\chi_\alpha \phi_{|U_\alpha}) \ .$$

One checks by calculations that ∇ is $\mathbb{R}$-linear and satisfies the Leibniz rule. $\square$

For $p \in \mathbb{Z}$ we consider the **p-forms on M with coefficients in V**

$$\Omega^p(M,V) := \Gamma(M, \Lambda^p T^*M \otimes V) \ .$$

The $\wedge$-product

$$\omega \wedge (\alpha \otimes v) := (\omega \wedge \alpha) \otimes v \ , \quad \omega, \alpha \in \Omega(M) \ , v \in \Gamma(M,V)$$

turns the sum

$$\Omega(M,V) := \bigoplus_{p \in Z} \Omega^p(M,V)$$

into a $\mathbb{Z}$-graded $\Omega(M)$ module. Note that a connection is an $\mathbb{R}$-linear map

$$\nabla : \Omega^0(M,V) \to \Omega^1(M,V) \ .$$

Lemma 4.3. *A connection ∇ on V has a unique extension*

$$\nabla' : \Omega(M,V) \to \Omega(M,V)$$

as an $\mathbb{R}$-linear degree one-map satisfying the Leibniz rule

$$\nabla'(\omega \wedge \phi) = (-1)^p \omega \wedge \nabla' \phi + d\omega \wedge \phi$$

for all $p \in \mathbb{N}$, $\omega \in \Omega^p(M)$ and $\phi \in \Omega(M,V)$.

Proof. We first show uniqueness. If ∇'_1, ∇'_2 are two extensions of ∇, then the Leibniz rule implies that their difference $\delta := \nabla'_1 - \nabla'_2$ satisfies

$$\delta(\omega \wedge \phi) = (-1)^p \omega \wedge \delta(\phi)$$

for $\omega \in \Omega^p(M)$ and $\phi \in \Omega(M,V)$. Furthermore, since both extend ∇ we have

$$\delta(\phi) = 0$$

for $\phi \in \Omega^0(M,V)$.

Let $\omega \in \Omega^p(M,V)$. In order to show that $\delta(\omega) = 0$ it suffices to show that $\chi\delta(\omega) = 0$ for every smooth function χ supported in a chart domain of M.

Assume that (x^i) are coordinates on an open subset $U \subseteq M$ and $\chi \in C^\infty_c(U)$. We can choose $\chi_1 \in C^\infty_c(U)$ such that $\chi\chi_1 = \chi$. We have

$$\omega_{|U} = \sum_I dx^I \otimes \phi_I$$

for a uniquely determined collection of sections $(\phi_I)_I$ in $\Omega^0(U,V)$, where the sum runs over the set of multi-indices $I = (i_1 < \dots < i_p)$. We have

$$\chi\omega = \chi \sum_I d(\chi_1 x^I) \wedge \chi_1\phi_I \ ,$$

where $\chi_1 x^I \in C^\infty(M)$ and $\chi_1\phi_I \in \Omega^0(M,V)$ are understod by extension by zero. We have

$$\begin{aligned} \chi\delta(\omega) &= \delta(\chi\omega) \\ &= \delta(\chi \sum_I d(\chi_1 x^I) \wedge \chi_1\phi_I) \\ &= (-1)^p \chi \sum_I d(\chi_1 x^I) \wedge \delta(\chi_1\phi_I) \\ &= 0 \ . \end{aligned}$$

The argument for the existence is similar. We choose an open covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ by chart domains, a subordinated partition of unity $(\chi_\alpha)_{\alpha\in A}$, a collection of smooth functions $(\chi_{\alpha,1})_{\alpha\in A}$ with $\chi_{\alpha,1} \in C^\infty_c(U_\alpha)$, $\chi_\alpha\chi_{\alpha_1} = \chi_\alpha$, and local coordinates $(x^i_\alpha)_{\alpha\in A}$. Given $\omega \in \Omega(M,V)$, then $\omega_{|U_\alpha} = \sum_I dx^I_\alpha \wedge \phi_{\alpha,I}$ for uniquely determined sections $\phi_{\alpha,I} \in \Omega(U_\alpha, V)$. We define

$$\nabla'\omega := (-1)^p \sum_{\alpha\in A} \chi_\alpha \sum_I dx^I_\alpha \wedge \nabla(\chi_{\alpha,1}\phi_I) \ .$$

One checks by a calculation that ∇' has the required properties. □
From now on we write $\nabla := \nabla'$ also for the extension.

We consider a vector field $X \in \Gamma(M,TM)$ on M. We define the operation of **insertion** of X

$$i_X : \Omega(M,V) \to \Omega(M,V)$$

of degree -1 such that on elementary tensors it is given by $i_X(\omega \otimes \phi) := i_X\omega \wedge \phi$.

We further define a version of the **Lie derivative**

$$\mathcal{L}^\nabla_X := \nabla i_X + i_X \nabla : \Omega(M,V) \to \Omega(M,V) \ .$$

Note that $\mathcal{L}_X^\nabla$ is given on elementary tensors by

$$\mathcal{L}_X^\nabla(\omega \otimes \phi) = \mathcal{L}_X \omega \otimes \phi + \omega \wedge \nabla_X \phi \ .$$

Indeed,

$$\begin{aligned}
\mathcal{L}_X^\nabla(\omega \otimes \phi) &= (\nabla i_X + i_X \nabla)(\omega \otimes \phi) \\
&= \nabla(i_X \omega \otimes \phi) + i_X(d\omega \otimes \phi + (-1)^{\deg(\omega)} \omega \wedge \nabla \phi) \\
&= d i_X \omega \otimes \phi + (-1)^{\deg(\omega)-1} i_X \omega \wedge \nabla \phi + i_X d\omega \otimes \phi \\
&\quad + (-1)^{\deg(\omega)} i_X \omega \wedge \nabla \phi + \omega \otimes \nabla_X \phi \\
&= ((d i_X + i_X d)\omega) \otimes \phi + \omega \otimes \nabla_X \phi \\
&= \mathcal{L}_X \omega \otimes \phi + \omega \otimes \nabla_X \phi
\end{aligned}$$

Lemma 4.4. *For vector fields $X, Y \in \Gamma(M, TM)$ and $\psi \in \Omega^1(M, V)$ we have*

$$[i_Y, \mathcal{L}_X^\nabla]\psi = -z i_{[X,Y]} \psi \ .$$

Proof. We calculate for $\omega \in \Omega^1(M)$ and $\phi \in \Gamma(M, V)$ and using

$$2d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

that

$$\begin{aligned}
[i_Y, \mathcal{L}_X^\nabla](\omega \otimes \phi) &= i_Y \mathcal{L}_X^\nabla(\omega \otimes \phi) - \mathcal{L}_X^\nabla i_Y(\omega \otimes \phi) \\
&= i_Y \mathcal{L}_X \omega \otimes \phi + \omega(Y) \nabla_X \phi - \nabla_X(\omega(Y)\phi) \\
&= (i_Y d i_X + i_Y i_X d)\omega \otimes \phi + \omega(Y) \nabla_X \phi - \nabla_X(\omega(Y)\phi) \\
&= Y(\omega(X)) + 2d\omega(X, Y) \otimes \phi - X(\omega(Y))\phi \\
&= -\omega([X, Y]) \otimes \phi
\end{aligned}$$

In general, the composition

$$\Omega^{p-1}(M, V) \xrightarrow{\nabla} \Omega^p(M, V) \xrightarrow{\nabla} \Omega^{p+1}(M, V)$$

does not vanish.

Definition 4.5. *We define the* ***curvature*** *of a connection as the degree 2-map $R^\nabla := \nabla \circ \nabla$.*

Lemma 4.6. *The $\mathbb{R}$-vector space $\Omega^p(M, \mathtt{End}(V))$ can be identified with the degree p-homomorphisms of $\Omega(M)$-modules $\Omega(M, V) \to \Omega(M, V)$.*

Proof. The result of the action of $\alpha \otimes \Phi \in \Omega^p(M, \mathtt{End}(V))$ on $\omega \otimes \phi \in \Omega(M, V)$ is given by $\alpha \wedge \omega \otimes \Phi(\phi)$. Here $\alpha, \beta \in \Omega(M)$, $\Phi \in \Gamma(M, \mathtt{End}(V))$ and $\phi \in \Gamma(M, V)$. Clearly, $\alpha \otimes \Phi$ acts as a degree p-homomorphism of $\Omega(M)$-modules.

Given a degree p-homomorphism of $\Omega(M)$-modules $\Omega(M, V) \to \Omega(M, V)$ one first uses the $C^\infty(M)$-linearity in order to see that it is given by a uniquely determined section of the bundle $\mathtt{End}(\Lambda T^*M \otimes V) \cong \mathtt{End}(\Lambda T^*M) \otimes \mathtt{End}(V)$. Since $\Lambda\mathbb{R}^k$ is a graded commutative unital algebra a degree p-homomorphism $\Lambda\mathbb{R}^k \to \Lambda\mathbb{R}^k$ of $\Lambda\mathbb{R}^k$-modules is given by multiplication by an element of $\Lambda^p\mathbb{R}^k$. Applying this fibrewise we conclude that a degree p-homomorphism of $\Omega(M)$-modules $\Omega(M, V) \to \Omega(M, V)$ must be given by a uniquely determined section of $\Lambda^p T^*M \otimes \mathtt{End}(V)$. □

Lemma 4.7. *We have $R^\nabla \in \Omega^2(M, \mathtt{End}(V))$.*

Proof. We must show that R^∇ is a morphism of $\Omega(M)$-modules $\Omega(M, V) \to \Omega(M, V)$ of degree 2. By definition R^∇ increases the degree by two. Let $\alpha \in \Omega^p(M)$. Then applying the Leibniz rule twice we get

$$\begin{aligned} R^\nabla \circ \alpha = \nabla \circ \nabla \circ \alpha &= (-1)^p \nabla \circ \alpha \circ \nabla + \nabla \circ d\alpha \\ = (-1)^{2p} \alpha \circ \nabla \circ \nabla + (-1)^p d\alpha \circ \nabla &+ (-1)^{p+1} d\alpha \circ \nabla + dd\alpha \\ &= \alpha \circ R^\nabla \end{aligned}$$

□

Lemma 4.8. *For vector fields $X, Y \in \Gamma(M, TM)$ and a section $\phi \in \Gamma(M, V)$ we have the following formula:*

$$R(X,Y)\phi = \frac{1}{2}\left(\nabla_X(\nabla_Y \phi) - \nabla_Y(\nabla_X \phi) - \nabla_{[X,Y]}\phi\right) .$$

Proof. We have

$$\begin{aligned} 2R(X,Y)\phi &= i_Y i_X (\nabla \circ \nabla)\phi \\ &= i_Y(-\nabla i_X \nabla \phi + \mathcal{L}^\nabla_X \nabla \phi) \\ &= -i_Y \nabla \nabla_X \phi + i_Y \mathcal{L}^\nabla_X \nabla \phi \\ &= -\nabla_Y \nabla_X \phi + \mathcal{L}^\nabla_X \nabla_Y \phi + [i_Y, \mathcal{L}^\nabla_X]\nabla \phi \\ &= -\nabla_Y \nabla_X \phi + \nabla_X \nabla_Y \phi + [i_Y, \mathcal{L}^\nabla_X]\nabla \phi \\ &\overset{Lemma 4.4}{=} \nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{[X,Y]}\phi . \end{aligned}$$

$\square$

Definition 4.9. *A connection ∇ on a bundle $V \to M$ is called **flat** if $R^\nabla = 0$.*

If ∇ is flat, then ∇ turns $\Omega(M,V)$ into a complex. We let $\mathbf{V} := (V,\nabla)$ be the notation of a bundle with a (flat) connection.

Definition 4.10. *We call*

$$H_{dR}(M,\mathbf{V}) := H(\Omega(M,V),\nabla)$$

the de Rham cohomology of M with coefficients in the flat bundle $\mathbf{V}$.

Remark 4.11. It follows from the Leibniz rule that the difference between two connections on V is a homomorphism $\Omega(M,V) \to \Omega(M,V)$ of degree one, i.e by Lemma 4.6 an element of $\Omega^1(M,\mathtt{End}(V))$. If we fix a connection ∇, then every other connection on V can uniquely be written in the form $\nabla + \alpha$ for $\alpha \in \Omega^1(M,\mathtt{End}(V))$.

If ∇ is a connection on V, then we get an induced connection ∇' on $\mathtt{End}(V)$ by

$$\nabla'\Phi = \nabla \circ \Phi + \Phi \circ \nabla : \Omega^0(M,\mathtt{End}(V)) \to \Omega^1(M,\mathtt{End}(V)) \ .$$

One checks the Leibniz rule by a calculation. From now one we will write $\nabla' := \nabla$. We get

$$R^{\nabla+\alpha} = R^\nabla + \nabla(\alpha) + \alpha \circ \alpha \ . \tag{23}$$

$\square$

Example 4.12. On a trivial bundle $V := M \times \mathbb{R}^n$ we have a **trivial connection** ∇^{triv}. If we identify $\Omega^0(M,V) \cong C^\infty(M) \otimes V$ using the trivialization, then $\nabla^{triv}\phi = d\phi$. We have $\Omega(M,V) \cong \Omega(M) \otimes V$ as complexes and $H_{dR}(M,V) \cong H_{dR}(M) \otimes V$ by Lemma 2.27.

If $\alpha \in \Omega^1(M,\mathtt{End}(V)) \cong \Omega^1(M) \otimes \mathtt{Mat}(n,\mathbb{R})$, then by (23) we have

$$R^{\nabla^{triv}+\alpha} = d\alpha + \alpha \circ \alpha \ .$$

So $\nabla^{triv} + \alpha$ is flat if and only if

$$d\alpha + \alpha \circ \alpha = 0 \ .$$

Let us write the term $\alpha \circ \alpha$ in local coordinates. Write $\alpha = dx^i \otimes A_i$. Then

$$\alpha \circ \alpha = dx^i \wedge dx^j \otimes A_i \circ A_j = \frac{1}{2} dx^i \wedge dx^j \otimes [A_i, A_j] \ .$$

If $\dim(V) = 1$, then this term vanishes and flatness is equivalent to $d\alpha = 0$. If $\dim(V) > 1$, then flatness is a nonlinear partial differential equation for α. $\square$

Example 4.13. In order to get a more interesting result we admit complex coefficients. We consider the trivial bundle $S^1 \times \mathbb{C} \to S^1$. Let $\alpha = \lambda dt$ for $\lambda \in C^\infty(S^1, \mathbb{C})$ and $\nabla := \nabla^{triv} - \lambda dt$. Note that ∇ is flat (as every connection on a bundle over a one-dimensional manifold).

We set $\mathbf{V} := (V, \nabla)$ and calculate the cohomology $H_{dR}(M, \mathbf{V})$ explicity. First of all

$$\mathtt{ker}(\nabla) = \{f \in C^\infty(S^1) \mid df = \lambda f\} \ .$$

We get $f(t) = f(0) \exp(\int_0^t \lambda(s) ds)$. We call

$$\mathrm{hol}_\nabla := \exp(\int_0^1 \lambda(s) ds) \in \mathbb{C}^*$$

the **holonomy** of ∇. If the holonomy is trivial (i.e. $= 1$), then f is determined by its value $f(0) \in \mathbb{C}$. Otherwise the solution does not close to a periodic function. We thus get

$$H^0_{dR}(S^1, \mathbf{V}) \cong \left\{ \begin{array}{cc} 0 & \mathrm{hol}(\nabla) \neq 1 \\ \mathbb{C} & \mathrm{hol}(\nabla) = 1 \end{array} \right. \ .$$

Let now $\omega dt \in \Omega^1(S^1, V)$. We try to solve $\nabla f = \omega dt$. This is the ordinary differential equation $f' - \lambda f = \omega$. We first consider the equation on $\mathbb{R}$ and discuss periodicity afterwards. The method of variation of constants gives the ansatz $f = C\Phi$ where $\Phi(t) = \exp(\int_0^t \lambda(s) ds)$. We get the equation $C'(t) = \Phi(t)^{-1}\omega(t)$ and hence

$$f(t) = f(0)\Phi(t) + \Phi(t) \int_0^t \Phi(s)^{-1} \omega(s) ds \ .$$

Periodicity requires that

$$0 = f(0) - f(1) = f(0)(1 - \Phi(1)) - \Phi(1) \int_0^1 \Phi(t)^{-1} \omega(t) dt = 0 \ .$$

If $\mathrm{hol}(\nabla) \neq 1$, then this equation has a unique solution for $f(0)$. If $\mathrm{hol}(\nabla) = 1$, then we can solve this equation if and only if $\int_0^1 \Phi(t)^{-1}\omega(t)dt = 0$. We get an isomorphism

$$H^1_{dR}(S^1, \mathbf{V}) \stackrel{\cong}{\to} \mathbb{C}\ , \quad [\omega dt] \mapsto \int_0^1 \Phi(t)^{-1}\omega(t)dt\ .$$

We conclude that

$$H^1_{dR}(S^1, \mathbf{V}) \cong \left\{ \begin{array}{cc} 0 & \mathrm{hol}(\nabla) \neq 1 \\ \mathbb{C} & \mathrm{hol}(\nabla) = 1 \end{array} \right. .$$

4.2 Geometry of flat vector bundles

In this section we discuss some elements of the global geometry of vector bundles equipped with a connection. We consider the parallel transport. We are in particular interested in consequences of the flatness of the connection.

Let V be a vector bundle with connection ∇.

Definition 4.14. *A section $\phi \in \Gamma(M, V)$ is called **parallel** if it satisfies the equation $\nabla\phi$.*

Lemma 4.15. *Let $U \subseteq \mathbb{R}$ be an interval, $t_0 \in U$ and $V \to U$ be a vector bundle with connection ∇. Then every vector $v \in V_{t_0}$ (the fibre of V at t_0) has a unique extension to a parallel section $\phi \in \Gamma(U, V)$.*

Proof. We first assume that V is trivial of dimension n. Then there exists a unique $\alpha \in C^\infty(U, \mathtt{Mat}(n, n, \mathbb{R}))$ such that $\nabla = \nabla^{triv} + \alpha dt$. The section ϕ is obtained by solving the differential equation

$$\phi' = -\alpha\phi\ , \quad \phi(t_0) = v\ .$$

This differential equation is linear with non-constant coefficients smoothly depending on t and therefore has a unique global solution.

For a general V this construction produces the section ϕ in the domain of a local trivialization of V. Because of the uniqueness these local solutions can be patched together. $\square$

Lemma 4.16. *A vector bundle on an interval is trivial.*

Proof. Let $V \to U$ be a vector bundle on an interval. We fix a point $t_0 \in U$. By Lemma 4.2 we can choose a connection. If we fix a basis (v_i) of V_{t_0}, then the parallel extensions (ϕ_i) of the basis vectors will give a trivialization of V.

We claim that these sections are linearly independent at each point of U. We consider a subinterval $U' \subseteq U$ which is a domain of a local trivialization of V and assume that $t_1 \in U'$ is such that $(\phi_i(t_1))_i$ is linearly independent in V_{t_1}. We consider the matrix $\Phi(t)$ formed by these sections defined using the local trivialization. We further write $\nabla_{|U'} = \nabla^{triv} + \alpha dt$. Then

$$\det(\Phi(t))' = \det(\Phi(t))\mathtt{Tr}(\Phi(t)^{-1}\Phi'(t)) = -\det(\Phi(t))\mathtt{Tr}(\Phi(t)^{-1}\alpha(t)\Phi(t)) \ .$$

This differential equation implies

$$\det(\Phi(t)) = \det(\Phi(t_1))\exp(-\int_{t_1}^{t}\mathtt{Tr}(\Phi(s)^{-1}\alpha(s)\Phi(s))ds)$$

for all $t \in U'$. Now $\det(\Phi(t_1)) \neq 0$ expresses the fact that $(\phi_i(t_1))_i$ is a basis. We conclude that $\det(\Phi(t)) \neq 0$ on U'.

We obtain the global statement by patching. □

In the following we show that a connection on a vector bundle gives rise to **local trivializations by radial parallel transport**. We consider a ball $U \subseteq \mathbb{R}^n$ and a vector bundle $V \to U$ with a connection ∇. Given $v \in V_0$ we can define a section $\phi \in \Gamma(U, V)$ such that for $x \in U$ the value $\phi(x)$ is the value of a parallel extension of v along the path $[0,1]x \cap U$. The smooth dependence of solutions of ordinary differential equations on parameters shows that ϕ is smooth. Again, if (v_i) is a basis of V_0, then the corresponding basis (ϕ_i) is a trivialization of V.

Let $v \in V_0$ and $\phi \in \Gamma(U, V)$ be the section obtained by radial parallel transport.

Lemma 4.17. *If ∇ is flat, then $\nabla\phi = 0$.*

Proof. We calculate for vectors $X, Y \in \mathbb{R}^n$ such that $X \in U$ (which we also consider as constant vector fields so that $[X, tY] = 0$)

$$\begin{aligned} (\nabla_X \nabla_{tY}\phi)(tX) &\overset{Lemma 4.8}{=} (\nabla_{tY}\nabla_X\phi)(tX) + (2R^{\nabla}(X, tY)\phi)(tX) \\ &= 0 \ . \end{aligned}$$

Here the first term vanishes since ϕ is parallel along the ray $\mathbb{R}^+X$, and the second term vanishes since $R^{\nabla} = 0$. The section $tX \mapsto \nabla_{tY}\phi(tX)$ on the ray $\mathbb{R}^+X$ is thus parallel. Since it vanishes for $t = 0$ it vanishes identically. Therefore, setting $t = 1$, we get $(\nabla_Y \phi)(X) = 0$. □

Corollary 4.18. *Let $V \to M$ be a vector bundle with a connection ∇. Then the following assertions are equivalent:*

1. *The connection ∇ is flat.*
2. *The bundle $V \to M$ admits local trivializations by parallel sections.*

Proof. If ∇ is flat, then by Lemma 4.17 the bundle $V \to M$ admits local trivializations by parallel sections obtained by radial parallel transport. Vice versa, if V admits local trivializations by parallel sections, then locally in such a trivialization $\nabla = \nabla^{triv}$. In particular it is flat. □

Remark 4.19. In this long remark we recall the relation between vector bundles and **cocycles**.

Let us assume that we have two local trivializations $(\phi_i)_i$ and $(\psi)_i$ of V by families of sections. Then we consider the matrix valued function $({g_i}^j)$ defined by

$$ {g_i}^j \phi_j = \psi_i \ . $$

If the sections are parallel with respect to a connection ∇, then we get

$$ 0 = \nabla {g_i}^j \phi_j = d{g_i}^j \phi_j $$

for all i. Since $(\phi_j)_j$ is a basis we see that $d{g_i}^j = 0$ for all i, j. Hence the matrix-valued function $({g_i}^j)$ is locally constant.

Let $V \to M$ be a k-dimensional real vector bundle, $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an open covering, $(\phi_{\alpha,j})$, $j \in \{1, \dots, k\}$, $\alpha \in A$, families of sections in $\Gamma(U_\alpha, V)$ which trivialize $V_{|U_\alpha}$. We then consider the associated cocycle, i.e. the family of $\mathtt{Mat}(k, k, \mathbb{R})$-valued functions

$$ (g_{\alpha\beta})_{\alpha\beta \in A^2} \ , \quad g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathtt{Mat}(k, k, \mathbb{R}) $$

characterized by

$$ {g_{\alpha\beta,i}}^j \phi_{\beta,j} = \phi_{\alpha,i} \ . $$

We have the cocycle relations

$$ g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta} \tag{24} $$

on $U_\alpha \cap U_\beta \cap U_\gamma$ for all $\alpha\beta\gamma \in A^3$.

Vice versa, given an open covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ and a collection of $\mathtt{Mat}(k,k,\mathbb{R})$-valued functions

$$(g_{\alpha\beta})_{\alpha\beta\in A^2}\ ,\quad g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathtt{Mat}(k,k,\mathbb{R})$$

as above satisfying the cocycle relations (24), then we can define a vector bundle as the quotient

$$W := \bigsqcup U_\alpha \times \mathbb{R}^k / \sim\ .$$

Here $(u,x) \in U_\alpha \times \mathbb{R}^k$ and $(v,x) \in U_\beta \times \mathbb{R}^k$ are related with respect to the relation $\sim$ if $u = v$ (in M) and $y = (g_{\alpha\beta})^t x$. The cocycle relation ensures that $\sim$ is an equivalence relation.

The canonical maps $U_\alpha \times \mathbb{R}^k \to W$ are local trivializations of W. The associated cocycle is exactly the one we started with.

Assume that the cocycle came from a collection of local trivializations of a vector bundle V as above. Then we can define an isomorphism $W \stackrel{\cong}{\to} V$ which sends the class of $(u,x) \in U_\alpha \times \mathbb{R}^k$ to $x^i\phi_{\alpha,i}(u) \in V$.

If $\mathbf{V}$ is a flat bundle, then we can find local trivializations $V_{|U_\alpha} := U_\alpha \times \mathbb{R}^n$ such that the associated cocycle $(g_{\alpha,\beta})$ consists of locally constant $\mathtt{Mat}(k,k,\mathbb{R})$-valued functions.

Conversely, if we are given a cocycle $(g_{\alpha\beta})_{\alpha\beta}$ consisting of locally constant functions, then the associated vector bundle W is equipped with a flat connection ∇. It is characterized by the property that for every $x \in \mathbb{R}^k$ and $\alpha \in A$ the section $U_\alpha \ni u \mapsto [u,x] \in W$ is parallel. $\square$

4.3 Properties of the de Rham cohomology with coefficients in a flat bundle

Let (V,∇) be a vector bundle with connection on M and $f : N \to M$ be a smooth map. We have induced maps $f^* : \Omega(M,V) \to \Omega(N, f^*V)$. First of all, the pull-back of $\psi \in \Omega(M,V) = \Gamma(M, \Lambda(T^*M) \otimes V)$ is naturally a section $\tilde{\psi} \in \Gamma(N, f^*\Lambda(T^*M) \otimes f^*V)$. We now apply the bundle map

$$\Lambda(df)^* \otimes \mathtt{id}_{f^*V} : f^*\Lambda(T^*M) \otimes f^*V \to \Lambda(T^*N) \otimes f^*V$$

in order to get the section

$$f^*\psi \in \Gamma(N, \Lambda(T^*N) \otimes V) = \Omega(N, f^*V) \ .$$

Note that $f^* : \Omega(M, V) \to \Omega(N, f^*V)$ is a homomorphism of $\Omega(M)$-modules, where the action of $\Omega(M)$ on the target is induced from the action of $\Omega(N)$ via $f^* : \Omega(M) \to \Omega(N)$.

Lemma 4.20. *The bundle $f^*V \to N$ has an* ***induced connection*** *$f^*\nabla$ which is uniquely characterized by the property that for $\phi \in \Gamma(M, V)$ we have*

$$(f^*\nabla)(f^*\phi) = f^*(\nabla\phi) \ .$$

The extension of $f^*\nabla$ to $\Omega(N, f^*V)$ satisfies a similar relation

$$f^*(\nabla\omega) = (f^*\nabla)(f^*\omega) \tag{25}$$

for every $\omega \in \Omega(M, V)$.

Proof. Given $x \in N$ we can find a trivializing family of sections $(\phi_i)_i$ of V in a neighborhood U_x of $f(x)$. Then $(f^*\phi_i)$ is a trivializing family of sections of f^*V on $f^{-1}(U_x)$. The condition $\nabla_x(f^*\phi_i) = f^*(\nabla\phi_i)$ for all i uniquely determines a connection ∇_x on $f^*V_{|f^{-1}(U_x)}$. One easily checks using the Leibnitz rule that it is independent of the choice of the trivializing family. Indeed, let $(\phi'_i)_i$ be a second trivializing family with induced connection ∇'_x. Then there exists a matrix valued function $({g_i}^j)$ such that $\phi'_i = {g_i}^j\phi_j$. We have

$$\begin{aligned}
\nabla_x(f^*\phi'_i) &= \nabla_x(f^*{g_i}^j)(f^*\phi_j) \\
&= d(f^*{g_i}^j) \otimes f^*\phi_j + f^*({g_i}^j)f^*(\nabla\phi_j) \\
&= f^*(d{g_i}^j \otimes \phi_j + {g_i}^j\nabla\phi_i) \\
&= f^*(\nabla {g_i}^j\phi_j) \\
&= f^*(\nabla\phi'_i) \\
&= \nabla'_x(f^*\phi'_i) \ .
\end{aligned}$$

We can perform this construction for each point $x \in N$, and for $x, y \in N$ the resulting connections ∇_x and ∇_y coincide on the intersections $f^{-1}(U_x) \cap f^{-1}(U_y)$ by the uniqueness statement. Hence we get a globally defined connection $f^*\nabla$. Technically one chooses a partition of unity $(\chi_x)_{x\in N}$ subordinated to the open covering $(U_x)_{x\in N}$ and sets $f^*\nabla := \sum_{x\in N}\chi_x\nabla_x$.

If $\omega = \alpha\otimes\phi$ for $\alpha\in\Omega^p(M)$ and $\phi\in\Gamma(M,V)$, then $f^*\omega = f^*\alpha\otimes f^*\phi$. We have

$$\begin{aligned}(f^*\nabla)(f^*\omega) &= df^*\alpha\otimes f^*\phi + (-1)^p f^*\alpha\otimes(f^*\nabla)(f^*\phi)\\ &= f^*d\alpha\otimes f^*\phi + (-1)^p f^*\alpha\otimes f^*(\nabla\phi)\\ &= f^*(\nabla\omega)\ .\end{aligned}$$

This implies (25). □

Lemma 4.21. *Let $f : N\to M$ be a smooth map and (V,∇) be a vector bundle with connection on M. Then we have $R^{f^*\nabla} = f^*R^\nabla$.*

Proof. We have for every $\phi\in\Gamma(M,V)$

$$\begin{aligned}(f^*R)(f^*\phi) &= f^*(R\phi)\\ &= f^*(\nabla\nabla\phi)\\ &= (f^*\nabla)(f^*(\nabla\phi))\\ &= (f^*\nabla)(f^*\nabla)(f^*\phi)\\ &= R^{f^*\nabla}(f^*\phi)\ .\end{aligned}$$

This implies the assertion. □

If ∇ is flat, then $f^*\nabla$ is flat, too. We define the **pull-back of a flat bundle** by $f^*\mathbf{V} := (f^*V, f^*\nabla)$. The relation (25) expresses the fact that

$$f^* : \Omega(M,\mathbf{V})\to\Omega(N,f^*\mathbf{V})$$

is a morphism of complexes. Hence we get an induced morphism in cohomology

$$f^* : H_{dR}(M,\mathbf{V})\to H_{dR}(N,f^*\mathbf{V})\ .$$

Lemma 4.22. 1. *If f_0 and f_1 are homotopic, then we have an isomorphism $\Psi_1 : f_0^*\mathbf{V}\overset{\cong}{\to} f_1^*\mathbf{V}$ which only depends on the homotopy.*

2. *The following diagram commutes*

$$\begin{tikzcd} H_{dR}(M,\mathbf{V}) \arrow[rr, "f_0^*"] \arrow[dr, "f_1^*"'] & & H_{dR}(M,f_0^*\mathbf{V}) \arrow[dl, "\Psi_1"] \\ & H_{dR}(N,f_1^*\mathbf{V}) & \end{tikzcd}$$

i.e. the de Rham cohomology with coefficients in a flat bundle is homotopy invariant.

Proof. Let $h : I \times N \to M$ be a homotopy from f_0 to f_1. Given

$$v \in (h^*V)_{(0,n)} \cong (f_0^*V)_n$$

we define $\Psi(t,n)(v) \in (h^*V)_{(t,n)}$ to be the parallel transport of v long the path $s \mapsto (st, n)$. We get a bundle map

$$\Psi_t : f_0^*V \to h_t^*V \ , \quad (f_0^*V)_n \ni v \mapsto \Psi(t,n)(v) \in (h^*V)_{(t,n)} \ .$$

Evaluating at $t = 1$ we get a bundle map $\Psi_1 : f_0^*V \to f_1^*V$.

We observe that for $\psi \in \Gamma(I \times N, h^*V)$ we have

$$\partial_t(\Psi_t^{-1} n_t^* \psi) = \Psi_t^{-1} n_t^* (\mathcal{L}_{\partial_t}^{h^*\nabla} \psi) \ , \tag{26}$$

where $n_t : N \to I \times N$, $n_t(n) := (t, n)$.

We show that $\Psi_t^{-1}(h_t^*\nabla)\Psi_t = f_0^*\nabla$. To this we show that the family of connections

$$t \mapsto \nabla_t := \Psi_t^{-1}(h_t^*\nabla)\Psi_t$$

on f_0^*V is constant. For a section $\phi \in \Gamma(N, f_0^*V)$ we define a section

$$\psi \in \Gamma(I \times N, h^*V) \ , \quad \psi(t,n) := \Psi(t,n)\phi(n) \ .$$

Then we have

$$\nabla_t \phi = \Psi_t^{-1} n_t^* (h^*\nabla)\psi \ .$$

We calculate the derivative with respect to t. Let $X \in \Gamma(N, TN)$. Then

$$\begin{aligned}
\partial_t \nabla_{t,X} \phi &= \partial_t[\Psi_t^{-1} n_t^* (h^*\nabla)_X \psi] \\
&\stackrel{(26)}{=} \Psi_t^{-1} n_t^* \mathcal{L}_{\partial_t}^{h^*\nabla} (h^*\nabla)_X \psi \\
&= \Psi_t^{-1} n_t^* (h^*\nabla)_{\partial_t} (h^*\nabla)_X \psi \\
&= \Psi_t^{-1} n_t^* \left[(h^*\nabla)_X (h^*\nabla)_{\partial_t} \psi + 2R^{h^*\nabla}(\partial_t, X)\psi\right] \\
&= 0,
\end{aligned}$$

since $(h^*\nabla)_{\partial_t}\psi = 0$ and $R^{h^*\nabla} = 0$. It follows that

$$\Psi_t^{-1}(h_t^*\nabla)\Psi_t = \nabla_t = \nabla_0 = f_0^*\nabla \ .$$

In particular, we get an isomorphism $\Psi_1 : f_0^*\mathbf{V} \to f_1^*\mathbf{V}$ of flat bundles.

We now claim that the maps of chain complexes

$$f_0^* : \Omega(M, \mathbf{V}) \to \Omega(N, f_0^*\mathbf{V})$$

and

$$\Omega(M, \mathbf{V}) \stackrel{f_1^*}{\to} \Omega(N, f_1^*\mathbf{V}) \stackrel{\Psi_1^{-1}}{\to} \Omega(N, f_0^*\mathbf{V})$$

are chain homotopic. To this end we define the degree -1-map

$$H : \Omega(M, \mathbf{V}) \to \Omega(N, f_0^*\mathbf{V})$$

by

$$H(\psi) := \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*\psi)dt \ .$$

We have

$$\begin{aligned}
&((f_0^*\nabla)H + H\nabla)\psi \\
&\quad = \quad (f_0^*\nabla)\int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*\psi)dt + \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt \\
&\quad = \quad \int_0^1 (f_0^*\nabla)\Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*\psi)dt + \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt \\
&\quad = \quad \int_0^1 \Psi_t^{-1} n_t^*(h^*\nabla)(\iota_{\partial_t} h^*\psi)dt + \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt \\
&\quad = \quad \int_0^1 \Psi_t^{-1} n_t^*\mathcal{L}_{\partial_t}^{h^*\nabla} h^*\psi dt - \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t}(h^*\nabla)(h^*\psi))dt + \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt \\
&\quad = \quad \int_0^1 \Psi_t^{-1} n_t^*\mathcal{L}_{\partial_t}^{h^*\nabla} h^*\psi dt - \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt + \int_0^1 \Psi_t^{-1} n_t^*(\iota_{\partial_t} h^*(\nabla\psi))dt \\
&\quad = \quad \int_0^1 \Psi_t^{-1} n_t^*\mathcal{L}_{\partial_t}^{h^*\nabla} h^*\psi dt \\
&\quad \stackrel{(26)}{=} \quad \int_0^1 \partial_t \Psi_t^{-1} n_t^* h^*\psi dt \\
&\quad = \quad \Psi_1^{-1} f_1^*\psi - f_0^*\psi
\end{aligned}$$

$\square$

4.4 Global structure and further examples of flat vector bundles

Definition 4.23. *A manifold M is called **simply connected** if every two paths with the same endpoints are homotopic through paths with these endpoints.*

Note that this differs from the usual definition of this notion. We do not assume that M is connected on the one hand. On the other hand, our definition involves smooth paths and homotopies instead of continuous ones.

Theorem 4.24. *If M is connected and simply connected, then every flat vector bundle $\mathbf{V}$ on M is trivial.*

Proof. Let $\mathbf{V} = (V, \nabla)$. We choose a point $m_0 \in M$. We are going to construct an isomorphism

$$\Psi : M \times V_{m_0} \to V$$

such that $\Psi \nabla^{triv} = \nabla \Psi$. Let $m \in M$. We choose a path γ from m_0 to m. We define $\Psi^{\gamma}_m : V_0 \to V_m$ by parallel transport along the path γ. We must verify that Ψ^{γ}_m does not depend on the choice of the path γ. Since any two paths are homotopic we will consider a homotopy $h : [0,1] \times [0,1]$ of paths from m_0 to m. We can trivialize $h^*V \cong [0,1] \times [0,1] \times V_{m_0}$ using the connection $h^*\nabla$ and the parallel transport along rays starting in $(0,0)$. The restriction of this trivialization to $\{i\} \times \{1\}$ is $\Psi^{h_i}_m$ for $i = 0,1$. Since $h(-,1)$ is constant with value m the parallel transport from $V_{(0,1)}$ to $V_{(1,1)}$ is the identity if one identifies nbthe fibres $h^*V_{(s,1)}$ for all $s \in [0,1]$ with V_m. We conclude that $\Psi^{h_0}_m = \Psi^{h_1}_m$.

We now show that Ψ is smooth and preserves the connection. Let $m_1 \in M$. We fix a path γ_1 from m_0 to m_1 and a coordinate neighbourhood of m_1 diffeomorphic to a ball in $\mathbb{R}^n$. For $m \in U$ let γ_m be the straight path (this uses the local coordinates) from m_1 to m and $\Psi^{\gamma_m}_m$ be the corresponding parallel transport. Then

$$\Psi^{\gamma}_m(v) = \Psi^{\gamma_m}_m(\Psi^{\gamma_1}_{m_1}(v)) \ ,$$

where γ is a smooth concatenation of γ_m and γ_1. The section $m \mapsto \Psi^{\gamma_m}_m(\Psi^{\gamma}_{m_1}(v))$ is smooth and parallel by Lemma 4.17. The last property is equivalent to the equality $\Psi \nabla^{triv} = \nabla \Psi$. $\square$

Example 4.25. The manifold $\mathbb{R}^n$ is simply connected. If γ_0 and γ_1 are two paths with the same endpoints, then $h(s,t) := s\gamma_1(t) + (1-s)\gamma_0(t)$ is a homotopy between them.

We now show that S^n is simply connected for $n \geq 2$. We use the following general fact which is a consequence of Sard's theorem.

Fact 4.26. *If* $f : M \to N$ *is a map between smooth manifolds and* $\dim(N) < \dim(M)$*, then* $f(N)$ *is a Lebesgue zero set in* M*.*

If γ_0 and γ_1 are paths in S^n and $n \geq 2$, then their joint image is a Lebesgue zero set. Hence there exists a point $o \in S^n$ which does not belong to the union of the images of the paths. Therefore the two paths are contained in $S^n \setminus \{o\} \cong \mathbb{R}^n$ and can be connected by a homotopy with constant endpoints as in the first example.

□

Example 4.27. The manifold S^1 is not simply connected. In Example 4.13 we have seen that the de Rham cohomology of S^1 with coefficients in a flat bundle depends non-trivially on the bundle. Hence these flat bundles can not all be isomorphic.

Example 4.28. In this example we show that the **fibre wise de Rham cohomology** of a locally trivial fibre bundle $E \to B$ is a vector bundle which has a natural flat connection. It is called the **Gauss-Manin connection**.

Let $E \to B$ be a locally trivial fibre bundle bundle. For $b \in B$ let E_b be the fibre at b. We fix an integer p and consider the set

$$\mathcal{H}^p(E/B) := \bigsqcup_{b \in B} H^p_{dR}(E_b) \ . \tag{27}$$

We assume that $H^p_{dR}(E_b)$ is finite-dimensional for all $b \in B$. In this case we equip $\mathcal{H}^p(E/B)$ with the structure of a vector bundle over B as follows. By definition, the bundle projection $\mathcal{H}^p(E/B) \to B$ maps the component $H^p_{dR}(E_b)$ to the point b. The fibres of this projection are vector spaces. The manifold structure on $\mathcal{H}^p(E/B)$ is defined in terms of local trivializations. It suffices to check that the associated cocycles are given by smooth functions. In the present case they turn out to by locally constant.

Let $\phi : U \times F \stackrel{\cong}{\to} E_{|U}$ be a local trivialization of $E \to B$. For $u \in U$ we write $\phi_u : F \to E_u$ for the inclusion of the fibre over u. Then we define an associated local trivialization of $\mathcal{H}^p(E/B)$ by

$$\mathcal{H}^p(E/B)_{|U} \stackrel{\cong}{\to} U \times H^p_{dR}(F)$$

by $\mathcal{H}^p(E/B)_u \ni x \mapsto (u, \phi_u^* x)$, $u \in U$. Given another trivialization ψ we consider the transition function

$$g : U \times F \to F \ , \quad (u,f) \mapsto g(u,f) := (\psi_u^{-1} \circ \phi_u)(f) \ .$$

We have

$$\phi_u^* \circ \psi_u^{-1,*} = (\psi_u^{-1} \circ \phi_u)^* = g(u,-)^* \in \mathtt{Aut}(H^p_{dR}(F)) \ .$$

By homotopy invariance of the de Rham cohomology this function with values in $\mathtt{Aut}(H^p_{dR}(F))$ is locally constant.

By the constructions in Example 4.19 it follows that $\mathcal{H}^p(E/B)$ has the structure of a locally trivial vector bundle with a flat connection $\nabla^{\mathcal{H}(E/B)}$. This connection is called the Gauss-Manin connection. $\square$

More generally we have:

Lemma 4.29. *Let $E \to B$ be a locally trivial fibre bundle and $\mathbf{V}$ be a flat bundle on E. Then the fibrewise de Rham cohomology $\mathcal{H}^p(E/B, \mathbf{V})$ (if it is finite dimensional) is a vector bundle on B with a flat Gauss-Manin connection $\nabla^{\mathcal{H}^p(E/B,\mathbf{V})}$.*

$\square$

Example 4.30. We consider a torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$. Let $A = (A^i{}_j) \in SL(2,\mathbb{Z})$. This matrix acts as a linear transformation on $\mathbb{R}^2$ and preserves the lattice $\mathbb{Z}^2$. Consequently it descends to a diffeomorphism $f_A : T^2 \to T^2$ such that

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\pi"] & \mathbb{R}^2 \arrow[d, "\pi"] \\ T^2 \arrow[r, "f_A"] & T^2 \end{tikzcd}$$

commutes. We now consider the action of $\mathbb{Z}$ on $\mathbb{R} \times T^2$ given by $(n,(t,x)) = (t + n, f_A^n(x))$ and let $T_{f_A} := \mathbb{R} \times T^2/\mathbb{Z}$ be the quotient. This is the **mapping torus** of the automorphism f_A of T^2.

A similar construction works for arbitrary pair of a manifold M and automorphism $f : M \to M$. We consider the induced cation of $\mathbb{Z}$ on $\mathbb{R}\times M$ and define the **mapping torus** of f by $T_f := (\mathbb{R} \times M)/\mathbb{Z}$. The projection to the first factor induces a map

$$p : T_f \to \mathbb{R}/\mathbb{Z} \cong S^1 \ , \quad [t, x] \mapsto [t] \ .$$

This is a locally trivial fibre bundle with fibre M.

We now determine the bundle $\mathcal{H}^1(T_{f_A}/S^1) \to S^1$ explicitly. It will again be a mapping torus. The cohomology $H^1_{dR}(T^2)$ has a basis $x^i = [\alpha^i]$, $i = 1, 2$, where $\alpha^i \in \Omega^1(T^2)$ is characterized by $\pi^*\alpha^i = dt^i$ and (t^1, t^2) are the coordinates on $\mathbb{R}^2$. We have $A^*t^i = A^i{}_j t^j$. This implies $H^1(f_A)(x^i) = A^i{}_j x^j$. We start with the trivial bundle $\mathbb{R} \times H^1_{dR}(T^2) \to \mathbb{R}$. The group $\mathbb{Z}$ acts on the total space by

$$(n, (t, x)) := (t + n, H^1(f_A)^{-1}(x)) \ .$$

The quotient is the vector bundle $T_{H^1(f_A)^{-1}} \to S^1$, again a mapping torus. We have a canonical identification of bundles

$$T_{H^1(f_A)^{-1}} \cong \mathcal{H}(T_{f_A}/S^1) \ .$$

On the fibre over $[t] \in [S^1]$ this identification is given by the map

$$(T_{H^1(f_A)^{-1}})_{[t]} \cong H^1_{dR}(T^2) \cong H^1_{dR}(\{t\} \times T^2) \cong \mathcal{H}^1(T_{f_A}/S^1)_{[t]} \ .$$

This is independent of the choice of the representative t of the class $[t]$. Indeed, we have

$$\begin{tikzcd}
(T_{H^1(f_A)^{-1}})_{[t]} \arrow[r] \arrow[d, equal] & H^1_{dR}(T^2) \arrow[r] \arrow[d, "H^1(f_A)^{-1}"] & H^1_{dR}(\{t\} \times T^2)] \arrow[r] & \mathcal{H}^1(T_{f_A}/S^1)_{[t]} \arrow[d, equal] \\
(T_{H^1(f_A)^{-1}})_{[t+n]} \arrow[r] & H^1_{dR}(T^2) \arrow[r] & H^1_{dR}(\{t+n\} \times T^2) \arrow[r] \arrow[u, "H_1((\cdots+n)\times f_A)"'] & \mathcal{H}^1(T_{f_A}/S^1)_{[t+n]}
\end{tikzcd}$$

where the left square commutes by the definition of $T_{H^1(f_A)^{-1}}$ and the right square by the definition of T_{f_A}.

The trivial connection ∇^{triv} on $\mathbb{R} \times H^1_{dR}(T^2) \to \mathbb{R}$ is $\mathbb{Z}$-invariant. In descends to the Gauss-Manin connection on the bundle $\mathcal{H}^1(T_{f_A}/S^1) \to S^1$. It is not trivial. Indeed, the parallel transport for $\nabla^{\mathcal{H}^1(T_{f_A}/S^1)}$ along the loop $s \mapsto [s]$ in S^1 is given by

$$H^1(f_A) \in \mathtt{Aut}(\mathcal{H}^1(T_{f_A}/S^1)_{[0]}) \cong H^1_{dR}(T^2) \ .$$

Indeed, if $[0,x] \in \mathcal{H}^1(T_{f_A}/S^1)_{[0]}$, then the parallel transport is $[1,x] = [0, H^1(f_A)(x)] \in \mathcal{H}^1(T_{f_A}/S^1)_{[0]}$.

In the basis (x^1, x^2) the linear map $H^1(f_A)$ is the multiplication by the matrix A. □

Example 4.31. Let Γ be a finite group which acts freely on a connected manifold E. We set $B := E/\Gamma$. The fibres of $f : E \to B$ are zero-dimensional. In this case $\mathcal{H}^q(E/B) \cong 0$ for $q \geq 1$. The bundle $\mathcal{H}^0(E/B) \to B$ is non-trivial. Let $\mathbb{R}[\Gamma]$ be the real vector space generated by Γ. The right multiplication of Γ on itself induces a linear action of Γ on $\mathbb{R}[\Gamma]$:

$$\gamma(\sum_{g\in\Gamma} r_g g) := \sum_{g\in\Gamma} r_g g\gamma^{-1} = \sum_{g\in\Gamma} r_{g\gamma} g \ .$$

For a point $e \in E$ we get an identification

$$\Gamma \xrightarrow{\sim} E_{f(e)} \ , \quad \gamma \mapsto \gamma e \ .$$

The pull-back along this map gives an identification

$$\mathcal{H}^0(E/B)_{f(e)} = H^0_{dR}(E_{f(e)}) \xrightarrow{\sim} \mathbb{R}[\Gamma] \ .$$

We now define an isomorphism

$$(E \times \mathbb{R}[\Gamma])/\Gamma \xrightarrow{\sim} \mathcal{H}^0(E/B) \ , \quad [e, \sum_{g\in G} n_g g] \mapsto \sum_{g\in\Gamma} n_g [ge] \ .$$

In order to see that this is well-defined we calculate

$$[\gamma e, \gamma \sum_{g\in G} n_g g] \mapsto \sum_{g\in\Gamma} n_g [g\gamma^{-1}\gamma e] = \sum_{g\in\Gamma} n_g [ge] \ .$$

This flat bundle $\mathcal{H}^0(E/B)$ is not trivial. Let σ be a path from e to γe. In B we have $f(e) = f(\gamma e)$ and $f \circ \sigma$ is a loop. The parallel transport of $[e,x] \in \mathcal{H}^0(E/B)_{f(e)}$ along this loop is $[\gamma e, x] = [e, \gamma^{-1} x]$. After identification $\mathcal{H}^0(E/B)_{f(e)} \cong \mathbb{R}[\gamma]$ the parallel transport $\mathcal{H}^0(E/B)$ along this loop is given by the action of γ^{-1} on $\mathbb{R}[\Gamma]$.

□

In order to apply Theorem 4.24 we must be able to decide whether manifolds are simply connected. We now state a theorem which we will show later after the discussion of the fundamental group.

Theorem 4.32. *Let $E \to B$ is a locally trivial fibre bundle with typical fibre F.*

1. *If F and B are connected and simply connected, then E is connected and simply connected.*
2. *If F is connected and E is connected and simply connected, then B is simply connected.*

Proof. (Sketch) This is a consequence of the long exact homotopy sequence

$$\cdots \to \pi_2(B, f(e)) \to \pi_1(F, e) \to \pi_1(E, e) \to \pi_1(B, f(e)) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) .$$

This sequence will be introduced in homotopy theory. □

Example 4.33. We can use Theorem 4.32 in order to show the following statements:

1. $\mathbb{CP}^n$ is simply connected. Indeed we have a fibre bundle $S^{2n+1} \to \mathbb{CP}^n$ with simply connected total space and connected fibre S^1.
2. $SU(n)$ is simply connected. We have a fibre bundle $SU(n+1) \to S^{2n+1}$ with fibre $SU(n)$. The base is simply connected for $n \geq 1$. We now argue by induction. We start with the observation that $SU(2) \cong S^3$ is simply connected.
3. The Grassmann manifold $Gr(k, \mathbb{C}^n)$ is simply connected. We have a fibre bundle $SU(n)/S(U(k) \times U(n-k)) \to Gr(k, \mathbb{C}^n)$ with simply connected total space and connected fibre.

5 The Leray-Serre spectral sequence and applications

5.1 Construction of the Leray-Serre spectral sequence

We consider a local trivial fibre bundle $f : E \to B$. In this section we construct a spectral sequence which converges against a graded version of $H_{dR}(E)$ and determine its second page E_2. It is called the **Leray-Serre spectral sequence**.

The spectral sequence is associated by the construction in Subsection 2.1 to a decreasing filtration $(\mathcal{F}^p\Omega(E))_{\in\mathbb{Z}}$ of the de Rham complex of the total space E of the bundle.

We start with a technical result which can be interpreted as the determination of the E_1-term of the spectral sequence for a trivial bundle. For two vector bundles $V_i \to M_i$, $i = 0, 1$, we define the vector bundle

$$V_0 \boxtimes V_1 := \mathtt{pr}_{M_0}^* V_0 \otimes \mathtt{pr}_{M_1}^* V_1$$

over $M_0 \times M_1$. For example, the $\wedge$-product induces a canonical isomorphism

$$\Lambda T^* M_0 \boxtimes \Lambda T^* M_1 \cong \Lambda T^*(M_0 \times M_1) \ .$$

This bundle has a bigrading given by

$$\Lambda^{s,t} T^*(M_0 \times M_1) := \Lambda^s T^* M_0 \boxtimes \Lambda^t T^* M_1 \ .$$

The bigrading of the bundle induces a bigrading of its space of sections

$$\Omega^{s,t}(M_0 \times M_1) := \Gamma(M_0 \times M_1, \Lambda^{s,t} T^*(M_0 \times M_1)) \ . \tag{28}$$

We can decompose the de Rham differential on $\Omega(M_0 \times M_1)$ as $d = d^{M_0} + d^{M_1}$, where d^{M_i} differentiate in the M_i-directions for $i = 0, 1$. Then $(\Omega^{*,*}(M_0 \times M_1), d^{M_1}, d^{M_2})$ is a double complex whose total complex is $(\Omega(M_0 \times M_1), d)$.

The exterior product

$$\times : \Omega(M_0) \otimes \Omega(M_1) \to \Omega(M_0 \times M_1) \ , \quad \alpha \otimes \omega := \mathtt{pr}_{M_0}^* \alpha \wedge \mathtt{pr}_{M_1}^* \omega$$

induces a map of double complexes. We stress that on the left-hand side we consider the algebraic tensor product of complexes. Hence, if M_0 and M_1 are both are not zero-dimensional, then $\times$ is not an isomorphism of complexes. Nevertheless, under suitable finiteness assumption, it is a quasi-isomorphism by the Künneth theorem 2.34. In the following we show a partial Künneth theorem.

Lemma 5.1. *If M_0 admits a finite good covering, then for every $p \in \mathbb{Z}$*

$$\times : (\Omega(M_0) \otimes \Omega^p(M_1), d^{M_0} \otimes \mathtt{id}_{\Omega^p(M_1)}) \to (\Omega^{*,p}(M_0 \times M_1), d^{M_0})$$

is a quasi-isomorphism. In particular,

$$H^q_{dR}(M_0) \otimes \Omega^p(M_1) \cong H^q(\Omega^{*,p}(M_0 \times M_1), d^{M_0}) \ .$$

Proof. We choose a finite good covering $\mathcal{V} := (V_\beta)_{\beta\in B}$ of M_0. We get an induced covering $\mathcal{V}\times M_1 = (V_\beta \times M_1)_{\beta\in B}$ of $M_0\times M_1$. We have a commutative square

$$\begin{tikzcd}
\Omega(M_0)\otimes\Omega^p(M_1) \arrow[r, "\times"] \arrow[d, "\iota_{M_0}\otimes\mathtt{id}"] & \Omega^{*,p}(M_0\times M_1) \arrow[d, "\iota_{M_0\times M_1}"] \\
\mathtt{tot}(\check{C}(\mathcal{V},\Lambda T^*M_0))\otimes\Omega^p(M_1) \arrow[r, "\check{\times}"] & \mathtt{tot}(\check{C}(\mathcal{V}\times M_1,\Lambda T^*M_0\boxtimes\Lambda^pT^*M_1))
\end{tikzcd} \quad . \tag{29}$$

We know by Lemma 2.12 that ι_{M_0} is a quasi-isomorphism. The same argument also shows that $\iota_{M_0\times M_1}$ is a quasi-isomorphisms. By Lemma 2.30 the left vertical map $\iota_{M_0}\otimes\mathtt{id}$ is a quasi-isomorphism.

We now use the finiteness of the covering in order to commute the tensor product by $\Omega^p(M_1)$ with the products involved in the construction of the Čech complex in the left lower corner. After this identification the lower horizontal map is induced by the exterior product maps

$$\Omega(V_\beta)\otimes\Omega^p(M_1)\to\Omega^{*,p}(V_\beta\times M_1) \tag{30}$$

for all $\beta\in B^{k+1}$ and $k\in\mathbb{N}$.

In order to show that $\times$ is a quasi-isomorphism it suffices to show that $\check{\times}$ is a quasi-isomorphism. To this end we compare the induced map of spectral sequences

$$(E_r(\check{\times}))_{r\geq 1} : ({}^IE_r^{alg}, d_r^{alg})_{r\geq 1}\to({}^IE_r, d_r)_{r\geq 1}$$

associated to the filtration by Čech degree. In particular, $E_1^{alg,k,*}\to E_1^{k,*}$ is the map induced in cohomology by the product of the maps (30).

Let $\beta\in B^{k+1}$, $v\in V_\beta$ and $i_v : v\to V_\beta$ be the inclusion. Then we have a commutative diagram

$$\begin{tikzcd}
\Omega(V_\beta)\otimes\Omega^p(M_1) \arrow[r, "\times"] \arrow[d, "i_v^*\otimes\mathtt{id}_{\Omega^p(M_1)}"] & \Omega^{*,p}(V_\beta\times M_1) \arrow[d, "(i_v\times\mathtt{id}_{M_1})^*"] \\
\Omega^p(M_1) \arrow[r, equal] & \Omega^p(M_1)
\end{tikzcd}$$

Since V_β is contractible to v, the map i_v^* and hence the left vertical map are quasi-isomorphisms. The right vertical map is a quasi-isomorphism, too. Indeed, a homotopy inverse is given by

$$\mathtt{pr}_{V_\beta}^* : \Omega^p(M_1)\to\Omega^{*,p}(V_\beta\times M_1)\ .$$

We have

$$(i_v \times \mathtt{id}_{M_1})^* \circ \mathtt{pr}_{V_\beta}^* = \mathtt{id}_{\Omega^p(M_1)}$$

and the usual homotopy $h : \Omega(V_\beta \times M_1) \to \Omega(V_\beta \times M_1)$ (associated to the product of the contraction of V_β with $\mathtt{id}_{M_1}$) from $\mathtt{pr}_{V_\beta}^* \circ (i_v \times \mathtt{id}_{M_1})^*$ to $\mathtt{id}_{\Omega(V_\beta \times M_1)}$ restricts to the complex $\Omega^{*,p}(V_\beta \times M_1)$.

We conclude that $\times$ in (29) is a quasi-isomorphism.

The second assertion of the Lemma follows from the first and Lemma 2.27. $\square$

We now turn back to a locally trivial fibre bundle $f : E \to B$. Let F denote the typical fibre. We first describe the filtration of $\Omega(E)$ leading to the Leray-Serre spectral sequence. The **vertical tangent bundle** of f is defined by

$$T^v f := \mathtt{ker}(df : TE \to f^*TB) \ .$$

Since f is a submersion, df is a surjective vector bundle map and its kernel $T^v f$ is indeed a vector subbundle of TE. Its sections are called vertical vector fields. We have an inclusion of bundles of algebras of multi vector fields

$$\Lambda T^v f \hookrightarrow \Lambda TE \ ,$$

and we define an decreasing filtration of ΛTE by subbundles

$$\mathcal{F}^p \Lambda TE = \mathtt{im}(\Lambda^p T^v f \wedge \Lambda TE \to \Lambda TE) \ .$$

For an inclusion of vector spaces $W \to V$ let

$$W^\perp := \{v' \in V^* \mid v'_{|W} = 0\} \subseteq V^*$$

be the **annihilator** of W. If $W' \subseteq W$, then we have $W^\perp \subseteq W'^{,\perp}$.

We have a canonical isomorphism

$$\Lambda T^* E \cong (\Lambda TE)^*$$

given by evaluation of a differential forms on multi vector fields. For every $n \in \mathbb{Z}$ we define the decreasing filtration of $\Lambda^n T^* E$ by subbundles by

$$\mathcal{F}^p \Lambda^n T^* E := (\mathcal{F}^{n-p+1} \Lambda^n TE)^\perp \subseteq \Lambda^n T^* E \ .$$

Finally we set

$$\mathcal{F}^p\Omega(E) := \Gamma(E, \mathcal{F}^p\Lambda T^*E) \subseteq \Omega(E) \ . \tag{31}$$

It is obvious that these subspaces form a decreasing filtration of the $\mathbb{Z}$-graded vector space $\Omega(E)$. Special cases are

$$\mathcal{F}^p\Omega^n(E) = \begin{cases} \Omega^n(E) & p \leq 0 \\ 0 & p \geq n+1 \end{cases} \ . \tag{32}$$

Example 5.2. If $E \to B$ is trivialized, i.e. $E \cong F \times B$, then we have

$$\mathcal{F}^p\Omega(E) \cong \bigoplus_{s \geq p} \Omega^{*,s}(F \times B) \ .$$

In this case it is clear that the filtration is compatible with the de Rham differential. We shall see next, this is true in general. □

Lemma 5.3. *The filtration of* $\Omega(E)$ *is compatible with the differential.*

Proof. We could refer to Example 5.2 and the fact that E is locally trivial. But we give an alternative proof which better explains the reason why the Lemma holds.

We start with giving an alternative description of the filtration. The lowest non-trivial step of the filtration of $\Omega^n(E)$ is $\mathcal{F}^n\Omega^n(E)$ and contains all n-forms pulled back from B and their products with functions. Indeed, these forms are exactly those annihilated by insertion of a vertical vector field. The next step is the space of forms which are annihilated by the insertion of two vertical fields, and so on.

Explicitly, we have all $p \in \mathbb{Z}$

$$\mathcal{F}^p\Omega^n(E) = \{\omega \in \Omega^n(E) \mid (\forall (X_i)_{i=1}^{n-p+1} \in \Gamma(E, T^vf)^{n-p+1} \mid i_{X_1} \dots i_{X_{n-p+1}}\omega = 0)\} \ .$$

Next we observe that $\Gamma(E, T^vf)$ is closed under the commutator of vector fields. Indeed, a vector field $X \in \Gamma(E, TE)$ is vertical exactly if $X(f^*\phi) = 0$ for all $\phi \in C^\infty(B)$. But for two vertical vector fields X, Y and any ϕ we have

$$[X, Y](f^*\phi) = X(Y(f^*\phi)) - Y(X(f^*\phi)) = 0 \ .$$

Hence the commutator $[X, Y]$ is vertical, too.

Let $\omega \in \mathcal{F}^p\Omega^n(E)$. We use the formula

$$\begin{aligned}(n+1)d\omega(Z_1,\ldots,Z_{n+1}) &= \sum_{i=1}^{n+1}(-1)^i Z_i\omega(Z_1,\ldots,\widehat{Z_i},\ldots,Z_{n+1}) \\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([Z_i,Z_j],Z_1,\ldots,\widehat{Z_i},\ldots,\widehat{Z_j},\ldots,Z_{n+1})\ .\end{aligned}$$

If $n+1-p+1$ of the vector fields Z_i are vertical, then every term contains an insertion of at least $n-p+1$ vertical vector fields and hence vanishes. □

Definition 5.4. *The* ***Leray-Serre spectral sequence (LSSS)*** $(E_r,d_r)_{r\geq 1}$ *of the locally trivial fibre bundle* $f : E \to B$ *is the spectral sequence associated to the filtration* $(\mathcal{F}^p\Omega(E))_{p\in\mathbb{Z}}$ *defined in* (31)*.*

For every $n \in \mathbb{N}$ the filtration $\mathcal{F}^*\Omega^n(E)$ is finite, exhaustive and separating. Consequently we have a finite exhaustive and separating filtration on $H^n(E)$ (called the **LSS-filtration**) and

$$E_\infty^{p,n-p} \cong \mathrm{Gr}^p H^n(E)\ .$$

In fact, by (32) the group $E_r^{p,n-p}$ stabilizes at the $n+1$ page, i.e. we have $E_{n+1}^{p,n-p} \cong E_\infty^{p,n-p}$.

We assume that the typical fibre F of the bundle admits a finite good covering. Then by Example 4.28 for every $q \in \mathbb{Z}$ we have the bundles $\mathcal{H}^q(E/B) \to B$ of fibrewise de Rham cohomology of degree q which has a flat Gauss-Manin connection $\nabla^{\mathcal{H}^q(E/B)}$.

Proposition 5.5. *For every $q \in \mathbb{Z}$ we have an isomorphism of complexes*

$$(E_1^{*,q},d_1) \cong (\Omega^*(B,\mathcal{H}^q(E/B)),(-1)^q\nabla^{\mathcal{H}^q(E/B)}).$$

In particular, we have isomorphisms

$$E_2^{p,q} \cong H^p(B,\mathcal{H}^q(E/B))\ .$$

Proof. We fix a good covering $\mathcal{U} := (U_\alpha)_{\alpha\in A}$ of the base B and trivializations

$$\Psi_\alpha : U_\alpha \times F \stackrel{\cong}{\to} E$$

of the fibre bundle. Then $f^{-1}\mathcal{U} := (f^{-1}(U_\alpha))_{\alpha\in A}$ is an open covering of E. Recall that $\mathcal{H}^q(E/B) \to B$ is described by the family of local trivializations

$$\psi_\alpha : \mathcal{H}^q(E/B)_{|U_\alpha} \cong U_\alpha \times H^q_{dR}(F) \tag{33}$$

and the locally constant transition maps

$$U_{\alpha\beta} \ni u \to g_{\alpha\beta}(u) = \psi_{\alpha,u} \circ \psi_{\beta,u}^{-1} := \Psi_{\alpha,u}^* \circ \Psi_{\beta,u}^{*,-1} \in \mathtt{End}(H_{dR}^q(F)) \tag{34}$$

where $\Psi_{\alpha,u}$ is the restriction of the trivialization to the fibre over $u \in U_{\alpha\beta}$. Then we have a natural isomorphism

$$\Omega(B, \mathcal{H}^q(E/B)) \cong \left\{ (\omega_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} H_{dR}^q(F) \otimes \Omega(U_\alpha) \mid \left(\forall \alpha\beta \in A^2 \mid g_{\alpha\beta}\omega_\beta = \omega_\alpha\right) \right\} . \tag{35}$$

The differential of $\omega = (\omega_\alpha)_{\alpha\in A}$ is given by

$$(\nabla^{\mathcal{H}^q(E/B)}\omega)_\alpha = d\omega_\alpha \ . \tag{36}$$

We will see that $(E_1^{*,q}, d_1)$ has the same description.

We define a filtration of the Čech complex

$$\mathcal{F}^p \, \mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^*E))^n := \bigoplus_s \check{C}^s(f^{-1}\mathcal{U}, \mathcal{F}^p\Lambda^{n-s}T^*E) \ .$$

The associacted spectral sequence with be denoted by (E'_r, d'_r). Note that the Čech differential preserves the filtration. The natural map

$$i : \Omega(E) \to \mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^*E))$$

is compatible with the filtrations.

Lemma 5.6. *The induced map of spectral sequences $E(i) : (E_r, d_r)_{r\geq 1} \to (E'_r, d'_r)_{r\geq 1}$ is an isomorphism.*

Proof. By the arguments of Lemma 2.11 and Lemma 2.12 (the main observation is that the filtration $\mathcal{F}^*\Omega(E)$ is induced by a filtration of the bundle ΛT^*E), for all $p \in \mathbb{Z}$ we have a quasi-isomorphisms

$$\mathcal{F}^p i : \mathcal{F}^p\Omega(E) \to \mathcal{F}^p \, \mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^*E)) \ . \tag{37}$$

Hence for all $p \in \mathbb{Z}$ we have quasi-isomorphisms

$$\mathrm{Gr}^p i : \mathrm{Gr}^p\Omega(E) \to \mathrm{Gr}^p \, \mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^*E)) \ .$$

The induced map in cohomology is an isomorphism of $E_1(i) : E_1 \to E'_1$. It follows that $E(i)$ is an isomorphism of spectral sequences.

$\square$

Consequently we must calculate $E_1'^{,*,q}$ and d_1'. Let $s \in \mathbb{N}$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in A^{s+1}$. Using Ψ_{α_0} we get an isomorphism of filtered chain complexes,

$$\Omega(f^{-1}U_\alpha) \cong \Omega(F \times U_\alpha) \ ,$$

where the filtration on the right-hand side is defined by

$$\mathcal{F}^p\Omega^n(F \times U_\alpha) := \bigoplus_{t \geq p} \Omega(F \times U_\alpha)^{n-t,t} \ ,$$

see Example 5.2. The total complex of the Čech complex is a sum of products of these pieces. Consequently we get an isomorphism of complexes

$$\mathrm{Gr}^p(\mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^*E))) \cong \bigoplus_{s,t} \prod_{\alpha \in A^{s+1}} \Omega^{t-p,p}(F \times U_\alpha) \tag{38}$$

with differential $d = \check{d} + (-1)^s d^F$ on the summand with index (s,t). This complex is again a total complex associated to a double complex with differentials $\check{d}$ and $(-1)^s d^F$ on the summand $\prod_{\alpha \in A^{s+1}} \Omega^{t-p,p}(F \times U_\alpha)$.

By definition, $E_1'^{,*,p}$ is the cohomology of (38). In order to compute it we will use again a spectral sequence. We consider its filtration by the Čech degree and the associated spectral sequence $(E_r'', d_r'')_{r \geq 1}$. Its first term is the cohomology of the differential d^F. By Lemma 5.1 we get

$$E_1''^{,s,t} \cong \prod_{\alpha \in A^{s+1}} H_{dR}^{t-p}(F) \otimes \Omega^p(U_\alpha) \ .$$

Now d_1'' is induced by the $\check{d}$. We shall give an explicit formula of $d_1 : E_1''^{,s,t} \to E_1''^{,s+1,t}$. We have $d_1'' = \sum_{i=0}^{s+1} (-1)^i d_i$, where

$$d_i : \prod_{\alpha \in A^{s+1}} H_{dR}^{t-p}(F) \otimes \Omega^p(U_\alpha) \to \prod_{\alpha \in A^{s+2}} H_{dR}^{t-p}(F) \otimes \Omega^p(U_\alpha)$$

is given by

$$(d_i\omega)_\alpha = \omega_{\alpha_0, \ldots, \check{\alpha}_i, \ldots, \alpha_{s+2} | U_\alpha}$$

for $i \geq 1$ and

$$(d_0\omega)_\alpha = (g_{\alpha_0\alpha_1} \otimes \mathtt{id})\omega_{\alpha_1, \ldots, \alpha_{s+2} | U_\alpha} \ ,$$

see (34). The distinction between the cases $i = 0$ and $i \geq 1$ encounters the fact that in the case $i = 0$ we must change the trivialization from Ψ_{α_1} to Ψ_{α_0}.

The complex $(E_1^{\prime\prime,*,t}, d_1)$ is canonically isomorphic to the complex

$$\check{C}(\mathcal{U}, \Lambda^p T^* B \otimes \mathcal{H}^{t-p}(E/B)) \ .$$

To this end we again use the trivialization ψ_{α_0} (see (33)) in order to define the isomorphism

$$\Omega^p(U_\alpha, \mathcal{H}^{t-p}(E/B)) \cong H^{t-p}_{dR}(F) \otimes \Omega^p(U_\alpha)$$

for all $\alpha \in A^{s+1}$ and $s \in \mathbb{N}$. Then

$$\check{C}^s(\mathcal{U}, \Lambda^p T^* B \otimes \mathcal{H}^{t-p}(E/B)) \cong \prod_{\alpha \in A^{s+1}} H^{t-p}_{dR}(F) \otimes \Omega^p(U_\alpha) \ .$$

The differential $\check{d}$ is given by the same formulas as d_1'' above. To this end we must observe that $g_{\alpha_0\alpha_1} = \psi_{\alpha_0} \circ \psi_{\alpha_1}^{-1}$.

By Lemma 2.11 we get

$$E_2^{\prime\prime,s,t} = \left\{ \begin{array}{cc} \Omega^p(B, \mathcal{H}^{t-p}(E/B)) & s = 0 \\ 0 & s \neq 0 \end{array} \right. \ .$$

Consequently

$$E_1^{p,q-p} \cong E_1^{\prime,p,q-p} \cong H^q \mathrm{Gr}^p(\mathtt{tot}(\check{C}(f^{-1}\mathcal{U}, \Lambda T^* E))) \cong E_2^{\prime\prime,0,q} \cong \Omega^p(B, \mathcal{H}^{q-p}(E/B)) \ .$$

It remains to identify the differential d_1 with $\nabla^{\mathcal{H}(E/B)}$. To this end we use the picture (35) which we compare with

$$E_2^{\prime\prime,0,t} \cong \mathtt{ker}(d_1'' : E_1^{\prime\prime,0,t} \to E_1^{\prime\prime,1,t}) \ .$$

Under this isomorphism we indeed have in view of

$$(\check{d}\omega)_{\alpha\beta} = \omega_{\alpha|U_{\alpha\beta}} - g_{\alpha\beta}\omega_{\beta|U_{\alpha\beta}}$$

$$E_2^{\prime\prime,0,t} \cong \left\{ (\omega_\alpha)_{\alpha\in A} \in \prod_{\alpha \in A} H^q_{dR}(F) \otimes \Omega^t(U_\alpha) \mid \left(\forall \alpha\beta \in A^2 \mid g_{\alpha\beta}\omega_\beta = \omega_\alpha\right) \right\} \ .$$

The differential d_1 is induced by $\check{d}+d^{II}$, where d^{II} is the product of $(-1)^{s+t-p}(\mathtt{id}_{H^{t-p}_{dR}(F)} \otimes d^{U_\alpha})$ on $\prod_{\alpha\in A^{s+1}} H^{t-p}_{dR}(F) \otimes \Omega^p(U_\alpha)$. If we apply this to $s = 0$ and elements in $\mathtt{ker}(\check{d})$ we see that $(d_1\omega)_\alpha = (-1)^{t-p} d\omega_\alpha$ as required in view of (36). □

5.2 Gysin sequence for S^n-bundles

We consider the Leray-Serre spectral sequence for a locally trivial fibre bundle $f : E \to B$ with fibre $F = S^1$.

Since the diffeomorphisms of S^1 act as identity on $H^0(S^1)$ the bundle $\mathcal{H}^0(E/B)$ is trivial and one-dimensional as a flat bundle. The constant function $1 \in \Omega^0(E)$ gives a global parallel section which we use to trivialize $\mathcal{H}^0(E/B)$.

The bundle $\mathcal{H}^1(E/B)$ may be non-trivial. Since $\mathcal{H}^i(E/B) = 0$ for $i \notin \{0, 1\}$ the second page of the LSSSs has only two lines.

$$\begin{tikzcd}
1 & H^0(B,\mathcal{H}^1(E/B)) \arrow[drr] & H^1(B,\mathcal{H}^1(E/B)) \arrow[drr] & H^2(B,\mathcal{H}^1(E/B)) & H^3(B,\mathcal{H}^1(E/B)) \\
0 & H^0(B) & H^1(B) & H^2(B) & H^3(B) \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

where we have only indicated the non-trivial differentials in the range pictured.

Therefore the LSSS degenerates at the third page and by Lemma 2.4 we have short exact sequences

$$0 \to E_3^{0,q} \to H^q_{dR}(E) \to E_3^{1,q-1} \to 0$$

for all $q \in \mathbb{Z}$. If we express the third page in terms of the data of the second page, then we obtain a long exact sequence which is called the **Gysin sequence** of the bundle $E \to B$

$$\cdots \to H^q_{dR}(B) \xrightarrow{f^*} H^q_{dR}(E) \xrightarrow{\sigma} H^{q-1}_{dR}(B, \mathcal{H}^1(E/B)) \xrightarrow{d_2} H^{q+1}_{dR}(B) \to \dots .$$

Example 5.7. In this example we use the Gysin sequence in order to calculate the de Rham cohomology of $\mathbb{CP}^n$. We use the bundle $S^{2n+1} \to \mathbb{CP}^n$ with fibre S^1. We know by Theorem 4.32 that $\mathbb{CP}^n$ is simply connected and therefore $\mathcal{H}^1(S^{2n+1}/\mathbb{CP}^n) \to \mathbb{CP}^n$ is a trivial one-dimensional flat bundle. If we fix a trivialization (the choice will be discussed later), then the LSSS has the form (with $B := \mathbb{CP}^n$)

$$
\begin{tikzcd}
1 & H^0(B) \arrow[drr] & H^1(B) \arrow[drr] & H^2(B) \arrow[drr] & H^3(B) \arrow[drr] & H^4(B) \arrow[drr] & H^5(B) & H^6(B) \\
0 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & H^6(B) \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

In the case of $\mathbb{CP}^3$ is the complete picture. Since $H^k_{dR}(S^{2n+1}) = 0$ for $1 \leq k \leq 2n$ we see that all indicated differentials must be isomorphisms. We inductively conclude:

1. $H^0_{dR}(\mathbb{CP}^n) \cong \mathbb{R}$
2. $H^1_{dR}(\mathbb{CP}^n) \cong 0$
3. $H^2_{dR}(\mathbb{CP}^n) \cong \mathbb{R}$
4. $\dots$
5. $H^{2n-2}_{dR}(\mathbb{CP}^n) \cong \mathbb{R}$
6. $H^{2n-1}_{dR}(\mathbb{CP}^n) \cong 0$
7. $H^{2n}_{dR}(\mathbb{CP}^n) \cong \mathbb{R}$.
8. $H^k_{dR}(\mathbb{CP}^n) \cong 0$ for $k \geq 2n+1$.

Hence

$$
H^k_{dR}(\mathbb{CP}^n) \cong \begin{cases} \mathbb{R} & k = 2i \ , \quad i \in \{0, 1, \dots, n\} \\ 0 & \qquad\qquad else \end{cases} .
$$

We further note that the map

$$
H^{2n+1}_{dR}(S^{2n+1}) \xrightarrow{\sigma} E_2^{2n,1} \cong H^{2n}_{dR}(\mathbb{CP}^n)
$$

is an isomorphism. The isomorphisms depend on the choice of the trivialization of $\mathcal{H}^1(S^{2n+1}/\mathbb{CP}^n)$. We discuss this in more detail when we analyze the multiplicative structure. $\square$

There is a Gysin sequence for bundles $f : E \to B$ with fibre S^n for all $n \geq 1$. To this end we consider the structure of the LSSS. It again has only two non-trivial rows. The only possible non-trivial differential after the E_2-term is d_{n+1}. So we have $E_2 \cong E_{n+1}$ and $E_{n+2} \cong E_\infty$. We write out the E_4-term the case $n = 3$ for simplicity

$$
\begin{tikzcd}[row sep=small, column sep=small]
3 & E_4^{0,3} \arrow[rrrrddd] & E_4^{1,3} \arrow[rrrrddd] & E_4^{2,3} & E_4^{3,3} & E_4^{4,3} & E_4^{5,3} \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_4^{0,0} & E_4^{1,0} & E_4^{2,0} & E_4^{3,0} & E_4^{4,0} & E_4^{5,0} \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}
$$

where $E_{n+1}^{p,0} \cong H^p(B)$ and $E_{n+1}^{p,n} \cong H^p(B, \mathcal{H}^n(E/B))$.

Hence we get a long exact **Gysin sequence**

$$\cdots \to H_{dR}^k(B, \mathcal{H}^n(E/B)) \overset{d_{n+1}}{\to} H_{dR}^{k+n+1}(B) \overset{f^*}{\to} H_{dR}^{k+n+1}(E) \overset{\sigma}{\to} H_{dR}^{k+1}(B, \mathcal{H}^n(E/B)) \to \dots .$$

Example 5.8. Let $n \geq 1$. The group of unit quaternions $Sp(1) \cong S^3$ acts on $\mathbb{H}^{n+1}$ by left multiplication. It preserves the quaternionic scalar product $\langle x, y \rangle := \sum_{i=1}^{n+1} x_i^* y_i$. Indeed, we have

$$\langle qx, qy \rangle = \sum_{i=1}^{n+1} x_i^* q^* q y_i = \sum_{i=1}^{n+1} x_i^* y_i = \langle x, y \rangle$$

since $q^*q = 1$ for a unit quaternion $q \in Sp(1)$. The real part of the scalar product is the usual euclidean scalar product on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)}$. Consequenty the action of $Sp(1)$ restricts to an action on the unit sphere S^{8n-1}. The quotient

$$\mathbb{HP}^n := S^{4n+3}/Sp(1)$$

is called the **quaternionic projective space**. We therefore have a fibre bundle $S^{4n+3} \to \mathbb{HP}^n$ with fibre $Sp(1) \cong S^3$.

Since S^{4n+3} is simply connected and S^3 is connected the manifold $\mathbb{HP}^n$ is simply connected by Theorem 4.32. We again conclude that differentials of the E_2-term of the LSSS between non-zero groups are isomorphisms. As in the case of the complex projective space we obtain

$$H_{dR}^k(\mathbb{HP}^n) \cong \begin{cases} \mathbb{R} & k = 4i \,, \quad i \in \{0, 1, \dots, n\} \\ 0 & \quad\quad\quad\quad else \end{cases} .$$

□

5.3 Functoriality of the Leray-Serre spectral sequence

In general it is very difficult to calculate higher differentials of the LSSS. Most calculations start from simple cases and then use the functoriality and the multiplicativity of the spectral sequence. In this subsection we discuss functoriality.

A map from a fibre bundle $f' : E' \to B'$ to a fibre bundle $f : E \to B$ is a commutative diagram

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ B' \arrow[r, "\bar{g}"] & B \end{tikzcd} \quad . \tag{39}$$

Note that $\bar{g}$ is determined by g.

Lemma 5.9. *The map $g^* : \Omega(E) \to \Omega(E')$ is filtration perserving. Consequently it induces a map of LSSS'es*

$$E(g^*) : (E_r, d_r)_{r\geq 1} \to (E'_r, d'_r)_{r\geq 1} \ .$$

Proof. Note that for $e' \in E'$ and $X \in T_{e'}E'$ we have $d\bar{g}(df'(X)) = df(dg(X))$. If X is vertical, then $df'(X) = 0$ and hence $df(dg(X)) = 0$. Consequently, if X is vertical, then $dg(X)$ is vertical, too.

Fix $n, p \in \mathbb{Z}$ and $\omega \in \mathcal{F}^p\Omega^n(E)$. Let $(X_i)_{1,\dots,n-p+1}$ be a collection of vertical tangent vectors at e'. Then we have

$$i_{X_1} \dots i_{X_{n-p+1}}(g^*\omega)(e') = i_{dg(X_1)} \dots i_{dg(X_{n-p+1})}\omega(g(e')) = 0 \ .$$

We conclude that $g^*\omega \in \mathcal{F}^p\Omega^n(E')$. □

Lemma 5.10. *We have a canonical map of flat vector bundles*

$$g^\sharp : \bar{g}^*\mathcal{H}(E/B) \to \mathcal{H}(E'/B') \ .$$

Proof. The map g induces for every $b' \in B'$ a smooth map $g_{b'} : E'_{b'} \to E_{\bar{g}(b')}$. We get a map $g^*_{b'} : H_{dR}(E_{\bar{g}(b')}) \to H_{dR}(E'_{b'})$. The collection of these maps give a map

$$g^\sharp : \bar{g}^*\mathcal{H}(E/B) \to \mathcal{H}(E'/B')$$

if we consider both sides as disjoint unions of real vector spaces, see (27). We must verify that this map is smooth and preserves connections.

Let $U \subseteq B$ and $\Psi : U \times F \to E_{|U}$ be a trivialization. We can assume (after shrinking U if necessary), that there exists a trivialization $\Psi' : \bar{g}^{-1}U \times F' \to E'_{|\bar{g}^{-1}U}$. The trivializations of the fibre bundles induce trivializations of the cohomology bundles $\Psi^* : \mathcal{H}(E/B)_{|U} \xrightarrow{\sim} U \times H_{dR}(F)$ and $\Psi'^* : \mathcal{H}(E'/B')_{|\bar{g}^{-1}U} \xrightarrow{\sim} \bar{g}^{-1}U \times H_{dR}(F')$. In these trivializations the Gauss-Manin connections are the trivial connections. For $u' \in U'$ let $G_{u'} : F' \to F$ be the composition $G_{u'} : (\Psi^{-1} \circ g \circ \Psi')(u', -) : F' \to F$. With respect to the trivializations Ψ'^* and $\bar{g}^*\Psi^*$ the map $g^\sharp$ is represented by $u' \mapsto G^*_{u'} : H_{dR}(F) \to H_{dR}(F')$. By the homotopy invariance of de Rham cohomology this map is locally constant. It is hence smooth and preserves the trivial connections. □

Lemma 5.11. *The following diagram commutes:*

$$\begin{tikzcd}
E_2^{p,q} \arrow[rr, "E_2(g)"] \arrow[d, "\cong"] & & E_2'^{,p,q} \arrow[d, "\cong"] \\
H^p_{dR}(B, \mathcal{H}^q(E/B)) \arrow[r, "\bar{g}^*"] & H^p_{dR}(B', \bar{g}^*\mathcal{H}^q(E/B)) \arrow[r, "g^\sharp"] & H^p_{dR}(B', \mathcal{H}^q(E'/B'))
\end{tikzcd}$$

Proof. We show that the corresponding diagram on the level of E_1-terms commute.

$$\begin{tikzcd}
E_1^{p,q} \arrow[rr, "E_1(g)"] \arrow[d, "\cong"] & & E_1'^{,p,q} \arrow[d, "\cong"] \\
\Omega^p(B, \mathcal{H}^q(E/B)) \arrow[r, "\bar{g}^*"] & \Omega^p(B', \bar{g}^*\mathcal{H}^q(E/B)) \arrow[r, "g^\sharp"] & \Omega^p(B', \mathcal{H}^q(E'/B'))
\end{tikzcd} .$$

We use the presentation (35) and the identification of the E_1-terms with the E_1'-terms given in Lemma 5.6.

We are reduced to the case of trivial bundles and a map

$$G : U' \times F' \to U \times F \ , \quad (u', f') \mapsto (\bar{G}(u'), G_{u'}(f')) \ ,$$

where $\bar{G} : U' \to U$ is the underlying map of base spaces. This map can be decomposed as

$$U' \times F' \to U' \times F \to U \times F \ , \quad (u', f') \mapsto (u', G_{u'}(f')) \mapsto (\bar{G}(u'), G_{u'}(f'))$$

The map $E_1(G)$ is induced by the map of complexes

$$(\mathrm{Gr}\ \Omega(U \times F), d^F) \stackrel{\bar{G}^*}{\to} (\mathrm{Gr}\ \Omega(U' \times F), d^F) \stackrel{G^\sharp}{\to} (\mathrm{Gr}\ \Omega(U' \times F'), d^F) \ .$$

In cohomology we get

$$\Omega(U, H_{dR}(F)) \stackrel{\bar{G}^*}{\to} \Omega(U', H_{dR}(F)) \stackrel{G^\sharp}{\to} \Omega(U', H_{dR}(F')) \ .$$

$\square$

Example 5.12. We can use the LSSS in order to study the kernel of the map $f^* : H^*_{dR}(B) \to H^*_{dR}(E)$.

Let $E \to B$ be a fibre bundle with connected fibres. Then we can consider $\mathtt{id} : B \to B$ as a fibre bundle and the map of fibre bundles

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "f"] & B \arrow[d, "\mathtt{id}_B"] \\ B \arrow[r, "\mathtt{id}_B"] & B \end{tikzcd} \quad . \tag{40}$$

We study the induced map of LSSS $E(f) : ({}^{\mathtt{id}_B}E_r, {}^{\mathtt{id}_B}d_r)_{r\ge 1} \to ({}^fE_r, {}^fd_r)_{r\ge 1}$. Note that $({}^{\mathtt{id}_B}E_r, {}^{\mathtt{id}_B}d_r)$ degenerates at the second page which has the form

$${}^{\mathtt{id}_B}E_2^{p,q} \cong \left\{ \begin{array}{cc} H^p(B) & q = 0 \\ 0 & else \end{array} \right. .$$

We also have

$${}^fE_2^{p,q} \cong \left\{ \begin{array}{cc} H^p(B) & q = 0 \\ * & else \end{array} \right. ,$$

and the map $E(f)$ induced exactly the identification of the zero lines.

Let us fix $p \in \mathbb{Z}$. Then we have a sequence of quotients

$${}^fE_2^{p,0} \to {}^fE_3^{p,0} \to \dots {}^fE_{p+1}^{p,0} = E_\infty^{p,0}$$

where ${}^fE_{k+1}^{p,0}$ is the cokernel of ${}^fd_k : {}^fE_k^{p-k,k-1} \to E_k^{p,0}$.

Corollary 5.13. *A class $x \in H^p(B) \cong {}^fE_2^{p,0}$ pulls back to zero on E if an only if its class in ${}^fE_\infty^{p,0}$ vanishes, i.e. if its class in ${}^fE_k^{p,0}$ is hit by ${}^fd_k$ for some $k \in \{2, \dots, q-1\}$.*

$\square$

Example 5.14. We consider the bundle $f : S^{2n+1} \to \mathbb{CP}^n$. If $x \in H^{2k}_{dR}(\mathbb{CP}^n) \cong \mathbb{R}$ for $k \in \{2, \dots, n\}$, then x is in the image of ${}^f d_2$ (which is an isomorphism). Hence $f^* x = 0$. This is of course also clear since $H^{2k}_{dR}(S^{2n+1}) = 0$. □

Example 5.15. Let $E \to B$ be a locally trivial fibre bundle over a connected base and $b \in B$. We identify the fibre E_b with F. Then we get a map fibre bundles

$$\begin{tikzcd} F \arrow[r, "\psi"] \arrow[d, "p"] & E \arrow[d, "f"] \\ p \arrow[r, "b"] & B \end{tikzcd} \quad . \tag{41}$$

Given a class $x \in H^q_{dR}(F)$ one can ask whether it extends to a class $\tilde{x} \in H^q_{dR}(E)$, i.e. $\psi^* \tilde{x} = x$. We consider the induced map of LSSS $E(\psi) : ({}^f E_r, {}^f d_r)_{r \geq 1} \to ({}^p E_r, {}^p d_r)_{r \geq 1}$. We have

$${}^p E_2^{p,q} \cong \begin{cases} H^q_{dR}(F) & p = 0 \\ 0 & else \end{cases} \quad .$$

This spectral sequence degerates at the second (actually the first) page. We also have

$${}^f E_2^{p,q} \cong \begin{cases} H^q_{dR}(F) & p = 0 \\ * & else \end{cases} \quad ,$$

and $E(\psi)$ induces the obvious identification of the zero column.

We fix $q \in \mathbb{Z}$. We have a decreasing chain of subspaces

$${}^f E_2^{0,q} \supseteq {}^f E_3^{0,q} \supseteq {}^f E_4^{0,q} \supseteq \cdots \supseteq {}^f E_{q+1}^{0,q} = {}^f E_\infty^{0,q} \ ,$$

where ${}^f E_{k+1}^{0,q} = \mathtt{ker}({}^f d_k : {}^f E_k^{0,q} \to {}^f E_k^{k+1,q-k})$.

Corollary 5.16. *A class $x \in H^q(F) \cong {}^f E_2^{0,q}$ extends to E if and only if it belongs to the kernel of all differentials ${}^f d_k$ for $k \in \{2, \dots, q\}$.*

□

Example 5.17. We consider again the bundle $S^{2n+1} \to \mathbb{CP}^n$ with fibre S^1. Let $\mathtt{or}_{S^1} \in H^1_{dR}(S^1)$ be the class of a normalized volume form. This class does not extend to S^{2n+1} since ${}^f d_2 \mathtt{or}_{S^1} \neq 0$ (again since ${}^f d_2$ is an isomorphism). Of course this is also a priori clear since $H^1_{dR}(S^{2n+1}) = 0$.

□

5.4 The multiplicative structure of the Leray-Serre spectral sequence

Let A be a differential graded algebra and $(\mathcal{F}^p A)_{p\in\mathbb{Z}}$ be a decreasing filtration of A as a chain complex. We say the filtration is **multiplicative** if the product restricts to maps

$$\mathcal{F}^p A \otimes \mathcal{F}^q A \to \mathcal{F}^{p+q} A \ .$$

We get induced maps

$$\mathrm{Gr}^p A \otimes \mathrm{Gr}^q(A) \to \mathrm{Gr}^{p+q} A$$

so that $\mathrm{Gr}(A)$ is a bigraded differential algebra.

We now consider the exact couple of a multiplicatively filtered differential graded commutative algebra. We observe that

$$W = \bigoplus_{p,q} H^q(\mathcal{F}^p A)$$

is a graded commutative algebra, graded by cohomological degree. Similarly,

$$E = \bigoplus_{p,q} H^{p+q}(\mathrm{Gr}^p(A))$$

is a graded commutative algebra graded by cohomological degree. Both algebras are actually bigheaded with p as an additional degree. The maps $i : W \to W$ and $\mathtt{pr} : W \to E$ are morphisms of graded algebras. Furthermore, $\partial : W \to E$ satisfies the Leibnitz rule, i.e. it is a derivation. Indeed, if $[x] \in E^{p,q}$ and $[y] \in E^{p',q'}$ with representatives $x \in \mathcal{F}^p A^{p+q}$ and $y \in \mathcal{F}^{p'} A^{p'+q'}$, then we have

$$\partial([x] \cup [y]) = \partial([x \cup y]) = [dx \cup y] + [(-1)^{p+q} x \cup dy] = \partial[x] \cup [y] + (-1)^{p+q}[x] \cup \partial[y] \ .$$

We call a bigraded exact couple $(E, W, i, \mathtt{pr}, \partial)$ **multiplicative** if E and W are graded commutative algebras, i and $\mathtt{pr}$ are morphisms of graded algebras, and ∂ satisfies the Leibniz rule as above.

Lemma 5.18. *If $(W, E, i, \mathtt{pr}, \partial)$ is a bigraded multiplicative exact couple, then the derived exact couple is bigraded multiplicative again.*

Proof. The image of a homomorphism of graded commutative algebras is again a graded commutative algebra. Hence $W' = i(W)$ is a graded commutative algebra and $i' : W' \to W'$ is a homomorphism of algebras since it just a restriction of a homomorphism.

The cohomology of a derivation with square zero is an algebra with multiplication defined on representatives. The composition $d := \mathtt{pr} \circ \partial$ is a derivation. Consequently $E' = H(d)$ is a graded commutative algebra. It is immediate from the definition $\partial'[e] = [\partial e]$ that ∂' is again a derivation. Finally the map $\mathtt{pr}' : W' \to E'$ is induced by the homomorphism $W \overset{\mathtt{pr}}{\to} \mathtt{ker}(d)$ by factorization over quotients and hence a homomorphism of algebras. $\square$

Lemma 5.19. *If $f : E \to B$ is a locally trivial fibre bundle, then the filtration $(\mathcal{F}^p\Omega(E))_p$ is multiplicative. Consequently, the Leray-Serre spectral sequence is bigraded multiplicative.*

Proof. Let $\omega \in F^p\Omega^n(E)$ and $\alpha \in F^q\Omega^m(E)$. We must show that $\omega\wedge\alpha \in F^{p+q}\Omega^{n+m}(E)$. Let $X_1, \dots, X_{n+m-p-q+1}$ be a collection of vertical vectors at $e \in E$. It follows from the derivation property of the insertion operation that $i_{X_{n+m-p-q+1}} \dots i_{X_1}(\omega \wedge \alpha)$ is a sum over pairs $(r, s) \in \mathbb{N}^2$ satisfying $r+s = n+m-p-q+1$ of terms where r vectors are inserted into ω and s vectors are inserted into α. The conditions $r < n + p - 1$ and $s < m+q-1$ together imply that $r+s < n+m-p-q-2$. Hence this case does not appear in the sum and we have for every term $r \geq n + p - 1$ or $s \geq m - q + 1$. Hence every term of the sums vanishes. $\square$

Corollary 5.20. *Let $E \to B$ be a locally trivial fibre bunds. Then for every $r \geq 1$ the r'th page E_r of the LSSS is a bigraded, graded-commutative algebra, and d_r is a derivation of total degree one and bidegree $(-r, r + 1)$.*

Example 5.21. We now determine the ring structure on $H_{dR}(\mathbb{CP}^n)$ and fix a preferred basis.

Let $1 := c_0 \in H^0_{dR}(\mathbb{CP}^n) \cong E_2^{0,0}$ be the canonical generator. We consider the inclusion of a point $\iota : * \to \mathbb{CP}^n$ and the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "g"] & S^{2n+1} \arrow[d, "f"] \\ * \arrow[r, "\iota"] & \mathbb{CP}^n \end{tikzcd} .$$

Then

$$ {}^fE_2^{0,1} \overset{E_2(\iota)}{\to} {}^gE_2^{0,1} \cong H^1_{dR}(S^1) \overset{\int_{S^1}}{\cong} \mathbb{R} $$

is an isomorphism. We fix a generator of $u \in {}^fE_2^{0,1}$ such that $\int_{S^1} E_2(\iota)(u) = 1$. Then we define the generator

$$x_2 := c_1 := d_2(u) \in {}^fE_2^{2,0} \cong H^2_{dR}(\mathbb{CP}^n) \ .$$

We now define inductively for $k = 1, \dots, n$

$$x_{2k+2} := {}^fd_2(u \cup x_{2k}) = c_1 \cup x_{2k} = c_1^k \ .$$

We claim that the multiplication $u : {}^fE_2^{p,0} \to {}^fE_2^{p,1}$ is an isomorphism. Note that we have a multiplicative isomorphism

$$({}^fE_1^{*,q}, d_1) \cong (\Omega^*(\mathbb{CP}^n) \otimes H^q_{dR}(S^1), d \otimes \mathtt{id}_{H^q_{dR}(S^1)}) \ ,$$

and that the multiplication $u \cup \cdots : H^0_{dR}(S^1) \to H^1_{dR}(S^1)$ is an isomorphism. Since the differentials ${}^fd_2 : {}^fE_2^{k,1} \to {}^fE_2^{k+2,1}$ are isomorphisms for $k = 0, \dots, n-1$ we conclude that $x_{2k} = c_1^k \neq 0$ as long as $k \leq n$. Hence $\{x_k\}$ is a basis of the one-dimensional $\mathbb{R}$-vector space $H^{2k}_{dR}(\mathbb{CP}^n)$ for $k = 0, \dots, n$. We thus have determined the ring structure:

Corollary 5.22. *We have an isomorphism of graded rings*

$$H_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[c_1]/(c_1^{n+1}) \ .$$

For $1 \leq k \leq n$ the inclusion $\mathbb{C}^{k+1} \hookrightarrow \mathbb{C}^{n+1}$ induces a diagram

$$\begin{tikzcd} S^{2k+1} \arrow[r, "g"] \arrow[d, "f_k"] & S^{2n+1} \arrow[d, "f_n"] \\ \mathbb{CP}^k \arrow[r, "i"] & \mathbb{CP}^n \end{tikzcd} \ .$$

We get an induced morphism of LSSS'es $E(g) : ({}^{f_n}E_r, {}^{f_n}d_r)_{r\geq 1} \to ({}^{f_k}E_r, {}^{f_k}d_r)_{r\geq 1}$. In particular, ${}^{f_n}E_r^{0,1} \to {}^{f_k}E_r^{0,1}$ is an isomorphism and compatible with the choices of u. Using the functoriality of the LSSS we conclude that $i^*c_{1,\mathbb{CP}^n} = c_{1,\mathbb{CP}^k}$ (with selfexplaining notation).

Corollary 5.23. *The restriction map $i^* : H^*_{dR}(\mathbb{CP}^n) \to H^*_{dR}(\mathbb{CP}^k)$ is just the canonical quotient map $\mathbb{R}[c_1]/(c_1^{n+1}) \to \mathbb{R}[c_1]/(c_1^{k+1})$.*

□

6 Chern classes

6.1 The first Chern class

A **complex line bundle** is a one-dimensional complex vector bundle. Let $L \to M$ be a complex line bundle. Then by Lemma 4.2 we can choose a connection ∇. The section $\mathtt{id}_L \in \Gamma(M, \mathtt{End}(L))$ gives a canonical identification of $\mathtt{End}(L)$ with the trivial line bundle. The curvature (see Definition 4.5) of ∇ (which as a priori an element $R^\nabla \in \Omega^2(M, \mathtt{End}(L))$) can therefore be interpreted as an element of $\Omega^2(M, \mathbb{C})$. We define the **first Chern form**

$$c_1(\nabla) := -\frac{1}{2\pi i} R^\nabla \in \Omega^2(M, \mathbb{C}) \ . \tag{42}$$

Lemma 6.1. *1. The first Chern form $c_1(\nabla)$ is closed.*

2. The cohomology class $c_1(L) := [c_1(\nabla)] \in H^2_{dR}(M, \mathbb{C})$ does not depend on the choice of ∇ (This justifies the notation!).

3. For line bundles L, L' on M we have $c_1(L \otimes L') = c_1(L) + c_1(L')$.

4. If L is trivial, then $c_1(L) = 0$.

5. We have $c_1(L^*) = -c_1(L)$, where L^* is the dual bundle.

6. For a map $f : M' \to M$ we have $c_1(f^*L) = f^*c_1(L)$.

Proof. 1. In general, the curvature of a connection ∇ on a vector bundle L satisfies the **Bianchi-identity**

$$\nabla^{\mathtt{End}(L)} R^\nabla = [\nabla, \nabla \circ \nabla] = \nabla \circ \nabla \circ \nabla - \nabla \circ \nabla \circ \nabla = 0 \ .$$

Since $\nabla^{\mathtt{End}(L)} \mathtt{id}_L = 0$ our trivialization of $\mathtt{End}(L)$ identifies the connection on $\mathtt{End}(L)$ with the trivial connection. Consequently, for a line bundle, we have $dR^\nabla = 0$.

2. Let ∇, ∇' be two connections. We consider the bundle $\tilde{L} := \mathtt{pr}_M^* L \to I \times M$ and the connection $\tilde{\nabla} := \mathtt{pr}_M^* \nabla + t(\mathtt{pr}_M^* \nabla' - \mathtt{pr}_M^* \nabla)$, where $t : I \to \mathbb{R}$ is the coordinate.

Note that $\tilde{\nabla}_{|\{0\} \times M} = \nabla$ and $\tilde{\nabla}_{|\{1\} \times M} = \nabla'$. This implies $R^{\tilde{\nabla}}_{|\{0\} \times M} = R^\nabla$ and $R^{\tilde{\nabla}}_{|\{1\} \times M} = R^{\nabla'}$. Using the the homotopy invariance of de Rham cohomology at the marked equality we get

$$[c_1(\nabla)] = [c_1(\tilde{\nabla})]_{\{0\}\times M} \stackrel{!}{=} [c_1(\tilde{\nabla})]_{\{1\}\times M} = [c_1(\nabla')] \ .$$

3. We have for homogeneous $\phi \in \Omega(M, L)$ and $\phi' \in \Omega(M, L')$ that

$$\nabla^{L\otimes L'}(\phi \wedge \phi') = \nabla\phi \wedge \phi' + (-1)^{\deg \phi}\phi \wedge \nabla'\phi'$$

   and hence

$$R^{\nabla^{L\otimes L'}}(\phi \wedge \phi') = R^{\nabla}\phi \wedge \phi' + \phi \wedge R^{\nabla'}\phi' = (R^{\nabla} + R^{\nabla'})(\phi \wedge \phi') \ .$$

   (the two mixed terms cancel each other).

4. If L is trivial, then we can take the trivial connection ∇^{triv}. We have $R^{\nabla^{triv}} = 0$ and hence $c_1(L) = 0$.
5. The tensor product $L \otimes L^*$ is trivialized by the evaluation. Hence $c_1(L) + c_1(L^*) = 0$.
6. We use the identity

$$f^*R^{\nabla} = R^{f^*\nabla} \ .$$

□

Definition 6.2. *The cohomology class $c_1(L) \in H^2_{dR}(M;\mathbb{C})$ is called the first **Chern class** of the line bundle L.*

We consider the functor

$$\overline{\mathtt{Vect}}_{\mathbb{C}} : \mathbf{Mf}^{op} \to \mathbf{Set}$$

which associates to every manifold M the set of isomorphism classes of complex vector bundles on M, and to $f : M' \to M$ the pull-back $f^* : \overline{\mathtt{Vect}}_{\mathbb{C}}(M) \to \overline{\mathtt{Vect}}_{\mathbb{C}}(M')$.

Definition 6.3. *A **characteristic class** of degree p for complex vector bundles is a natural transformation*

$$\overline{\mathtt{Vect}}_{\mathbb{C}} \to H^p_{dR}(\dots, \mathbb{F}) \ ,$$

where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Lemma 6.4. *Let $c : \overline{\mathtt{Vect}}_{\mathbb{C}} \to H^p_{dR}(\dots;\mathbb{F})$ be a characteristic class of degree p for vector bundles and $p \geq 1$. Then we have $c(E) = 0$ if E is trivializable.*

Proof. If $E \to M$ is trivializable, then we have an isomorphism $E \cong f^*E'$, where $E' \in \overline{\mathtt{Vect}}_{\mathbb{C}}(*)$ and $f : M \to *$. Then $c(E) = f^*c(E') = 0$ since already $c(E') \in H^p_{dR}(*;\mathbb{F}) = 0$. □

Example 6.5. The transformation

$$\overline{\mathtt{Vect}}_{\mathbb{C}}(M) \ni E \mapsto \dim(E) \in H^0_{dR}(M)$$

is a characteristic class of degree zero.

For every vector bundle $E \to M$ we can define the line bundle $\det(E) := \Lambda^{max} E$. If E is one-dimensional, then $\det(E) \cong E$. The transformation

$$\overline{\mathtt{Vect}}_{\mathbb{C}}(M) \ni E \mapsto c_1(\det(E)) \in H^2_{dR}(M;\mathbb{C})$$

is a characteristic class of degree one.

Characteristic classes can be used to distinguish vector bundles or to descide wether they are trivializable.

□

Example 6.6. One $\mathbb{CP}^n$ we have the tautological line bundle $L^{taut} \to \mathbb{CP}^n$. It is a subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$. We consider the chart

$$\mathbb{C}^n \ni z = (z_1, \dots, z_n) \mapsto [1 : z_1 : \dots : z_n] \in \mathbb{CP}^n$$

of $\mathbb{CP}^n$. On this chart we can trivialize L^{taut} using the section $s(z) := (1, z_1, \dots, z_n)$. We let P be the orthogonal projection from the trivial $n+1$-dimensional bundle onto L^{taut} and define the connection ∇ on L^{taut} by

$$\nabla\phi := P\nabla^{triv}\phi, \quad \phi \in \Gamma(\mathbb{CP}^n, L^{taut}) .$$

On the right-hand side of this formula ϕ is considered as a section of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$ in the natural way. We have

$$\nabla s = Pds = \frac{\bar{z}^t dz}{1 + \bar{z}^t z} \otimes s .$$

For the curvature we get

$$\begin{aligned} R^\nabla &= d\frac{\bar{z}^t dz}{1+\bar{z}^t z} \\ &= \frac{d\bar{z}^t \wedge dz}{1+\|z\|^2} - \frac{(d\bar{z}^t z + \bar{z}^t dz)\wedge \bar{z}^t dz}{(1+\|z\|^2)^2} \\ &= \frac{(1+\|z\|^2)d\bar{z}^t \wedge dz - d\bar{z}^t z \wedge \bar{z}^t dz}{(1+\|z\|^2)^2} \ . \end{aligned}$$

We first assume that $n=1$. Then

$$R^\nabla = \frac{(1+|z|^2)d\bar{z}\wedge dz - |z|^2 d\bar{z}\wedge dz}{(1+|z|^2)^2} = \frac{1}{(1+|z|^2)^2} d\bar{z}\wedge dz = \frac{2i}{(1+r^2)^2}\mathrm{vol}_{\mathbb{C}} \ ,$$

where $r=|z|$. We calculate

$$\int_{\mathbb{CP}^1} R^\nabla = 2\pi i \int_0^\infty \frac{2r dr}{(1+r^2)^2} = 2\pi i \int_0^\infty \frac{ds}{(1+s)^2} = 2\pi i \ .$$

This implies:

Corollary 6.7.

$$\int_{\mathbb{CP}^1} c_1(L^{taut}) = -1 \ .$$

We now consider the higher-dimensional case. Note that $(d\bar{z}^t z \wedge \bar{z}^t dz)^2 = 0$. Therefore

$$(R^\nabla)^n = \frac{(1+\|z\|^2)^n (d\bar{z}^t\wedge dz)^n - n(1+\|z\|^2)^{n-1}(d\bar{z}^t \wedge dz)^{n-1}\wedge (d\bar{z}^t z\wedge \bar{z}^t dz)}{(1+\|z\|^2)^{2n}} \ .$$

We have the identities

$$(d\bar{z}^t\wedge dz)^n = (2i)^n n! \mathrm{vol}_{\mathbb{C}^n} \ , \quad (d\bar{z}^t\wedge dz)^{n-1}\wedge (d\bar{z}^t z\wedge \bar{z}^t dz) = (2i)^n (n-1)! \|z\|^2 \mathrm{vol}_{\mathbb{C}^n} \ .$$

We get

$$(R^\nabla)^n = \frac{(2i)^n n! \mathrm{vol}_{\mathbb{C}^n}}{(1+r^2)^{n+1}} \ .$$

Using that $\mathrm{vol}_{\mathbb{C}^n} = \mathrm{vol}_{S^{2n-1}} r^{2n-1} dr$ we get

$$\int_{\mathbb{CP}^n} (R^\nabla)^n = (2i)^n n! \mathrm{vol}(S^{2n-1}) \int_0^\infty \frac{r^{2n-1}dr}{(1+r^2)^{n+1}} \ .$$

We simplify the integral to

$$\frac{1}{2}\int_0^\infty \frac{r^{2n-2}}{(1+r^2)^{n+1}} 2rdr = \frac{1}{2}\int_0^\infty \frac{s^{n-1}}{(1+s)^{n+1}} ds = \frac{1}{2}\frac{\Gamma(n)}{\Gamma(n+1)} = \frac{1}{2n} \ .$$

Here we use the general formula (for $p, q, r > 0$)

$$\int_0^\infty \frac{s^{p-1}ds}{(1+qs)^{p+r}} = \frac{\Gamma(p)\Gamma(r)}{q^p\Gamma(p+r)} \ .$$

We further have

$$\mathrm{vol}(S^{2n-1}) = \frac{2\pi^n}{\Gamma(n)} = \frac{2\pi^n}{(n-1)!} \ .$$

So

$$\int_{\mathbb{CP}^n} (R^\nabla)^n = (2i)^n n! \frac{2\pi^n}{(n-1)!}\frac{1}{2n} = (2\pi i)^n \ .$$

We get:

Corollary 6.8.

$$\int_{\mathbb{CP}^n} c_1(L^{taut})^n = (-1)^n \ .$$

In particular, we conclude using Lemma 6.4 that for every $n \geq 1$ the bundle $L^{taut} \to \mathbb{CP}^n$ is not trivializable.

□

For every manifold M we have a natural inclusion $H_{dR}(M) \hookrightarrow H_{dR}(M;\mathbb{C})$, where we consider the target as a real vector space by restriction of structure. It induces an isomorphism $H_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C} \stackrel{\cong}{\to} H_{dR}(M;\mathbb{C})$.

Recall that $H_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[c_1]/(c_1^{n+1})$ for the generator $c_1 \in H^2_{dR}(\mathbb{CP}^n)$ fixed in Example 5.21. We get an induced isomorphism $H_{dR}(\mathbb{CP}^n;\mathbb{C}) \cong \mathbb{C}[c_1]/(c_1^{n+1})$.

Lemma 6.9. *We have $c_1 = -c_1(L^{taut})$. In particular, the first Chern class of the tautological bundle is real, i.e. $c_1(L^{taut}) \in H_{dR}(\mathbb{CP}^n)$.*

Proof. In view of Corollary 5.23 it suffices to show this on $\mathbb{CP}^1$. We consider the LSSS $(E_r, d_r)_{r\geq 1}$ for the bundle $f : S^3 \to \mathbb{CP}^1$. On S^3 we consider the form

$$\theta := (\frac{1}{2\pi i}\bar{z}^t dz)_{|S^3} \in \Omega^1(S^3) \tag{43}$$

defined by restricting a form defined on all of $\mathbb{C}^2$. We parametrize a fibre of f through $z \in S^3$ by $\mathbb{R}/\mathbb{Z} \ni t \mapsto \exp(2\pi i t)z \in \mathbb{C}^2$. The push-forward of ∂_t at the point t is $2\pi i \exp(2\pi i t)z$. The pull-back of θ is thus given by dt, i.e. the normalized volume form of the fibres. The class $[\theta] \in \mathrm{Gr}^0\Omega^1(S^3)$ is closed and represents a class $u \in E_1^{0,1}$. We now calculate

$$d\theta = (\frac{1}{2\pi i} d\bar{z}^t \wedge dz)_{|S^3} \in \Omega^2(S^3) \ .$$

If we insert the vertical vector iz at the point $z \in S^3$, then we get $-\frac{1}{2\pi} d\bar{z}^t z$. If we insert in this the tangent vector $X \in T_z S^3$, then we get

$$-\frac{1}{2\pi} d\bar{z}^t(X) z = -\frac{1}{2\pi} \bar{X}^t z = 0$$

since the tangent space $T_z S^3$ is the orthogonal complement of the line through z. Consequently, $d\theta \in F^1\Omega^2(S^3)$. This means that $d_1 u = 0$. Hence $u \in E_2^{0,1}$ is exactly the normalized generator fixed in Example 5.21. It follows that $d\theta$ represents $d_2 u$. To this end we must consider $d\theta$ as a representative for a section of $\Omega^2(\mathbb{CP}^1, \mathcal{H}^0(S^3/\mathbb{CP}^1, \mathbb{C}))$. A local section s of the bundle $S^3 \to \mathbb{CP}^1$ induces a local trivialization of $\mathcal{H}^0(S^3/\mathbb{CP}^1;\mathbb{C}) \to \mathbb{CP}^1$. We consider the section given by

$$\mathbb{CP}^1 \ni [1:z] \mapsto s(z) := \frac{(1,z)}{\sqrt{1+|z|^2}} \in S^3 \ .$$

We must calculate the pull-back

$$\begin{aligned} s^* d\theta &= d s^*\theta \\ &= d(\frac{1}{2\pi i} \frac{\bar{z}}{\sqrt{1+|z|^2}} d\frac{z}{\sqrt{1+|z|^2}}) + d(\frac{1}{2\pi i} \frac{1}{\sqrt{1+|z|^2}} d\frac{1}{\sqrt{1+|z|^2}}) \\ &= \frac{1}{2\pi i} d\frac{\bar{z}}{\sqrt{1+|z|^2}} \wedge d\frac{z}{\sqrt{1+|z|^2}} \ . \end{aligned}$$

We have

$$d\frac{z}{\sqrt{1+|z|^2}} = \frac{dz}{\sqrt{1+|z|^2}} - \frac{z(\bar{z}dz + d\bar{z}z)}{2\sqrt{1+|z|^2}^3} = \frac{(2+|z|^2)dz - z^2 d\bar{z}}{2\sqrt{1+|z|^2}^3}$$

Hence

$$\begin{aligned} s^*d\theta &= \frac{1}{2\pi i}\frac{(2+|z|^2)d\bar{z}-\bar{z}^2dz}{2\sqrt{1+|z|^2}^3}\wedge\frac{(2+|z|^2)dz-z^2d\bar{z}}{2\sqrt{1+|z|^2}^3} \\ &= \frac{(4+4|z|^2+|z|^4)d\bar{z}\wedge dz-|z|^4d\bar{z}\wedge dz}{8\pi i(1+|z|^2)^3} \\ &= \frac{d\bar{z}\wedge dz}{2\pi i(1+|z|^2)^2} \\ &= \frac{\mathrm{vol}_{\mathbb{C}}}{\pi(1+|z|^2)^2} \end{aligned}$$

We have

$$\int_{\mathbb{CP}^1} c_1 = \int_{\mathbb{C}} s^*d\theta = \int_{\mathbb{C}} \frac{\mathrm{vol}_{\mathbb{C}}}{\pi(1+|z|^2)^2} = 1 \ . \tag{44}$$

Since $\int_{\mathbb{CP}^1} c_1(L^{taut}) = -1$ we see that $c_1 = -c_1(L^{taut})$. □

Lemma 6.10. *Let $L \to M$ be a line bundle over a compact manifold M. Then there exists $n \in \mathbb{N}$ and a map $s : M \to \mathbb{CP}^n$ such that $(s^*L^{taut})^* \cong L$. Consequently we have*

$$c_1(L) = s^*c_1$$

and $c_1(L)$ is real.

Proof. For every $m \in M$ we choose a section $s_m \in \Gamma(M,L)$ such that $s_m(m) \neq 0$. Then we find a finite sequence of points $\{m_0,\dots,m_n\} \subset M$ such that

$$\bigcup_{k=0}^{n}\{s_{m_k} \neq 0\} = M \ .$$

We define the map

$$s := [s_{m_0} : \cdots : s_{m_n}] : M \to \mathbb{CP}^n \ .$$

We obtain the isomorphism of line bundles $L \xrightarrow{\sim} (s^*L^{taut})^*$ such that for every $m \in M$ the vector $x = \sum_{i=0}^n a_i s_{m_i}(m) \in L_m$ is mapped to the element of $(s^*L^{taut}_m)^*$ given by

$$s^*L^{taut}_m \ni (m,(b_0,\dots,b_n)) \mapsto \sum_{i=0}^{n} a_i b_i \ .$$

Note that this is well-defined independently of the choice of the representation of x since $(b_0,\dots,b_n) \sim (s_{m_0}(m),\dots,s_{m_n}(m))$. □

Remark 6.11. The last conclusion of Lemma 6.10 that $c_1(L)$ is real is true in general without any assumption on the compactness of M. Here is an argument. There exists an increasing sequence of compact submanifolds (with boundary) $M_1 \subseteq M_2 \subseteq \ldots$ such that $M = \bigcup_i M_i$. We have $H_{dR}(M) = \mathtt{lim}_{i \in \mathbb{N}^{op}} H_{dR}(M_i)$. A class in $H_{dR}(M)$ is real iff its restriction to M_i is real for all $i \in \mathbb{N}$. Since $c_1(L)_{|M_i} = c_1(L_{|M_i})$is real by Lemma 6.10 we conclude that $c_1(L)$ is real, too.

$\square$

Example 6.12. We continue to use the notation of Lemma 6.10. We can use the isomorphism $s^*L^{taut} \cong L^*$ in order to induce a metric on L^*. Let $\pi : E \to M$ be the unit-sphere bundle of L^*. We get the map of fibre bundles

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[d, "\pi"] & S^{2n+1} \arrow[d, "f"] \\ M \arrow[r, "s"] & \mathbb{CP}^n \end{tikzcd}$$

which induces diffeomorphisms on fibres. We get an induced map of LSSS'es $E(r) : ({}^fE_r, {}^fd_r)_{r\geq 1} \to ({}^\pi E_r, {}^fd_r)_{r\geq 1}$. We trivialize $\mathcal{H}^1(E/M) \cong s^*\mathcal{H}^1(S^{2n+1}/\mathbb{CP}^n)$ by pulling back the trivialization of $\mathcal{H}^1(S^{2n+1}/\mathbb{CP}^n)$. Hence the spectral sequence has the form

$$\begin{tikzcd} 1 & H^0(M) \arrow[drr] & H^1(M) \arrow[drr] & H^2(M) \arrow[drr] & H^3(M) \arrow[drr] & H^4(M) \arrow[drr] & H^5(B) & H^6(M) \\ 0 & H^0(M) & H^1(M) & H^2(M) & H^3(M) & H^4(M) & H^5(M) & H^6(M) \\ & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{tikzcd}$$

where the differential is given by multiplication by $c_1(L)$ and $H^*(M)$ is a shorthand for $H^*_{dR}(M)$.

Example 6.13. In this example we calculate $c_1(T\mathbb{CP}^n) := c_1(\det(T\mathbb{CP}^n))$. We define a map of complex vector bundles over $\mathbb{CP}^n$

$$a : T\mathbb{CP}^n \to \mathtt{Hom}(L, \mathbb{C}^{n+1}/L)$$

as follows. Let $X \in T_x\mathbb{CP}^n$ and $\ell \in L_x$. Then we choose a local holomorphic section ϕ of L such that $\phi(x) = \ell$. We define

$$a(X)(\ell) := [\nabla^{triv}_X \phi] \ .$$

Holomorphy of the section is necessary in order to get a complex linear map in X. The right-hand side is independent of the choice of the extension ϕ. This follows from the Leibnitz rule. Namely, any other extension can be written in the form $f\phi$, where f is holomorphic and $f(1) = 1$. Then we have

$$[\nabla_X^{triv}(f\phi)] = [\nabla_X^{triv}\phi + X(f)\phi] = [\nabla_X^{triv}\phi] \ .$$

We now show that a is an isomorphism. It suffices to show injectivity. Using the local section s in Example 6.6 we can choose $\phi = \lambda s$ for a suitable constant λ. We get

$$a(X)(\ell) = \lambda[(1-P)\nabla_X^{triv}s] = \lambda[(0,X) - \frac{\bar{z}^t X}{1+\|z\|^2}(1,z)] \ .$$

If $X - \frac{\bar{z}^t X}{1+\|z\|^2} z = 0$ and $\frac{\bar{z}^t X}{1+\|z\|^2} = 0$, then $X = 0$.

In other words, we have an exact sequence

$$0 \to \mathbb{C} \to \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1\times} \to T\mathbb{CP}^n \to 0 \ . \tag{45}$$

This implies

$$\det(T\mathbb{CP}^n) \cong (L^*)^{n+1} \ .$$

We conclude:

Corollary 6.14. $c_1(\mathbb{CP}^n) = (n+1)c_1$.

□

Example 6.15. We consider an oriented surface Σ_g of genus g. If we choose a Riemannian metric on Σ_g, then we can define a complex structure on $T\Sigma_g$ such that multiplication by i in $T_x\Sigma_g$ is the positive $\pi/2$-turn, i.e. for $X \in T_x\Sigma_g$ we have $\langle X, iX\rangle = 0$ and the family (X, iX) is an oriented basis. We let $S(\Sigma_g) \to \Sigma_g$ be the unit-sphere bundle. We are interested in the topology of this bundle.

Let $c_1(T\Sigma_g) \in H^2_{dR}(\Sigma_g)$ be the first Chern class of $T\Sigma_g$ (considered as a complex vector bundle). The number

$$d(T\Sigma_g) := \int_{\Sigma_g} c_1(T\Sigma_g)$$

is called the **degree** of the $T\Sigma_g$.

We start with $\Sigma_0 \cong S^2 \cong \mathbb{CP}^1$. In this case we get

$$d(T\Sigma_g) = \int_{\mathbb{CP}^1} c_1(T\mathbb{CP}^1) \overset{Cor.6.14}{=} \int_{\mathbb{CP}^1} 2c_1 \overset{(44)}{=} 2 \ .$$

In the case $g = 1$ we have $\Sigma_1 \cong T^2$. Since $T\Sigma_1$ is trivializable we have

$$d(T\Sigma_1) = \int_{\Sigma_1} c_1(T\Sigma_1) = 0 \ .$$

Recall that we can obtain Σ_g from S^2 by attaching g handles. Further recall that

$$d(T\Sigma_g) = \int_{\Sigma_g} c_1(\nabla)$$

for any connection ∇ on $T\Sigma_g$.

We can arrange the connection ∇ on $T\Sigma_g$ such that the situation near every handle looks the same. The attachment of one handle changes the integral by $H - D = -2$, where H is the contribution of the handle and D is the contribution of the two discs removed. Therefore attaching g handles gives a change of the integral by $-2g$. We conclude that the degree of the tangent bundle of a genus g-surface is

$$d(T\Sigma_g) = 2 - 2g = \chi(\Sigma_g) \ .$$

Corollary 6.16 (Gauss-Bonnet)**.** *We have*

$$\int c_1(T\Sigma_g) = \chi(\Sigma_g) \ .$$

We can now calculate the cohomology of the unit sphere bundle $S(\Sigma_g)$. Since the base Σ_g and the fibre S^1 of $S(\Sigma_g)$ are connected the manifold $S(\Sigma_g)$ is connected, too. Thus $H^0(S(\Sigma_g)) \cong \mathbb{R}$. For the higher degree cohomology we use the Gysin sequence

$$0 \to H^1_{dR}(\Sigma_g) \to H^1_{dR}(S(\Sigma_g)) \overset{\sigma}{\to} H^0_{dR}(\Sigma_g) \overset{-d\mathrm{vol}_{\Sigma_g}}{\to} H^2_{dR}(\Sigma_g) \to H^2_{dR}(S(\Sigma_g)) \overset{\sigma}{\to} H^1_{dR}(\Sigma_g) \to 0 \ .$$

and

$$H^3_{dR}(S(\Sigma_g)) \overset{\sigma}{\cong} H^2_{dR}(\Sigma_g) \ .$$

If $d \neq 0$, then σ induces isomorphisms

$$H^1_{dR}(S(\Sigma_g)) \cong H^1_{dR}(\Sigma_g) \cong H^2_{dR}(S(\Sigma_g)) \ .$$

We have

$$H^k_{dR}(S(\Sigma_g)) \cong \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R}^{2g} & k=1,2 \\ \mathbb{R} & k=3 \\ 0 & else \end{cases} \ .$$

If $d = 0$, then $g = 1$ and

$$H^k_{dR}(S(\Sigma_1)) \cong \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R}^{3} & k=1,2 \\ \mathbb{R} & k=3 \\ 0 & else \end{cases} \ .$$

6.2 Cohomology of bundles over spheres

In this subsection we consider a locally trivial fibre bundle $E \to S^n$. The LSSS has only two non-trivial columns. Its only non-trivial differential after the E_2-term is d_n. Therefore $E_2 \cong E_n$ and $E_{n+1} \cong E_\infty$.

We assume that $n \geq 2$. In this case S^n is simply connected and we can trivialize the bundles $\mathcal{H}^q(E/S^n)$. This trivialization induces the first isomorphism in

$$H^n_{dR}(S^n, \mathcal{H}^q(E/S^n)) \cong H^n_{dR}(S^n) \otimes \mathcal{H}^q(F) \cong \mathcal{H}^q(F) \ ,$$

where the second uses the choice of a generator $\mathrm{vol}_{S^n} \in H^n_{dR}(S^n)$ which we normalize such that $\int_{S^n} \mathrm{vol}_{S^n} = 1$. We therefore get

$$H^k(S^n, \mathcal{H}^q(E/S^n)) \cong \begin{cases} H^q(F) & k=0,n \\ 0 & else \end{cases}$$

Here we write out the the E_4-term for $n = 4$.

$$
\begin{tikzcd}
4 & H^4(F) \arrow[rrrrddd] & 0 & 0 & 0 & H^4(F) \\
3 & H^3(F) \arrow[rrrrddd] & 0 & 0 & 0 & H^3(F) \\
2 & H^2(F) & 0 & 0 & 0 & H^2(F) \\
1 & H^1(F) & 0 & 0 & 0 & H^1(F) \\
0 & H^0(F) & 0 & 0 & 0 & H^0(F) \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

We get a long exact sequence

$$H^{k-n}_{dR}(F) \to H^k_{dR}(E) \xrightarrow{\sigma} H^k_{dR}(F) \xrightarrow{d_n} H^{k-n+1}_{dR}(F) \to \dots$$

Example 6.17. Let G be a Lie group and recall that a class $x \in H_{dR}(G)$ is called primitive if $\mu^* x = x \otimes 1 + 1 \otimes x$, where $\mu : G \times G \to G$ is the group multiplication and we use the Künneth formula $H_{dR}(G \times G) \cong H_{dR}(G) \otimes H_{dR}(G)$.

For $n \geq 3$ will show inductively:

Lemma 6.18. *We have an isomorphism*

$$H_{dR}(SU(n)) \cong \mathbb{R}[u_3, u_5, \dots, u_{2n-1}] \ ,$$

where the generators u_i of degree i are primitive.

The case $n = 2$ is clear since

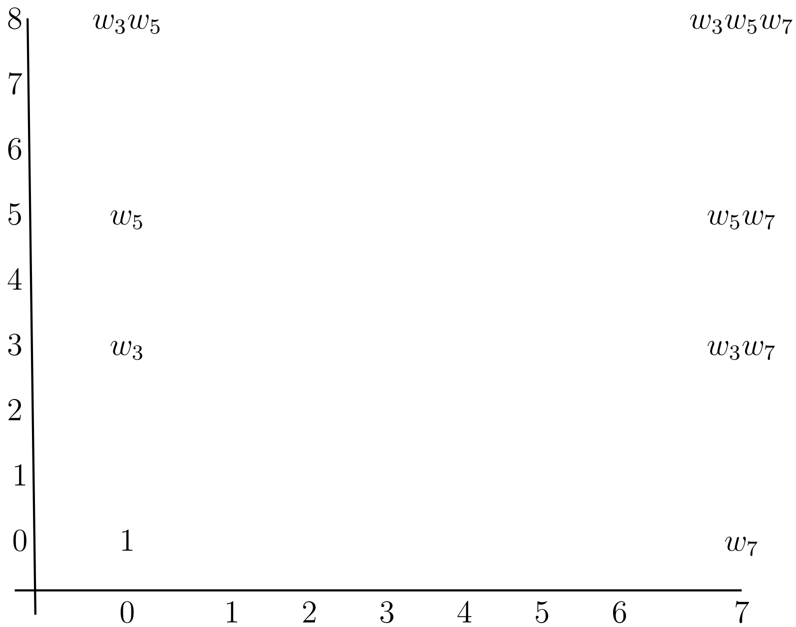
$$SU(2) \cong S^3 \ , \quad H_{dR}(SU(2)) \cong \mathbb{R}[u_3] \ ,$$

where u_3 corresponds to the orientation class of S^3 which we normalize such that $\int_{S^3} u_3 = 1$.

For the induction step from n to $n+1$ we consider the bundle $SU(n+1) \to S^{2n+1}$ with fibre $SU(n)$. By induction assumption we get

$$E_{2n+1} \cong \mathbb{R}[w_3, w_5, \dots, w_{2n-1}, w_{2n+1}]$$

as rings, where the elements w_{2k-1} generate $E_{2n+1}^{0,2k-1}$ for $k = 2, \ldots, n$, and the last element w_{2n+1} generates $E_{2n+1}^{0,2n+1}$. Here is the picture for $n = 3$.



We have

$$d_{2n+1} w_{2k-1} \in E_{2n+1}^{2n+1,2k-2n} = 0 \ , \qquad k = 2, \ldots, n \ .$$

Therefore also all differentials of the products of the generators vanish. Consequently

$$E_2 \cong E_{2n+1} \cong E_\infty \ .$$

We choose classes

$$u_{2k-1} \in H_{dR}^{2k-1}(SU(n+1)) \ , \qquad k = 2, \ldots, n$$

and

$$u_{2n+1} \in \mathcal{F}^{2n+1} H_{dR}^{2n+1}(SU(n+1))$$

which are detected by the classes w_k. We define a multiplicative filtration (actually a grading) of $\mathbb{R}[u_3, \ldots, u_{2n+1}]$ such that $u_{2k-1} \in \mathcal{F}^0$ for all $k = 2, \ldots, n$ and $u_{2n+1} \in \mathcal{F}^{2n+1}$. Then we get a map of filtered rings

$$\mathbb{R}[u_3, \ldots, u_{2n+1}] \to H_{dR}(SU(n+1))$$

which induces is an isomorphism of associated graded rings. Consequently it is an isomorphism.

We also get an explicit understanding of the classes. For $k \le n$ the classes

$$u_{2k-1} \in H^{2k-1}_{dR}(SU(n+1))$$

are characterized by the property that they restrict to classes with the same name in $H^{2k-1}_{dR}(SU(n))$ under the inclusion $SU(n) \hookrightarrow SU(n+1)$. Furthermore, the class u_{2n+1} is the pull-back of the normalized generator $\mathrm{vol}_{S^{2n+1}} \in H^{2n+1}_{dR}(S^{2n+1})$ under the projection $SU(n+1) \to S^{2n+1}$.

It remains to show that u_{2k-1} is primitive. Note that the inclusion $SU(n) \to SU(n+1)$ is a group homomorphism. For $k = 2, \dots, n$ the generator $u_{2k-1} \in H_{dR}(SU(n+1))$ is primitive if and only if its restriction to $SU(n)$ is primitive. So by induction it suffices to show that u_{2n+1} is primitive.

Let $a : SU(n+1) \times S^{2n+1} \to S^{2n+1}$ be the action. The associativity relation is

$$a \circ (\mathtt{id} \times a) = a \circ (\mu \times \mathtt{id}) : SU(n+1) \times SU(n+1) \times S^{2n+1} \to S^{2n+1} \ .$$

By definition we have

$$a^*\mathrm{vol}_{S^{2n+1}} = u_{2n+1} \otimes 1 + 1 \otimes \mathrm{vol}_{S^{2n+1}} \ .$$

We now compare the identities

$$(\mu \times \mathtt{id})^* a^*\mathrm{vol}_{S^{2n+1}} = (\mu^* \otimes \mathtt{id})(u_{2n+1} \otimes 1 + 1 \otimes \mathrm{vol}_{S^{2n+1}}) = \mu^* u_{2n+1} \otimes 1 + 1 \otimes \mathrm{vol}_{S^{2n+1}}$$

and

$$(\mathtt{id} \times a)^* a^*\mathrm{vol}_{S^{2n+1}} = (\mathtt{id} \otimes a^*)(u_{2n+1} \otimes 1 + 1 \otimes \mathrm{vol}_{S^{2n+1}}) = u_{2n+1} \otimes 1 \otimes 1 + 1 \otimes u_{2n+1} \otimes 1 + 1 \otimes 1 \otimes \mathrm{vol}_{S^{2n+1}}$$

in order to conclude that

$$\mu^* u_{2n+1} = u_{2n+1} \otimes 1 + 1 \otimes u_{2n+1} \ .$$

$\square$

Example 6.19.

Lemma 6.20. *We have an isomorphism*

$$H_{dR}(U(n)) \cong \mathbb{R}[u_1, u_3, \ldots, u_{2n-1}]$$

where the generators u_i of degree i are primitive. Furthermore, the inclusion $SU(n) \to U(n)$ induces a map in cohomology which sends u_1 to zero and otherwise identifies the generators with the same names.

Proof. We argue as in the proof of Lemma 6.18. We use the bundles $U(n+1) \to S^{2n+1}$ with fibre $U(n)$ and induction. We start with $U(1) = S^1$.

In order to get the restriction map to $SU(n)$ we use the comparison of spectral sequences for the bundle map

$$\begin{tikzcd} SU(n+1) \arrow[r] \arrow[d] & U(n+1) \arrow[d] \\ S^{2n+1} \arrow[r, equal] & S^{2n+1} \end{tikzcd} .$$

□

This calculation is compatible with the fact that we have a diffeomorphism (not a group homomorphism)

$$U(1) \times SU(n) \xrightarrow{\cong} U(n) .$$

It maps a pair (λ, g) to the product of $\mathtt{diag}(\lambda, 1, \ldots, 1)$ and g.

6.3 The Leray-Hirsch theorem and higher Chern classes

We consider a fibre bundle $f : E \to B$. Then the cohomology of the total space $H_{dR}(E)$ becomes a graded commutative algebra over the graded commutative ring $H_{dR}(B)$ such that $x \in H_{dR}(B)$ acts on $H_{dR}(E)$ by left multiplication with f^*x.

Let us now assume that the bundle has a connected base B. We choose a base point in B and consider the fibre $F := f^{-1}(\{b\})$ as the concrete model of the fibres of the bundle. We assume that F is compact.

Proposition 6.21 (Leray-Hirsch Theorem)**.** *If the restriction $H_{dR}(E) \to H_{dR}(F)$ is surjective, then we have an isomorphism $H_{dR}(E) \cong H_{dR}(F) \otimes H_{dR}(B)$ as graded $H_{dR}(B)$-modules.*

Proof. We first observe that the LSSS is a spectral sequence of differential graded $H_{dR}(B)$-algebras. We can choose a split $s : H_{dR}(F) \to H_{dR}(E)$ of the restriction map as graded vector spaces. This induces a trivialization of the fibrewise cohomology bundle $\mathcal{H}(E/B) \to B$ by

$$B \times H_{dR}(F) \ni (b, x) \mapsto s(x)_{|E_b} \in \mathcal{H}(E/B)_b \subseteq \mathcal{H}(E/B)$$

We further have the isomorphism

$$E_2 \cong H_{dR}(F) \otimes H_{dR}(B)$$

of $H_{dR}(B)$-algebras. By assumption and Corollary 5.16 the classes in $H_{dR}(F) \otimes 1$ are annihilated by all differentials. Consequently the LSSS degenerates at the second term. We get a morphism

$$H_{dR}(F) \otimes H_{dR}(B) \to H_{dR}(E) \ , \quad f \otimes b \mapsto s(f) \cup b$$

of filtered $H_{dR}(B)$-modules, where we set

$$\mathcal{F}^p(H_{dR}(F) \otimes H_{dR}(B)) = \bigoplus_{t \geq p} H_{dR}(F) \otimes H^t_{dR}(B) \ .$$

This map induces an isomorphism of graded groups and is therefore an isomorphism. □

Remark 6.22. If one works with other multiplicative cohomology theories, then for the Leray-Hirsch theorem one must assume that the restriction map from the cohomology of the total space to the cohomology of the fibre is **split** surjective. Since in our case the cohomology takes values in real vector spaces, every surjection is split. □

Let $E \to B$ be a complex vector bundle over a connected base space B. We set $n := \dim(E)$. Then we define the **projective bundle** $f : \mathbb{P}(E) \to B$ with fibre $\mathbb{CP}^{n-1}$. By definition, point in $\mathbb{P}(E)$ is a line in E. We furthermore have a **tautological bundle** $L \to \mathbb{P}(E)$. A point in L is a pair of a line in E and a point in this line. A local trivialization of $E \to B$ induces a local trivialization of $\mathbb{P}(E) \to B$ naturally. In this way we can define the manifold structure on the projective bundle.

We set $x := c_1(L) \in H^2_{dR}(\mathbb{P}(E))$. By Example 5.21 the restriction to the fibre of the collection of classes $(1, x, x^2, \dots, x_{n-1})$ generates the cohomology of the fibre

$\mathbb{P}(E) \to B$ as a vector space. So the assumption of the Leray-Hirsch theorem 6.21 is satisfied. We take the split

$$s : H_{dR}(\mathbb{CP}^{n-1}) \to H_{dR}(\mathbb{P}(E)) \tag{46}$$

of the restriction to the fibre given by $s(c_1^k) := x^k$ for $k = 0, \ldots, n-1$ in order to define the isomorphism of graded $H_{dR}(B)$-modules

$$\mathbb{R}[c_1]/(c_1^n) \otimes H_{dR}(B) \stackrel{\cong}{\to} H_{dR}(\mathbb{P}(E)) \ . \tag{47}$$

In general, this is **not an isomorphism of rings** since x^n does not necessarily vanish in $H^*_{dR}(\mathbb{P}(E))$. This observation is the starting point for the definition of Chern classes. Note that x^n can be expressed as a unique linear combination of the x^k for $k = 0, \ldots, n-1$ with coefficients in $H_{dR}(B)$. More precisely, there are uniquely determined classes $c_i(E) \in H^{2i}(B)$, $i = 1, \ldots, n$ such that

$$\sum_{i=0}^{n} (-1)^i f^* c_{n-i}(E) x^i = 0 \ , \tag{48}$$

where we set $c_0(E) := 1$.

Definition 6.23. *For $i = 1, \ldots, n = \dim(E)$ the class $c_i(E) \in H^{2i}_{dR}(B)$ is called the i'th **Chern class** of the bundle E. For $i > n$ we set $c_i(E) := 0$.*

Remark 6.24. The Chern classes measures the deviation of the map (47) from a homomorphism of algebras, i.e. the deviation of the split (46) from being a morphism of graded rings. Indeed, s is a ring homomorphism if and only if $c_i(E) = 0$ for all $i = 1, \ldots, n$.

The following is easy to check:

Corollary 6.25. *The association $E \mapsto c_i(E)$ is a characteristic class of degree $2i$ for complex vector bundles (see Definition 6.3).*

Example 6.26. Let us check that the first Chern class c_1 coincides with our previous construction (Definition 6.2) of the first Chern class for line bundles. If $E \to B$ is a line bundle, then $\mathbb{P}(E) = B$ and $L = E$. The defining relation for the new Chern classes is $c_1(E) - c_0(E)x = 0$. In view of $c_0(E) = 1$ and $x = c_1(L)$ (old definition) we conclude that the new definition reproduces the old one. □

The projective bundle $\mathbb{P}(E) \to B$ is the bundle of flags of length one of E. More generally we can define the bundle $\mathbb{F}_k(E) \to E$ of **flags** of length k. A point in $\mathbb{F}_k(E)$ is an increasing sequence of subspaces $V_1 \subset \cdots \subset V_k$ of a fibre of E such that the quotient $V_{\ell+1}/V_\ell$ of two consecutive ones is one-dimensional. The manifold $\mathbb{F}(E) := \mathbb{F}_{n-1}(E)$ is the manifold of **complete flags**.

For $k = 1, \ldots, n$ we have fibre bundles $\mathbb{F}_k(E) \to \mathbb{F}_{k-1}(E)$ with fibre $\mathbb{CP}^{n-k}$. Note that on $\mathbb{F}_{k-1}(E)$ we have a bundle $\mathbb{V}_{k-1}(E) \to \mathbb{F}_{k-1}(E)$ such that the fibre of $\mathbb{V}_{k-1}(E)$ over the flag $V_1 \subset \cdots \subset V_{k-1}$ in E_b is V_{k-1}. To give an extension $V_1 \subset \cdots \subset V_k$ of this flag is equivalent to give a line in E_b/V_{k-1}. Hence we have a canonical isomorphism $\mathbb{F}_k(E) \cong \mathbb{P}(E/\mathbb{V}_{k-1}(E))$ of bundles over $\mathbb{F}_{k-1}(E)$.

The bundle $\mathbb{F}_k(E) \to \mathbb{F}_{k-1}(E)$ satisfies the assumption of the Leray-Hirsch theorem. for all $k = 1, \ldots, n$. Consequently, this theorem can be applied to the bundle $\mathbb{F}(E) \to B$ as well.

On $\mathbb{F}(E)$ we have line bundles $L_k \to \mathbb{F}(E)$ for $k = 1, \ldots, n$ such that the fibre of L_k on the flag $(V_1 \subset \cdots \subset V_n) \in \mathbb{F}(E)$ is V_k/V_{k-1}. We define classes

$$x_k := c_1(L_k) \in H^2_{dR}(\mathbb{F}(E)) \ , \quad k = 1, \ldots, n \ .$$

We know that $H_{dR}(\mathbb{F}(E))$ is a free $H_{dR}(B)$-module generated by the monomials

$$x_1^{i_1} \cup x_2^{i_2} \cup \cdots \cup x_n^{i_n} \ , \quad 0 \le i_k \le n-k \ , \quad k = 1, \ldots, n \ .$$

Let $f : \mathbb{F}(E) \to B$ be the projection. If we choose a metric on E, then we get a natural map $\rho_k : \mathbb{F}(E) \to \mathbb{P}(E)$ which sends the flag $(V_1 \subset \cdots \subset V_n)$ to the line $V_k \cap V_{k-1}^\perp$. We have a natural isomorphism $\rho_k^* L \cong L_k$ and hence $\rho_k^* x = x_k$. From (48) we get

$$\sum_{i=0}^n (-1)^i f^* c_{n-i}(E) x_k^i = 0 \ , \quad k = 1, \ldots, n \ .$$

This implies the identity of polynomials in t

$$\sum_{i=0}^n (-1)^i t^i f^* c_{n-i}(E) = \prod_{k=1}^n (x_k - t) \ .$$

In other words, we can express the pull-back of the Chern classes through the elementary symmetric functions in the x_k. We have

$$f^* c_i(E) = \sigma_i(x_1, \ldots, x_k) =: \sigma_i(x) \ .$$

For example

$$\begin{aligned}
\sigma_1(x_1,\ldots,x_k) &= x_1+\cdots+x_n \ , \\
\sigma_2(x_1,\ldots,x_k) &= x_1x_2+\cdots+x_{n-1}x_n \ , \\
\sigma_3(x_1,\ldots,x_k) &= x_1x_2x_3+\cdots+x_{n-2}x_{n-1}x_n \ .
\end{aligned}$$

Corollary 6.27. *We have an isomorphism*

$$H_{dR}(\mathbb{F}(\mathbb{C}^n) \cong \frac{\mathbb{R}[x_1,\ldots,x_n]}{(\sigma_1(x),\ldots,\sigma_n(x))} \ .$$

We define the **total Chern class**

$$c(E) := \sum_{i=0}^{n} c_i(E)$$

such that

$$f^*c(E) = \prod_{k=1}^{n}(1+x_i) \ . \tag{49}$$

Lemma 6.28. *For bundles $E \to B$ and $E' \to B$ we have the identity*

$$c(E \oplus E') = c(E) \cup c(E') \ .$$

Proof. The method of the proof of this Lemma is called **the splitting principle**.

We define a map

$$d : \mathbb{F}(E) \times_B \mathbb{F}(E') \to \mathbb{F}(E \oplus E')$$

such that

$$d((V_1 \subset \cdots \subset V_n),(W_1 \subset \cdots \subset W_{n'})) = (V_1 \subset \cdots \subset V_n \subset V_n \oplus W_1 \subset \cdots \subset V_n \oplus W_{n'}) \ .$$

Then

$$d^*(x_k) = \begin{cases} y_k & k=1,\ldots,n \\ y'_{k-n} & k=n+1,\ldots,n+n' \end{cases} \ ,$$

where y_k, y'_k are the classes on $\mathbb{F}(E) \times_B \mathbb{F}(E')$ pulled back from the corresponding classes on $\mathbb{F}(E)$ and $\mathbb{F}(E')$ using the projections $\mathtt{pr} : \mathbb{F}(E) \times_B \mathbb{F}(E') \to \mathbb{F}(E)$ and

$\mathtt{pr}' : \mathbb{F}(E) \times_B \mathbb{F}(E') \to \mathbb{F}(E')$. Let $g : \mathbb{F}(E) \to B$ and $g' : \mathbb{F}(E) \to B$ be the projections. Then we get

$$\begin{aligned}
(\mathtt{pr}, \mathtt{pr}')^*(g, g')^* c(E \oplus E') &= d^* f^* c(E \oplus E') \\
&= d^* \prod_{k=1}^{n+n'} (1 + x_k) \\
&= \prod_{k=1}^{n} (1 + y_k) \cup \prod_{k=1}^{n'} (1 + y'_k) \\
&= \mathtt{pr}^* g^* c(E) \cup \mathtt{pr}'^* g'^* c(E') \\
&= (\mathtt{pr}, \mathtt{pr}')^*(g, g')^* c(E) \cup c(E')
\end{aligned}$$

□

For example, if $E = L \oplus L'$ is a sums of two line bundles, then we have

$$c(E) = 1 + c_1(L) + c_1(L') + c_1(L)c_1(L') \ ,$$

i.e. in particular

$$c_2(E) = c_1(L)c_1(L') \ .$$

Example 6.29. We calculate the higher Chern classes of $T\mathbb{CP}^n$. The exact sequence (45) gives

$$c(T\mathbb{CP}^n) = c((n+1)L^*) = c(L^*)^{n+1} = (1 + c_1)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} c_1^k \ .$$

We read off that

$$c_k(T\mathbb{CP}^n) = \binom{n+1}{k} c_1^k$$

for $k = 1, \ldots, n$. □

Example 6.30. We consider a complex vector bundle $E \to B$ of dimension n.

Lemma 6.31. *If E admits a nowhere vanishing section then $c_n(E) = 0$.*

Proof. If E admits such a section, then we get a decomposition $E = \underline{\mathbb{C}} \oplus E'$, where the trivial summand $\underline{\mathbb{C}}$ is generated by the section. Since $c_1(\underline{\mathbb{C}}) = 0$ we get

$$c(E) = c(\underline{\mathbb{C}})c(E') = c(E') = 1 + \cdots + c_{n-1}(E') \ .$$

$\square$

For example, in view of Example 6.29 the bundle $T\mathbb{CP}^n \to \mathbb{CP}^n$ does not admit any nowhere vanishing section. $\square$

Example 6.32. In this example we provide a formula for the Chern classes of a tensor product $E \otimes F$ of two complex vector bundles of dimensions e and f over B. We work in the polynomial ring $\mathbb{Z}[x_1, \dots, x_e, y_1, \dots, y_f]$. We let $\sigma_i(x) := \sigma_i(x_1, \dots, x_e)$ and $\sigma_i(y) := \sigma_i(y_1, \dots, y_f)$, where σ_i are the elementary symmetric functions. These polynomials generate a polynomial subring

$$\mathbb{Z}[\sigma_1(x), \dots, \sigma_e(x), \sigma_1(y), \dots, \sigma_f(y)] \subseteq \mathbb{Z}[x_1, \dots, x_e, y_1, \dots, y_f] \ .$$

We now observe that

$$u := \prod_{i=1}^{e} \prod_{j=1}^{f} (1 + x_i + y_j) \in \mathbb{Z}[\sigma_1(x), \dots, \sigma_e(x), \sigma_1(y), \dots, \sigma_f(y)] \ .$$

More precisely we write $u = u(\sigma_1(x), \dots, \sigma_e(x), \sigma_1(y), \dots, \sigma_f(y))$.

Lemma 6.33. *We have*

$$c(E \otimes F) = u(c_1(E), \dots, c_n(E), c_1(F), \dots, c_f(F)) \ .$$

Proof. We use the splitting principle. We consider the pull-back diagram

$$\begin{tikzcd} \mathbb{F}(E) \times_B \mathbb{F}(F) \arrow[r, "r"] \arrow[d, "s"] \arrow[dr, "h"] & \mathbb{F}(F) \arrow[d, "q"] \\ \mathbb{F}(E) \arrow[r, "p"] & B \end{tikzcd} \ .$$

The Leray-Hirsch theorem holds for all maps in this diagram. In particular, the pull-back

$$h^* : H_{dR}(B) \to H_{dR}(\mathbb{F}(E) \times_B \mathbb{F}(F))$$

is injective.

We have decompositions in to sums of line bundles

$$p^*E \cong L_1 \oplus \cdots \oplus L_e \ , \quad q^*F \cong H_1 \oplus \cdots \oplus H_f \ .$$

This gives

$$h^*(E \otimes F) = \bigoplus_{i=1}^{e} \bigoplus_{j=1}^{f} s^* L_i \otimes r^* H_j \ .$$

We let $x_i := s^* c_1(L_i)$ and $y_j := r^* c_1(H_j)$. Then

$$h^* c(E \otimes F) = \prod_{j=1}^{e} \prod_{j=1}^{f} (1 + x_i + y_j) = h^* u(c_1(E), \ldots, c_e(E), c_1(F), \ldots, c_f(F)) \ .$$

Since h^* is injective the assertion of the Lemma follows. □

Let us make the formula explicit for two 2-dimensional bundles.

$$\begin{aligned}
& c(E \otimes F) \\
= \ & 1 + \Big(2c_1(E) + 2c_1(F)\Big) \\
& + \Big(3c_1(E)c_1(F) + c_1(E)^2 + c_1(F)^2 + 2c_2(E) + 2c_2(F)\Big) \\
& + \Big(2c_1(E)c_2(F) + 2c_2(E)c_1(F) + c_1(E)^2 c_1(F) + c_1(E)c_1(F)^2 \\
& \qquad + 2c_1(E)c_2(E) + 2c_1(F)c_2(F)\Big) \\
& + \Big(c_2(E)^2 + c_2(F)^2 + c_1(E)^2 c_2(F) + c_2(E)c_1(F)^2 \\
& \qquad + c_1(E)c_2(E)c_1(F) + c_1(E)c_1(F)c_2(F) - c_2(E)c_2(F)\Big) \ .
\end{aligned}$$

Similar formulas exists in general, but they are not easy. In order to deal with tensor products of bundles a better adapted choice of characteristic classes are the components of the **Chern character**, see Subsection 8.3.

□

6.4 Grassmannians

In this section we calculate the de Rham cohomology of the Grassmann manifold $Gr(k, \mathbb{C}^n)$. Let $L \to Gr(k, \mathbb{C}^n)$ be the k-dimensional tautological bundle. It is a subbundle of the n-dimensional trivial bundle $\mathbb{C}^n \times \mathrm{Gr}(k, \mathbb{C}^n) \to \mathrm{Gr}(k, \mathbb{C}^n)$ and we let $L^\perp \to Gr(k, \mathbb{C}^n)$ be the orthogonal complement of L.

We define cohomology classes of $Gr(k,\mathbb{C}^n)$ by $c_i := c_i(L)$, $i=1,\dots,k$ and $d_j = c_j(L^\perp)$ for $j=1,\dots,n-k$. Since $L\oplus L^\perp$ is trivial we have the relations

$$\sum_{i+j=k} c_i d_j = 0\ , \quad k=1,\dots,n\ ,$$

where we set $c_i := 0$ for $i>k$ and $d_j := 0$ for $j>n-k$. In the following Proposition we show that $H_{dR}(Gr(k,\mathbb{C}^n))$ is generated as a ring by these characteristic classes c_i and d_j with exactly these relations.

Proposition 6.34. *We have an isomorphism*

$$H_{dR}(\mathrm{Gr}(k,\mathbb{C}^n)) \cong \frac{\mathbb{R}[c_1,\dots,c_k,d_1,\dots,d_{n-k}]}{(\sum_{i+j=k} c_i d_j = 0\ , k=1,\dots,n)}\ .$$

Proof. We consider the bundles of complete flags $\mathbb{F}(L)\to \mathrm{Gr}(k,\mathbb{C}^n)$ and $\mathbb{F}(L^\perp)\to \mathrm{Gr}(k,\mathbb{C}^n)$ and denote the Chern classes of the canonical bundles by $x_i \in H^2_{dR}(\mathbb{F}(L))$, $i=1,\dots,k$, and by $y_j\in H_{dR}(\mathbb{F}(L^\perp))$, $j=1,\dots,n-k$. We now observe that

$$\mathbb{F}(L)\times_{\mathrm{Gr}(k,\mathbb{C}^n)}\mathbb{F}(L^\perp)\cong \mathbb{F}(\mathbb{C}^n)\ ,$$

and we use the same notation x_i and y_j for the pull-back of these characteristic classes along the projection from the fibre product to its factors. We write $\sigma_i(x,y)$ for the ith elementary symmetric function on the variables x_i and y_j. Then we have the relations

$$\sigma_k(x,y) = \sum_{i+j=k}\sigma_i(x)\sigma_j(y)\ ,\quad k=1,\dots,n\ .$$

We now use that by Corollary 6.27

$$H_{dR}(\mathbb{F}(\mathbb{C}^n)) \cong \frac{\mathbb{R}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]}{(\sigma_1(x,y),\dots,\sigma_n(x,y))}\ .$$

The Leray-Hirsch theorem for the bundle

$$\mathbb{F}(\mathbb{C}^n)\cong \mathbb{F}(L)\times_{\mathrm{Gr}(k,\mathbb{C}^n}\mathbb{F}(L^\perp)\to \mathrm{Gr}(k,\mathbb{C}^n) \tag{50}$$

induces an isomorphism

$$\frac{H_{dR}(Gr(k,\mathbb{C}^n))[x_1,\dots,x_k,y_1,\dots,y_{n-k}]}{((c_i=\sigma_i(x), i=1,\dots,k)\ ,(d_j=\sigma_j(y), j=1,\dots,n-k))} \xrightarrow{\cong} \frac{\mathbb{R}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]}{(\sigma_1(x,y),\dots,\sigma_n(x,y))}\ .$$

If we restrict to the subspace of $H_{dR}(\mathrm{Gr}(k,\mathbb{C}^n))$ generated by the classes c_i and d_j, then we get a map

$$\frac{\frac{\mathbb{R}[c_1,\dots,c_k,d_1,\dots,d_{n-k}]}{(\sum_{i+j=k} c_i d_j = 0, k=1,\dots,n)}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]}{((c_i=\sigma_i(x), i=1,\dots,k)\ ,(d_j=\sigma_j(y), j=1,\dots,n-k))} \to \frac{\mathbb{R}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]}{(\sigma_1(x,y),\dots,\sigma_n(x,y))}$$

which sends c_i to $\sigma_i(x)$ and d_j to $\sigma_j(y)$. By inspection we see that it is an isomorphism. It follows that

$$H_{dR}(\mathrm{Gr}(k,\mathbb{C}^n)) \cong \frac{\mathbb{R}[c_1,\dots,c_k,d_1,\dots,d_{n-k}]}{(\sum_{i+j=k} c_i d_j = 0\ , k=1,\dots,n)}\ .$$

Example 6.35. In this example we consider the Stiefel manifold $V(k,\mathbb{C}^n)$ of k-tuples of orthonormal vectors.

Lemma 6.36. *We have*

$$H^\ell_{dR}(V(k,\mathbb{C}^n)) = 0\ , \quad \ell = 1\dots,2(n-k)\ .$$

Proof. We have a presentation of the Stiefel manifold as a homogeneous space

$$V(k,\mathbb{C}^n) \cong U(n)/U(n-k)\ .$$

Indeed, $U(n)$ acts transitively on k-tuples of orthonormal vectors in $\mathbb{C}^n$, and the stabilizer of the tuple $(e_1,\dots,e_k)$ (the beginning of the standard basis) is the subgroup $U(n-k) \subseteq U(n)$ embedded as lower right block. We consider the LSSS spectral sequence for the bundle $U(n) \to V(k,\mathbb{C}^n)$ with fibre $U(n-k)$. Note that the polynomial generators (see Lemma 6.20)

$$u_{2i-1} \in E_2^{0,2i-1} \cong H^{2i-1}_{dR}(U(n-k))\ , \quad i = 1,\dots,n-k$$

extend to $U(n)$ and are therefore (Corollary 5.16) permanent cycles (i.e. they are annihilated by all higher differentials). The polynomials of their extension to $U(n)$ generate the cohomology of $U(n)$ in degrees $\le 2(n-k)$. For $\ell \in \{1,\dots,2(n-k)\}$ the restriction $H^\ell_{dR}(U(n)) \to H^\ell_{dR}(U(n-k))$ is injective

Assume that $\ell \in \{1,\dots,2(n-k)\}$ is minimal such that there is a non-trivial class $x \in H^\ell_{dR}(V(k,\mathbb{C}^n)) = E_2^{0,\ell}$. Then this class can not be hit by any differential and therefore survives to the E_∞-page. In other words, it lifts non-trivially to $U(n)$. But then its restriction to the fibre $U(n-k)$ would be non-trivial. This is contradicts the observation made in the previous paragraph. $\square$

Example 6.37. Here we generalize Example 6.10 to higher-dimensional bundles.

Let $V \to B$ be a k-dimensional vector bundle. We say that V is **globally generated** if there exists a family of sections $(s_i)_{i=1}^n$ of V such that for every $b \in B$ the collection of values $(s_i(b))_{i=1,\dots,n}$ generates the fibre V_b as a complex vector space. If B is compact, then V is globally generated. Indeed, one can choose for every point $b \in B$ a collection of sections $(s_{i,b})_{i=1,\dots,k}$ of V such that the values $(s_{i,b}(b))_{i=1,\dots,k}$ generate V_b. Then

$$U_b := \{b' \in B \mid V_{b'} = \langle s_{1,b}(b'), \dots, s_{k,b}(b')\rangle\}$$

is an open neighborhood of b. By the compactness of B there is a finite subset $A \subseteq B$ such that $\bigcup_{a\in A} U_a = B$. Then the finite collection $(s_{i,a})_{i\in\{1,\dots,k\},a\in A}$ of sections globally generates V.

Note that one can show more generally, that a vector bundle over a connected manifold B is always globally generated, independently of compactness of B.

Assume now that $(s_i)_{i=1}^n$ generates the bundle $V \to B$ globally. Such a collection of sections gives rise to a map $s : B \to Gr(k, \mathbb{C}^n)$ as follows. For $b \in B$ we define a surjective map

$$f(b)^* : \mathbb{C}^{n,*} \to V_b \ , \quad x \mapsto \sum_{i=1}^n x^i s_i(b) \ .$$

Its adjoint is the injective map

$$f(b) : V_b^* \to \mathbb{C}^n \ ,$$

and we set

$$s(b) := \mathtt{im}(f(b)) \subseteq \mathbb{C}^n \ .$$

We have a pull-back diagram

$$\begin{tikzcd} V^* \arrow[r, "s"] \arrow[d] & L \arrow[d] \\ B \arrow[r, "s"] & Gr(k, \mathbb{C}^n) \end{tikzcd} \quad ,$$

or equivalently, an isomorphism $V \cong s^*L^*$. We conclude that

$$c_i(V) = (-1)^i s^* c_i(L) \ , \quad i = 1, \dots, k \ .$$

$\square$

Example 6.38. We consider isomorphism of complex bundles

$$T\mathrm{Gr}(k, \mathbb{C}^n) \to \mathtt{Hom}(L, \mathbb{C}^n/L) \ , \quad T_x Gr(k, \mathbb{C}^n) \ni X \mapsto \Big([\phi] \mapsto [\nabla^{triv}_X \tilde{\phi}(x)]\Big) \ ,$$

where $\tilde{\phi}$ is a section of L such that $\tilde{\phi}(x) = \phi$, or

$$T\mathrm{Gr}(k, \mathbb{C}^n) \cong \mathtt{Hom}(L, \mathbb{C}^n/L) \cong L^* \otimes \mathbb{C}^n/L \ .$$

This allows to calculate, but the answer is complicated.

□

7 Geometric applications of cohomology - degree and intersection numbers

7.1 The mapping degree

We consider two closed connected oriented manifolds M and N of the same dimension k and a smooth map $f : M \to N$. Note that $H^k_{dR}(M)$ and $H^k_{dR}(N)$ are one-dimensional real vector spaces. We fix basis vectors by choosing the normalized volume classes $[\mathrm{vol}_M] \in H^k_{dR}(M)$ and $[\mathrm{vol}_N] \in H^k_{dR}(N)$ such that $\int_M [\mathrm{vol}_M] = 1$ and $\int_N [\mathrm{vol}_N] = 1$.

Definition 7.1. *The* ***mapping degree*** $\deg(f) \in \mathbb{R}$ *of f is defined such that*

$$f^*[\mathrm{vol}_N] = \deg(f)[\mathrm{vol}_M] \ ,$$

or equivalently, by

$$\deg(f) := \int_M f^*[\mathrm{vol}_N] \ .$$

It is clear that $\deg(f)$ only depends on the homotopy classes of f. Moreover, it changes its sign if the orientation of exactly one of M or N is flipped.

Example 7.2. We consider a closed curve $f : S^1 \to \mathbb{C} \setminus \{0\}$. Such a map has a winding number n_f. It can be calculated by

$$n_f = \frac{1}{2\pi i} \int_{S^1} \frac{df}{f} \ .$$

By normalizing we get a map $f_1 := f/|f| : S^1 \to S^1$. We claim that $\deg(f_1) = n_f$.

We parametrize S^1 by $t \in [0,1]$ using $t \mapsto e^{2\pi i t}$. Then dt is the normalized volume form in this chart. We have

$$\begin{aligned}
\deg(f_1) &= \int_0^1 f_1^* dt \\
&= \frac{1}{2\pi i} \int_0^1 \ln\Big(\frac{f(e^{2\pi i t})}{|f(e^{2\pi i t})|}\Big)' dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{2} \left(\ln(f(e^{2\pi i t})) - \ln(\bar{f}(e^{2\pi i t}))\right)' dt \\
&= \frac{1}{2} \int_0^1 \Big(\frac{f'(e^{2\pi i t})}{f(e^{2\pi i t})} - \frac{\bar{f}'(e^{2\pi i t})}{\bar{f}(e^{2\pi i t})}\Big) dt \\
&= \int_0^1 \frac{f'(e^{2\pi i t})}{f(e^{2\pi i t})} dt \\
&= \frac{1}{2\pi i} \int_{S^1} \frac{df}{f} \\
&= n_f \ .
\end{aligned}$$

□

Recall that a point $m \in M$ is called **regular** if $df(m) : T_m M \to T_{f(m)} N$ is surjective (i.e. an isomorphism in our case since both manifolds have the same dimension).

Definition 7.3. *We define the **sign** of f at a regular point $m \in M$ of f by*

$$s_f(m) := \begin{cases} 1 & df(m) \text{ preserves the orientation} \\ -1 & df(m) \text{ does not preserve the orientation} \end{cases} .$$

A point $n \in N$ is called **a regular value** of f, if every point $m \in f^{-1}(\{n\})$ is a regular point of f. It is good to know the following theorem from differential topology:

Theorem 7.4 (Sard)**.** *The set of regular values of a smooth map has full Lebesgue measure. In particular it is dense.*

Theorem 7.5. *If $n \in N$ is a regular value of f, then*

$$\deg(f) = \sum_{m \in f^{-1}(\{n\})} s_f(m) \ .$$

Proof. Every $m \in f^{-1}(\{n\})$ admits an open neighbourhood U_m such that $f_{|U_m} : U_m \to f(U_m)$ is a diffeomorphism of U_m onto an open neighbourhood $f(U_m)$ of n. In particular, for $m \in f^{-1}(\{n\})$ we have $f^{-1}(\{n\}) \cap U_m) = \{m\}$. Consequently, the preimage $f^{-1}(\{n\})$ of the regular point $n \in N$ is discrete. Since f is continuous, this preimage is also closed. Since M is compact we see that $f^{-1}(\{n\})$ is finite. Therefore we can find a closed neighbourhood $B \subseteq N$ of n which is diffeomorphic to a ball with smooth boundary $S := \partial B$ such that $B \subset f(U_m)$ for all $m \in f^{-1}(\{n\})$. We consider the end of the long exact sequence of the pair $(N \setminus \mathrm{Int}(B), S)$.

$$\cdots \to H^{k-1}_{dR}(S) \stackrel{\partial}{\to} H^k_{dR}(N \setminus \mathrm{Int}(B), S) \to H^k_{dR}(N \setminus \mathrm{Int}(B)) \to 0 \ .$$

By Example 1.45 we know that ∂ is surjective and therefore $H^k_{dR}(N \setminus \mathrm{Int}(B)) = 0$. We can find a form $\alpha \in \Omega^{k-1}(N \setminus \mathrm{Int}(B))$ such that $d\alpha = \mathrm{vol}_{N|N\setminus\mathrm{Int}(B)}$. By Stoke's theorem we have

$$1 = \int_N \mathrm{vol}_N = \int_{N\setminus\mathrm{Int}(B)} \mathrm{vol}_N + \int_B \mathrm{vol}_N = \int_{N\setminus\mathrm{Int}(B)} d\alpha + \int_B \mathrm{vol}_N = \int_S \alpha + \int_B \mathrm{vol}_N \ . \tag{51}$$

Here we orient S as the boundary of $N \setminus \mathrm{Int}(B)$.

We have

$$\int_M f^*\mathrm{vol}_N = \int_{M\setminus f^{-1}(\mathrm{Int}(B))} f^*\mathrm{vol}_N + \int_{f^{-1}(B)} f^*\mathrm{vol}_N \ .$$

By Stoke's theorem

$$\int_{M\setminus f^{-1}(\mathrm{Int}(B))} f^*\mathrm{vol}_N = \int_{M\setminus f^{-1}(\mathrm{Int}(B))} f^* d\alpha = \int_{\partial f^{-1}(\mathrm{Int}(B))} f^*\alpha = \sum_{m\in f^{-1}(\{n\})} \int_{f^{-1}(S)\cap U_m} f^*\alpha \ .$$

Furthermore,

$$\int_{f^{-1}(B)} f^*\mathrm{vol}_N = \sum_{m\in f^{-1}(\{n\})} \int_{f^{-1}(B)\cap U_m} f^*\mathrm{vol}_N \ .$$

We thus have

$$\begin{aligned}
\deg(f) &= \int_M f^*\mathrm{vol}_N \\
&= \sum_{m\in f^{-1}(\{n\})} \left(\int_{f^{-1}(S)\cap U_m} f^*\alpha + \int_{f^{-1}(B)\cap U_m} f^*\mathrm{vol}_N\right) \\
&= \sum_{m\in f^{-1}(\{n\})} s_f(m)\left(\int_S \alpha + \int_B \mathrm{vol}_N\right) \\
&\overset{(51)}{=} \sum_{m\in f^{-1}(\{n\})} s_f(m)\ .
\end{aligned}$$

□

Observe that Theorem 7.5 implies that the mapping degree $\deg(f)$ is an integer.

Example 7.6. Let G be a finite group acting on an oriented closed connected manifold in an orientation preserving way. Then we can consider the projection $f : M \to M/G$. In this case every point $m \in M$ is regular and $s_f(m) = 1$ if we equip M/G with the induced orientation. We get $\deg(f) = \sharp G$.

Let $L \subset L'$ be two lattices of full rank in $\mathbb{R}^n$. Then we have a map of tori $\mathbb{R}^n/L \to \mathbb{R}^n/L'$ which has degree $[L' : L] = \sharp(L'/L)$.

The mapping degree of the projection $S^{2n-1} \to L(p,q)$ is p, see (8) for notation.

□

Example 7.7. Every closed oriented manifold M of dimension n admits a map $f : M \to S^n$ of degree 1. Let $m \in M$ and $\phi : U \to \mathbb{R}^n$ be an orientation preserving chart at m. Let $\chi \in C_c^\infty(U)$ be such that $\chi(m) \equiv 1$ near m and $0 \le \chi < 1$ otherwise. We identify $S^n \cong \mathbb{R}^n \cup \{\infty\}$ using the coordinate $\frac{x}{\|x\|^2}$ in a neighbourhood of ∞. We define f by

$$f(m) := \begin{cases} \frac{\phi(m)}{\chi(m)} & m \in U \\ \infty & m \notin U \end{cases} .$$

Indeed,

$$m \to \frac{f(m)}{\|f(m)\|^2} = \frac{\phi(m)\chi(m)}{\|\phi(m)\|^2}$$

extends smoothly by zero from $U \setminus \{m\}$ to $M \setminus \{m\}$. Note that $f^{-1}(\{0\}) = \{m\}$ is a regular point and $s_f(m) = 1$. Hence $\deg(f) = 1$.

Let $k \in \mathbb{Z}$. Then we can perform this construction near a collection of k points of M and also use orientation reversing charts if $k < 0$. Consequently, there exists maps of degree k from M to S^n for every $k \in \mathbb{Z}$.

$\square$

Example 7.8. Here is a typical application of the mapping degree as a tool for showing that an equation has a solutions. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map which is the identity near ∞.

Corollary 7.9. *For every $b \in \mathbb{R}^n$ the equation*

$$f(x) = b \ , \quad x \in \mathbb{R}^n$$

has a solution.

Proof. We identify $\mathbb{R}^n$ with $S^n \setminus \{N\}$ (the complement of the north pole) as usual. Then f uniquely extends to a smooth map $\tilde{f} : S^n \to S^n$ which is regular at N. Since $f^{-1}(\{N\}) = \{N\}$ we see that $|\deg(f)| = 1$.

Assume by contradiction that $b \in \mathbb{R}^n$ is such that $f^{-1}(\{b\}) = \emptyset$. Since S^n is compact the image of f is closed. Hence $f^{-1}(\{b'\}) = \emptyset$ for all b' in a neighbourhood of b. In particular, by Sard's theorem 7.4 we can find a regular value b' with empty preimage and conclude that $\deg(f) = 0$. A contradiction.

$\square$

Example 7.10. Let again M be a closed oriented and connected manifold of dimension n. In general there is no map of non-vanishing degree from S^n to M.

Assume that there is $\ell \in \{1, \ldots, n-1\}$ such that $H^\ell_{dR}(M) \neq 0$. Let $x \in H^\ell_{dR}(M)$ be a non-vanishing element. By Poincaré duality there exists $y \in H^{n-\ell}_{dR}(M)$ such that $\int_M x \cup y = 1$. Hence $x \cup y = [\mathrm{vol}_M]$. Let $f : S^n \to M$ be any map. It follows

$$\deg(f) = \int_{S^n} f^*[\mathrm{vol}_M] = \int_{S^n} f^*x \cup f^*y = 0$$

since $f^*x \in H^\ell_{dR}(S^n) = 0$.

For example, for $n \geq 2$ the degree of every map $S^n \to T^n$ or $S^{2n} \to \mathbb{CP}^n$ vanishes.

$\square$

Example 7.11. Let $n \in \mathbb{N}$, $n \geq 2$ and $f : S^{4n} \to \mathbb{HP}^n$ a smooth map. Then for every regular value $x \in \mathbb{HP}^n$ the number $\sharp f^{-1}(\{x\})$ is even. $\square$

Example 7.12. Let $f : M \to N$ be a map between closed connected oriented manifolds of the same dimension. Then$f^* : H^*_{dR}(N) \to H^*_{dR}(M)$ is injective if and only if $\deg(f) \neq 0$. $\square$

Example 7.13. For a product of maps we have $\deg(g \times f) = \deg(g)\deg(f)$. $\square$

Example 7.14. Let $p \in \mathbb{C}[z]$ be a polynomial of degree n. We can form the homogeneous polynomial $z_0^n p(\frac{z_1}{z_0}) \in \mathbb{C}[z_0, z_1]$ of degree n. The latter can be considered as a map

$$f : \mathbb{CP}^1 \to \mathbb{CP}^1 \ , \quad [z_0 : z_1] \mapsto [z_0^n : z_0^n p(\frac{z_1}{z_0})] \ .$$

We first calculate the degree of this map.

We consider L^{taut} as a subbundle of the trivial 2-dimensional bundle $\mathbb{CP}^1 \times \mathbb{C}^2$. In particular, for $[z_0 : z_1] \in \mathbb{CP}^1$ we have $(z_0, z_1) \in L^{taut}_{[z_0,z_1]}$.

We define sections $s_0, s_1 \in \Gamma(\mathbb{CP}^1, L^{taut,*})$ given by $s_i([z_0 : z_1])(z_0, z_1) = z_i$ for $i = 0, 1$. Then we define the sections s_0^n and $s_0^n p(\frac{s_1}{s_0})$ of $(L^{taut,*})^n$. These sections do not vanish simultaneously. By Lemma 6.10 this pair of sections gives rise to the map f and we have $f^*L^{taut,*} = (L^{taut,*})^n$. We conclude that $f^*c_1 = nc_1$ and consequently $\deg(f) = n$.

We consider a point $x \in \mathbb{C}$ such that $z \mapsto p(z) - x$ has only simple zeros. If y is such a zero, then $df([1 : y]) = dp(y)$ is an isomorphism. We have $f^{-1}(\{[1 : x]\}) \cong \{p = x\}$. Since f is holomorphic df preserves the orientation at every regular point $m \in \mathbb{CP}^1$. Consequently, $s_f(m) = 1$. We conclude that

$$\sharp\{p = x\} = \sharp\{f = [1 : x]\} = \deg(f) = n$$

as expected since the polynomial $p(z) - x \in \mathbb{C}[z]$ of degree n has exactly n zeros if they are all simple.

7.2 Integration over the fibre and the edge homomorphism

We consider a locally trivial fibre bundle $f : E \to B$ with closed fibre F of dimension n which is fibrewise oriented. By definition, a **fibrewise orientation** is an orientation of the vertical tangent bundle $T^v f = \mathtt{ker}(df)$. For every $b \in B$ we therefore have $T^v f_{|E_b} \cong TE_b$ and an induced orientation of the fibre E_b of the bundle over b. In this subsection we construct a natural map

$$\int_{E/B} : H^*_{dR}(E) \to H^{*-n}_{dR}(B)$$

called **integration over the fibre**. It generalizes the integration map

$$\int_F : H^n_{dR}(F) \to \mathbb{R} \ .$$

In fact, if F is a closed oriented manifold, then we can consider the bundle $F \to *$ which has a fibrewise orientation. With the identification $\mathbb{R} \cong H^0_{dR}(*)$ we will have the equality $\int_{F/*} = \int_F$.

Integration over the fibre is induced by an integration map on the level of de Rham complexes which we describe first. The integral over the fibre of a form $\omega \in \Omega^k(E)$ is the form

$$\int_{E/B} \omega \in \Omega^{k-n}(B)$$

defined as follows. Let $b \in B$ and $X_1, \ldots, X_{k-n}$ be vectors in T_bB. We choose lifts $\tilde{X}_i \in \Gamma(E_b, TE)$ such that $df(e)(\tilde{X}_i(e)) = X_i$ for all $i = 1, \ldots, k-n$ and $e \in E_b$. Then

$$i_{\tilde{X}_{k-n}} \ldots i_{\tilde{X}_1} \omega_{|E_b} \in \Omega^n(E_b) \ .$$

Note that this form does not depend on the choice of the lifts $\tilde{X}_i$. We define the evaluation of $\int_{E/B} \omega$ at b and on the vectors $X_1, \ldots, X_{k-n}$ by

$$(\int_{E/B} \omega)(b)(X_1, \ldots, X_{k-n}) := \int_{E_b} i_{\tilde{X}_{k-n}} \ldots i_{\tilde{X}_1} \omega_{|E_b} \ .$$

In order to define the integral we use the orientation of E_b induced by the fibrewise orientation. Using local trivializations one can check that this construction produces a smooth form. More details follow below. We get a map

$$\int_{E/B} : \Omega^k(E) \to \Omega^{k-n}(B) \ .$$

We now show compatibility with the differential.

In order to incorporate the homotopy formula and a fibrewise Stokes theorem into the story we consider generally a bundle of compact manifolds with boundary. The typical fibre F is then compact, oriented, and has a boundary ∂F. We let $\partial E \to B$ be the corresponding bundle with fibre ∂F and induced fibrewise orientation.

Lemma 7.15. *For $\omega \in \Omega(E)$ we have the identity*

$$(-1)^n d \int_{E/B} \omega + \int_{\partial E/B} \omega = \int_{E/B} d\omega \ . \tag{52}$$

Proof. Since this equality can be checked locally in B we can assume that $E = F \times \mathbb{R}^\ell$ and $B = \mathbb{R}^\ell$. We have $\omega = \sum_{p+q=n} \omega^{p,q}$ with $\omega^{p,q} \in \Omega^{p,q}(F \times \mathbb{R}^\ell)$ (see (28) for notation), and we can write $\omega^{p,q} = \sum_{J \in I^q} \omega^{p,q}_J \wedge dx^J$, where $I^q := \{(1 \le i_1 < \cdots < i_q \le \ell)\}$ is the index set for the standard basis of $\Lambda^q \mathbb{R}^{\ell,*}$ and $\omega^{p,q}_J$ is a section of $\mathtt{pr}_F^* \Lambda^p T^* F \to F \times \mathbb{R}^\ell$ considered as a subbundle of $\Lambda^p T^*(F \times \mathbb{R}^\ell)$, i.e $\omega^{p,q}_J \in \Omega^{p,0}(F \times \mathbb{R}^\ell)$ in the notation of (28). Then

$$\int_{E/B} \omega = \sum_{J \in I^{k-n}} \int_F \omega^{n,k-n}_J dx^J \ .$$

We calculate

$$\begin{aligned} d \int_{E/B} \omega &= d \sum_{J \in I^{k-n}} \int_F \omega^{n,k-n}_J dx^J \\ &= \sum_{i=1}^{\ell} \partial_i \sum_{J \in I^{k-n}} \int_F \omega^{n,k-n}_J dx^i \wedge dx^J \\ &= \sum_{i=1}^{\ell} \sum_{J \in I^{k-n}} \int_F \partial_i \omega^{n,k-n}_J dx^i \wedge dx^J \end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{E/B} d\omega &= \int_{E/B} (d^F + d^B)\omega \\
&= (-1)^n \sum_{J\in I^{n-k}} \int_F \partial_i \omega_J^{n,k-n} dx^i \wedge dx^J \\
&\quad + \sum_{J\in I^{n-k+1}} \int_{E/B} d^F \omega_J^{n-1,k-n} dx^j \\
&= (-1)^n \sum_{J\in I^{n-k}} \int_F \partial_i \omega_J^{n,k-n} dx^i \wedge dx^J \\
&\quad + \sum_{J\in I^{n-k+1}} \int_{\partial F} \omega_J^{n-1,k-n} dx^J \\
&= (-1)^n d \int_{E/B} \omega + \int_{\partial F} \omega \ .
\end{aligned}$$

□

Example 7.16. The formula (52) can be considered as a generalization of the homotopy formula (1). In fact, if $\omega \in \Omega([0,1]\times M)$, then by specializing (52) to the bundle $[0,1]\times M \to M$ we get

$$\omega_{|\{1\}\times M} - \omega_{|\{0\}\times M} = \int_{I\times M/M} d\omega + d\int_{I\times M/M} \omega \ .$$

Here we have used that $\int_{\partial([0,1]\times M)/M} \omega = \omega_{|\{1\}\times M} - \omega_{|\{0\}\times M}$. □

Corollary 7.17. *If F is closed, then we have a map of complexes*

$$\int_{E/B} : \Omega(E) \to \Omega(B)[n] \ .$$

We get an induced integration map in cohomology

$$\int_{E/B} : H^k_{dR}(E) \to H^{k-n}_{dR}(B)$$

for all $k \in \mathbb{Z}$.

Proof. Note that the sign $(-1)^n$ in the formula (52) takes care of the same sign in the definition of the differential of the shifted complex $\Omega(B)[n]$, see (4). $\square$

Lemma 7.18 (Naturality of integration)**.** *For a pull-back diagram*

$$\begin{tikzcd} E' \arrow[r, "h"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ B' \arrow[r, "g"] & E \end{tikzcd}$$

we have the identity

$$g^* \circ \int_{E/B} = \int_{E'/B'} \circ h^*$$

of maps on forms as well as in cohomology.

Proof. Immediate from the definitions. $\square$

Lemma 7.19 (Projection formula)**.** *We consider a fibre bundle $f : E \to B$ with closed oriented n-dimensional fibres. For $x \in H_{dR}(E)$ and $y \in H_{dR}(B)$ we have*

$$\int_{E/B} (x \cup f^* y) = (\int_{E/B} x) \cup y \ .$$

Proof. This follows immediately from the corresponding identity on the level of forms. $\square$

Example 7.20. We consider an iterated bundle $E \to G \to B$ with closed fibres. The choice of fibrewise orientations for two of the three bundles $E \to G$, $G \to B$ and $E \to B$ induces an orientation on the third such that

$$\int_{E/B} = \int_{G/B} \circ \int_{E/G}$$

holds. $\square$

Note that $\int_{E/B}$ annihilates $F^1 H_{dR}(E)$ and therefore induces a map

$$\mathrm{Gr}^0 H^k_{dR}(E) \to H^{k-n}_{dR}(B) \ .$$

We consider the LSSS of $E \to B$. The orientation of the fibres induces a trivialization of $\mathcal{H}^n(E/B)$. We get

$$E_2^{p,n} \cong H^p_{dR}(B) \ .$$

Here is a picture for $n = 3$.

$$\begin{array}{c|cccccc}
3 & H^0_{dR}(B) & H^1_{dR}(B) & H^2_{dR}(B) & H^3_{dR}(B) & H^4_{dR}(B) & H^5_{dR}(B) \\
2 & E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & E_2^{4,2} & E_2^{5,2} \\
1 & E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & E_2^{5,1} \\
0 & H^0_{dR}(B) & H^1_{dR}(B) & H^2_{dR}(B) & H^3_{dR}(B) & H^4_{dR}(B) & H^5_{dR}(B) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}$$

The line marks the entries contributing to $H^4_{dR}(E)$.

A class $x \in \mathrm{Gr}^0(H^k_{dR}(E))$ is detected by an element $\xi \in E_\infty^{k-n,n} \subseteq E_2^{k-n,n}$ (the encircled entry in the picture above). Under the identification $E_2^{k-n,n} \cong H^{n-k}_{dR}(B)$ we have $\xi = \int_{E/B} x$. In order to see this let ω be a representative for x. This form represents a section $\xi \in \Omega^k(B, \mathcal{H}^n(E/B)) = E_1^{k-n,n}$. Under the trivialization $\Omega^{k-n}(B, \mathcal{H}^n(E/B)) \cong \Omega^{n-k}(B)$ it corresponds to $\int_{E/B} \omega$.

Example 7.21. We consider the Hopf bundle $h : S^{2n-1} \to \mathbb{CP}^{n-1}$. We know that the normalized volume class $[\mathrm{vol}_{S^{2n-1}}] \in H^{2n-1}_{dR}(S^{2n-1})$ is detected by the element $\lambda u \cup c_1 \in E_2^{2n-2,1}$ for a suitable non-vanishing $\lambda \in \mathbb{R}$. We determine the factor λ. The form (compare with (43))

$$\theta := (\frac{1}{2\pi i} \bar{z}^t dz)_{|S^{2n-1}} \in \Omega^1(S^{2n-1})$$

represents $u \in E_2^{0,1}$.

We can choose the representative $\omega \in \Omega^2(\mathbb{CP}^{n-1})$ of $c_1 \in H^2_{dR}(\mathbb{CP}^{n-1}) \cong E_2^{2,0}$ such that $d\theta = h^*\omega$. By assumption the product $\lambda\theta \wedge h^*\omega^{n-1}$ represents $[\mathrm{vol}_{S^{2n-1}}]$. Note that $\int_{S^{2n-1}/\mathbb{CP}^{n-1}} \theta = 1$. Hence, by the projection formula 7.19

$$1 = \int_{S^{2n-1}} \lambda\theta \wedge h^*\omega^{n-1} = \int_{S^{2n-1}/\mathbb{CP}^{n-1}} \lambda\theta \wedge \int_{\mathbb{CP}^{n-1}} \omega^{n-1} \stackrel{6.8}{=} \lambda \ .$$

□

7.3 Transgression

Let $f : E \to B$ be a fibre bundle with fibre F over a connected base B. We consider a positive integer k and a cohomology class $x \in H^k_{dR}(B)$ such that $f^*x = 0$. Let $x = [\omega]$ for a closed form $\omega \in \Omega^k(B)$. Then there exists $\alpha \in \Omega^{k-1}(E)$ such that $d\alpha = f^*\omega$. Since k is positve we have $(d\alpha)_{|F} = (f^*\omega)_{|F} = 0$.

Lemma 7.22. *The class $[\alpha_{|F}] \in H^{k-1}_{dR}(F)$ only depends on x up to the image of the restriction $H^{k-1}_{dR}(E) \to H^{k-1}_{dR}(F)$.*

Proof. The form α is determined by ω up to closed forms in $\Omega^k(E)$. Therefore $[\alpha_{|F}] \in H^{k-1}_{dR}(F)$ is determined by ω to the image of the restriction map as asserted. If we choose a different representative ω', then $\omega' - \omega = d\beta$ for some $\beta \in \Omega^{k-1}(B)$. We can take $\alpha' = \alpha + f^*\beta$. If $k \geq 2$, then $(f^*\beta)_{|F} = 0$ and therefore $[\alpha_{|F}] = [\alpha'_{|F}]$ in $H^{k-1}_{dR}(F)$. If $k = 1$, then $[\beta_{|F}] \in \mathtt{im}(H^{k-1}_{dR}(E) \to H^{k-1}(F))$. Indeed, $\beta_{|F}$ is a constant and can be extended as a constant to E. □

In the following definition α is as above.

Definition 7.23. *Let $E \to B$ be a fibre bundle with fibre F over a connected base B, k a positive integer, and $x \in \mathtt{ker}(f^* : H^k_{dR}(B) \to H^k_{dR}(E))$. The class*

$$T(x) := [\alpha_{|F}] \in \frac{H^{k-1}_{dR}(F)}{\mathtt{im}(H^{k-1}_{dR}(E) \to H^{k-1}_{dR}(F))}$$

*is called the **transgression** of x.*

We now consider the LSSS for the bundle $f : E \to B$. The transgression inverts, in some sense, the differential $d_k : E^{0,k-1}_k \to E^{k,0}_k$. Indeed, if $x \in \mathtt{ker}(f^*)$ and $y \in T(x) \subseteq E^{k-1,0}_2$, then $d_\ell y = 0$ for all $\ell \in \mathbb{N}$ with $2 \leq \ell \leq k-1$, and $d_k y = [x]$ in $E^{k,0}_k$, where $[x]$ denotes the class of x under the quotient map $E^{0,k}_2 \to E^{0,k}_k$.

The transgression annihilates decomposable elements.

Lemma 7.24. *If $x = a \cup b$ for classes of non-zero degree and $a \in \mathtt{ker}(f^*)$, then $0 \in T(x)$.*

Proof. We can choose $\alpha := \gamma \cup f^*\beta$, where β represents b and $d\gamma = f^*\omega$ for a representative ω of a. But $(f^*\beta)_{|F} = 0$ and therefore $\alpha_{|F} = 0$. $\square$

Example 7.25. We consider the bundle $S^{2n+1} \to \mathbb{CP}^n$. Then $T(c_1) = u$, the normalized volume form of the fibre. Since $H^1_{dR}(S^{2n-1}) = 0$ the transgression is unique. $\square$

Example 7.26. We consider the bundle $f : V(k, \mathbb{C}^n) \to Gr(k, \mathbb{C}^n)$ with fibre $U(k)$ and let $L \to Gr(k, \mathbb{C}^n)$ be the tautological bundle. We assume that $i \in \mathbb{N}$ is such that $i \le 2(n-k)$. Recall the calculation of the cohomology of $U(k)$ in 6.20.

Lemma 7.27. *The transgressive classes in $H^{2i-1}_{dR}(U(k)) \cong {}^fE_2^{0,2i-1}$ are multiples of the generator u_{2i-1}.*

Proof. In order to see this let $x \in H^{2i-1}_{dR}(U(k))$ be a polynomial in the generators $u_1, \dots, u_{2i-3}$. We must show that x is not transgressive.

Assume by contradiction that x is transgressive. We consider the standard embedding $\mathbb{C}^{n-(k-i)} \to \mathbb{C}^n$. We get a bundle map

$$\begin{tikzcd} V(i, \mathbb{C}^{n-(k-i)}) \arrow[r] \arrow[d, "g"] & V(k, \mathbb{C}^n) \arrow[d, "f"] \\ Gr(i, \mathbb{C}^{n-(k-i)}) \arrow[r, "h"] & Gr(k, \mathbb{C}^n) \end{tikzcd} ,$$

where $V(i, \mathbb{C}^{n-(k-i)}) \to V(k, \mathbb{C}^n)$ maps the orthonormal system $(v_1, \dots, v_i)$ in $\mathbb{C}^{n-(k-i)}$ to the system $(v_1, \dots, v_i, e_{n-(k-i)+1}, \dots, e_n)$ in $\mathbb{C}^n$ obtained by adding the last $k-i$ basis vectors. On the level of fibres this bundle map is given by standard the embedding $U(i) \to U(k)$ up to conjugation. We have an associated map of LSSS'es. If $x \in H^{2i-1}_{dR}(U(k)) = {}^fE_2^{0,2i-1}$ is a polynomial in the generators $u_1, \dots, u_{2i-3}$ and transgressive, then its restriction $E(h)(x) \in {}^gE_2^{0,2i-1} \cong H^{2i-1}_{dR}(U(i))$ would be non-trivial (by Lemma 6.20) and transgressive, too. By Lemma 5.16 it would extend to a non-trivial class in $H^{2i-1}_{dR}(V(i, \mathbb{C}^{n-(k-i)}))$. Since $2i - 1 \le 2(n - (k-i) - i) = 2(n-k)$ this is impossible in view of Lemma 6.36. $\square$

The transgression

$$T(c_i(L)) \in H^{2i-1}_{dR}(U(k))$$

is the well-defined since $H^{2i-1}_{dR}(V(k,\mathbb{C}^n)) = 0$ by Lemma 6.36. This class is transgressive. By Lemma 7.27 there exists $\lambda_i \in \mathbb{R}$ such that

$$d_{2i}(\lambda_i u_{2i-1}) = [c_i(L)] \in E^{0,2i}_{2i} \ , \quad T(c_i) = \lambda_i u_{2i-1} \ .$$

Of course we have $\lambda_1 = 1$. It seems to be complicated to determine the numbers λ_i for $i \geq 2$ without further theory.

□

7.4 The Thom class of a sphere bundle and Poincaré-Hopf

We consider a fibre bundle $f : E \to B$ with fibre S^{n-1} over a connected base B. A trivialization of $\mathcal{H}^{n-1}(E/B) \to B$ is called **orientation**. Equivalenty one can give an orientation as a fibrewise orientation as defined in Subsection 7.2. We refer to Subsection 5.2 for spectral sequence calculations in this case.

We have a class

$$u \in E^{0,n-1}_n = E^{0,n-1}_2 \cong H^{n-1}_{dR}(S^{n-1})$$

which we normalize such that $\int_{S^{n-1}} u = 1$.

Definition 7.28. *We define the* ***Euler class*** *of the oriented sphere bundle by*

$$\chi(E) := d_n(u) \in E^{n,0}_n = E^{n,0}_2 \cong H^n_{dR}(B) \ .$$

Note that $\chi(E)$ changes its sign if we switch the orientation of the sphere bundle. Sometimes we write $\chi(E \to B)$ in order to indicate that the Euler class is an invariant of the bundle.

Remark 7.29. The Euler class is a characteristic class for oriented sphere bundles. If

$$\begin{tikzcd} g^*E \arrow[r] \arrow[d, "g^*f"] & E \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

is a pull-back diagram of bundles with fibre S^{n-1} and the orientation of g^*f is induced from the orientation of f, then we have the relation

$$g^*\chi(E) = \chi(g^*E) \ .$$

This immediately follows from the functoriality of the LSSS shown in Lemma 5.9.□

The Gysin sequence has the form

$$\dots \stackrel{\dots\cup\chi(E)}{\to} H^k_{dR}(B) \stackrel{f}{\to} H^k_{dR}(E) \stackrel{\sigma}{\to} H^{k-n+1}_{dR}(B) \stackrel{\dots\cup\chi(E)}{\to} H^{k+1}_{dR}(B) \to \dots$$

In particular we have $f^*\chi(E) = 0$.

We have two cases with different behaviour:

1. If $\chi(E) \neq 0$, then $\dots \cup \chi(E) : H^0_{dR}(B) \to H^n_{dR}(B)$ is injective. Hence

   $$H^{n-1}_{dR}(E) \cong \mathtt{im}(f^* : H^{n-1}_{dR}(B) \to H^{n-1}_{dR}(E)) \ .$$

   The restriction $H^{n-1}_{dR}(E) \to H^{n-1}_{dR}(S^{n-1})$ therefore vanishes. The transgression

   $$T : \mathtt{ker}(f^* : H^n_{dR}(B) \to H^n_{dR}(E)) \to H^{n-1}_{dR}(S^{n-1})$$

   is well-defined. We have $T(\chi(E)) = u$.

2. If $\chi(E) = 0$, then there exists a class $\mathtt{Th}(E) \in H^{n-1}_{dR}(E)$ such that $\mathtt{Th}(E)_{|S^{n-1}} = u$. Such a class is called a **Thom class** of the sphere bundle. Vice versa, if the sphere bundle bundle admits a Thom class, then $\chi(E) = 0$. See Corollary 5.16. In this case, by the Leray-Hirsch theorem, $H_{dR}(E)$ is a free module over $H_{dR}(B)$ with generators 1 and $\mathtt{Th}(E)$. Note that $\int_{E/B} \mathtt{Th}(E) = 1$.

Example 7.30. We consider the S^1-bundle $S^{2n+1} \to \mathbb{CP}^n$. In this case

$$\chi(S^{2n+1} \to \mathbb{CP}^n) = c_1 \ .$$

This class does not vanish. Hence the S^1-bundle $S^{2n+1} \to \mathbb{CP}^n$ does not admit a Thom class. □

Example 7.31. We consider the trivial bundle $S^{n-1} \times B \to B$. In this case the LSSS degenerates at the second term and $\chi(S^{n-1} \times B \to B) = 0$. A Thom class is given by $[\mathrm{vol}_{S^{n-1}}] \times 1 \in H^{n-1}_{dR}(S^{n-1} \times B)$. The Leray-Hirsch theorem is equivalent to the Künneth theorem.

If $E \to B$ is a oriented bundle with fibre S^{n-1} and $\chi(E \to B) \neq 0$, then the bundle is not trivial. □

We now consider a real n-dimensional vector bundle $V \to B$. We choose a metric so that we can form the unit sphere bundle $S(V) \to B$. The vector bundle V is oriented if and only if the sphere $S(V) \to B$ is oriented.

Definition 7.32. *We define the Euler class of the oriented vector bundle V by $\chi(V) := -\chi(S(V)) \in H^n_{dR}(B)$.*

Exercise: Show that $\chi(V)$ does not depend on the choice of the metric in V. □

Lemma 7.33. *If V admits a nowhere vanishing section, then $\chi(V) = 0$.*

Proof. Let $f : S(V) \to B$ be the sphere bundle. If X is a nowhere vanishing section, then we can define the normalized section $Y := \frac{X}{\|X\|} \in \Gamma(B, S(V))$. Note that $f^*\chi(V) = 0$. Since $\mathtt{id} = f \circ Y$ we have

$$\chi(V) = Y^* f^* \chi(V) = 0 \ .$$

□

We consider a real n-dimensional vector bundle $V \to B$. If $X \in \Gamma(B, V)$ is transverse to the zero section 0_V, then $Z(X) := \{X = 0\} \subseteq B$ is a codimension-n submanifold.

Definition 7.34. *We say that X has* **non-degenerated zeros** *if X and 0_V are transverse.*

If B and V are oriented, then $Z(X)$ has an induced orientation. If $\dim(B) = n$, then $Z(X)$ is zero dimensional. In this case an orientation of $Z(X)$ is a function

$$\deg_X : Z(X) \to \{1, -1\} \ .$$

It associates to $b \in Z(X)$ the **local degree**.

Remark 7.35. In the following we describe how one can check in local coordinates that X has non-degenerated zeros and how one can calculate the local degree. We identify B with the zero section of V. If $b \in B$, then we write $0_b \in V_b \subseteq V$ for the corresponding point in the zero section. We have a natural decomposition

$$TV_{|B} \cong TB \oplus V \ .$$

Let us describe this decomposition at a point $b \in B$. We have a natural inclusion $\iota_b : V_b \to T_{0_b}V$ which maps $A \in V_b$ to $\iota_b(A) = \frac{d}{dt}_{|t=0} tA \in T_{0_b}V$. Moreover, the differential of the zero section 0_V gives an inclusion $d0_V(b) : T_bB \to T_{0_b}V$. This gives the isomorphism $d0_V(b) \oplus \iota_b : T_bB \oplus V_b \stackrel{\cong}{\to} T_{0_b}V$.

Assume that $b \in Z(X)$. Then $dX(b) - d0_V(b) : T_bB \to T_{0_b}(V)$ has values in $\iota_b(V_b)$. In order to see this we use local coordinates x of B centered at b and a trivialization of V. We write points in V as pairs (x, ξ) with $\xi \in \mathbb{R}^n$. The section X is then given by $x \mapsto (x, \xi(x))$. The differential of this map is $(\mathtt{id}, d\xi(x))$. The zero section 0_V is given by $x \mapsto (x, 0)$. So finally, $dX(b) - d0_V(b)$ is $(0, d\xi(0))$.

Corollary 7.36. *We see that X and 0_V are transverse at 0_b of $d\xi(0)$ is surjective.*

Let us now assume that $\dim(B) = n = \dim(V)$. Then $b \in Z(X)$ is non-degenerated if and only if $d\xi(0)$ is an isomorphism. In this case

$$\iota_b^{-1} \circ (dX(b) - d0_V(b)) : T_bB \to V_b$$

is an isomorphism.

Let us now assume that B and V are oriented. Then we get induced orientations of the vector spaces T_bB and V_b for all $b \in B$.

Corollary 7.37. *The local degree is given by W*

$$\deg_X(b) := \begin{cases} 1 & \iota_b^{-1} \circ (dX(b) - d0_V(b)) \text{ perserves orientations} \\ -1 & \text{else} \end{cases}$$

□

Example 7.38. The non-degenerated zeros of a gradient vector field ξ on $\mathbb{R}^2$ are classified into tree types:

source: $\xi(x, y) := (x, y)$. We have $s_\xi(0) = 1$.

saddle: $\xi(x, y) := (x, -y)$. We have $s_\xi(0) = -1$.

sink: $\xi(x, y) := (-x, -y)$. We have $s_\xi(0) = 1$.

The rotation field $\xi(x, y) = (-y, x)$ has $s_\xi(0) = 1$. □

Theorem 7.39 (Poincaré-Hopf). *Let $V \to B$ be an n-dimensional real oriented vector bundle over a closed oriented manifold B of dimension n. If the zeros of a* *section $X \in \Gamma(B,V)$ are non-degenerated, then we have the equality*

$$\int_B \chi(V) = \sum_{b\in Z(X)} \deg_X(b)\ .$$

Proof. Ldet $f : S(V) \to B$ be the sphere bundle of V for some choice of a metric. We choose a closed form $\omega \in \Omega^n(B)$ such that $\chi(V) = [\omega]$. Since $f^*\chi(V) = 0$ we can choose $\alpha \in \Omega^{k-1}(S(V))$ such that $f^*\omega = d\alpha$. Note that $-\alpha_{|S(V_b)}$ represents the normalized volume form for all $b \in B$.

For every zero $b \in Z(X)$ we let U_b be a small oriented coordinate ball with smooth boundary centered at b not containing any other zero of X. For $r \in (0,1]$ we let $rU_b \subset U_b$ be the scaled neighbourhood.

We calculate for every $r \in (0,1]$:

$$\begin{aligned}
\int_B \chi(V) &= \int_B \omega \\
&= \sum_{b\in Z(X)} \left(\int_{B\setminus \mathrm{Int}(rU_b)} \omega + \int_{rU_b} \omega\right) \\
&= \sum_{b\in Z(X)} \left(\int_{B\setminus \mathrm{Int}(rU_b)} Y^*f^*\omega + \int_{rU_b} \omega\right) \\
&= \sum_{b\in Z(X)} \left(\int_{B\setminus \mathrm{Int}(rU_b)} Y^*d\alpha + \int_{rU_b} \omega\right) \\
&= \sum_{b\in Z(X)} \left(-\int_{\partial(rU_b)} Y^*\alpha + \int_{rU_b} \omega\right)\ .
\end{aligned}$$

The minus sign comes from the fact that we orient $\partial(rU_b)$ as the boundary of rU_b and not of $B \setminus \mathrm{Int}(rU_b)$. We now consider the limit $r \to 0$. We clearly have

$$\mathtt{lim}_{r\to 0} \int_{rU_b} \omega = 0\ .$$

We write $Y_r := Y_{|\partial(rU_b)}$. We use an orientation preserving trivialization of V near b. We parametrize $\partial(rU_b)$ by $\xi \in S^{n-1}$. Then we have $Y_r(\xi) = (r\xi, \frac{X(r\xi)}{\|X(r\xi)\|})$. The Taylor expansion of X at 0 gives $X(x) = dX(0)(x) + O(x^2)$. Then we have

$$Y_r(\xi) = (r\xi, \frac{dX(0)(r\xi) + O(r^2)}{\|dX(0)(r\xi) + O(r^2)\|}) = (O(r), \frac{dX(0)(\xi) + O(r)}{\|dX(0)(\xi) + O(r)\|}) = (0, \frac{dX(0)(\xi)}{\|dX(0)(\xi)\|}) + O(r)\ .$$

The map

$$X^o : S^{n-1} \to S^{n-1} \ , \quad \xi \mapsto \frac{dX(0)(\xi)}{\|dX(0)(\xi)\|}$$

has degree $\deg_X(b)$. Since $-\alpha_{|S(V_b)}$ represents the normalized volume class it follows that

$$\mathtt{lim}_{r\to 0} \int_{\partial(rU_b)} Y^*\alpha = \int_{S^{n-1}} X^{o,*}\alpha_{|S(V_b)} = -\deg_X(b) \ .$$

Therefore

$$\int_B \chi(V) = \sum_{b\in Z(X)} \deg_X(b) \ .$$

□

Example 7.40. Let M be a closed oriented n-dimensional manifold. Then we can consider the Euler class $\chi(TM) \in H^n_{dR}(M)$. The following numbers are equal:

1. The Euler characteristic $\chi(M)$.
2. The number $\int_M \chi(TM)$.
3. The number $\sum_{m\in M} \deg_X(m)$ for every vector field X on M with non-degenerated zeros.

In Theorem 7.39 we have shown the equality of 2. and 3. In order to relate these numbers with the Euler characteristic 1. one usually employs **Morse theory**. This goes beyond the scope of this course.

□

We now assume that V is a complex vector bundle of complex dimension n and $V_{|\mathbb{R}}$ is the underlying real bundle.

Lemma 7.41. *We have the equality*

$$c_n(V) = \chi(V_{|\mathbb{R}}) \ .$$

Proof. We choose a hermitean metric on V. We factorize the projection $f : S(V) \to B$ as $S(V) \xrightarrow{h} \mathbb{P}(V) \xrightarrow{g} B$. Note that $h : S(V) \to \mathbb{P}(V)$ is the orthonormal frame bundle of the tautological bundle $L \to \mathbb{P}(V)$. We choose a closed form $\omega \in \Omega^2(\mathbb{P}(V))$ representing $-c_1(L) \in H^2_{dR}(\mathbb{P}(V))$. Since $h^*c_1(L) = 0$ we can choose $\theta \in \Omega^1(S(V))$

such that $d\theta = h^*\omega$. We further choose closed forms $w_i \in \Omega^{2i}(B)$ representing the Chern classes $c_i(V) \in H^{2i}_{dR}(B)$.

By definition of the Chern classes (see (48)) on $\mathbb{P}(V)$ we have the relation

$$0 = \sum_{i=0}^{n} (-1)^i c_1(L)^i g^* c_{n-i}(V) \ .$$

Hence there exists a form $\alpha \in \Omega^{2n-1}(\mathbb{P}(V))$ such that

$$d\alpha = \sum_{i=0}^{n} \omega^i \wedge g^* w_{n-i} \ .$$

We get

$$f^* w_n = d\left(h^*\alpha - \sum_{i=1}^{n} \theta \wedge h^*\omega^{i-1} \wedge f^* w_{n-i}\right) \ . \tag{53}$$

We have

$$\left(h^*\alpha - \sum_{i=1}^{n} \theta \wedge h^*\omega^{i-1} \wedge f^* w_{n-i}\right)_{|S^{2n-1}} = -\left(\theta \wedge h^*\omega^{n-1}\right)_{S^{2n-1}} \ .$$

Now by Example 7.21 we have

$$\int_{S^{2n-1}} \theta \wedge h^*\omega^{n-1} = 1 \ .$$

Hence $(h^*\alpha - \sum_{i=1}^{n} \theta \wedge h^*\omega^{i-1} \wedge f^* w_{n-i})$ is an extension $\theta \wedge h^*\omega^{n-1}$ from the fibre S^{2n-1} to $S(V)$. The relation (53) now asserts that $[\mathrm{vol}_{S^{2n-1}}] \in {}^fE_2^{0,2n-1} \cong H^{2n-1}_{dR}(S^{2n-1})$ is transgressive and ${}^fd_{2n}[\mathrm{vol}_{S^{2n-1}}] = -c_n$. On the other hand, by definition of the Euler class of V, we have ${}^fd_{2n}[\mathrm{vol}_{S^{2n-1}}] = -\chi(V_{|\mathbb{R}})$. □

Example 7.42. For two complex vector bundles V and V' we have

$$\chi(V_{\mathbb{R}} \oplus V'_{\mathbb{R}}) = \chi(V) \cup \chi(V') \ .$$

This is because the same relation holds for the highest Chern classes. □

Example 7.43. Let Σ_g be a closed oriented surface of genus $g \in \mathbb{N}$ and X be a vector field on Σ_g with non-degenerated zeros. Then we have

$$\sum_{b\in\Sigma_g} \deg_X(b) = 2-2g\ .$$

Indeed, we have

$$\int_{\Sigma_g} \chi(T\Sigma_{g|\mathbb{R}}) = \int_{\Sigma_g} c_1(T\Sigma_g) \stackrel{Cor. 6.16}{=} 2-2g$$

where $T\Sigma_g$ is the tangent bundle of Σ_g considered as a complex vector bundle. If the vector field is holomorphic, then it has exactly $2-2g$ zeros since in this case $\deg_X(b) = 1$ for all zeros.

For example, ∂_1 provides a holomorphic vector field on T^2 without zeros. This is compatible with the calculation since T^2 has genus one.

On $\mathbb{CP}^1$ we have a vector field X given by $\mathtt{Re}(z\partial_z)$ in coordinates $[1:z]$. Note that in the coordinate $u = z^{-1}$ near ∞ it is given by $-\mathtt{Re}(u\partial_u)$. This vector field has two non-degenerated zeros, namely at 0 and ∞. Since it is holomorphic we have $\deg_X(0) = \deg_X(\infty) = 1$. Since $\mathbb{CP}^1$ has genus 0 we see that this is again compatible with the Poincaré-Hopf theorem.

$\square$

Example 7.44. We consider a holomorphic vector field with non-degenerated zeros on $\mathbb{CP}^n$. Then it has exactly $n+1$ zeros. Indeed, by holomorphy the degrees are all positive and by Example 6.29 we have

$$\chi((T\mathbb{CP}^n)_{|\mathbb{R}}) = c_n(T\mathbb{CP}^n) = \binom{n+1}{n} = (n+1)c_1^n$$

and hence

$$\sharp Z(X) = \sum_{x\in Z(X)} \deg_X(x) = \int_{\mathbb{CP}^n} \chi((T\mathbb{CP}^n)_{|\mathbb{R}}) = (n+1)\int_{\mathbb{CP}^n} c_1^n = n+1\ .$$

Here is an example. We let X be the vector field given by $\mathtt{Re}(\sum_{i=1}^n z_i\partial_{z_i})$ in the coordinates $[1:z_1:\dots:z_n]$. The non-degenerated zeros of this vector field are the $n+1$ points

$$[1:0:\dots:0]\ , [0:1:0:\dots:0]\ , \dots\ , [0:\dots:0:1]\ .$$

$\square$

Exercise: Calculate the number of zeros of a non-generated vector field on $Gr(2, \mathbb{C}^4)$. $\square$

Exercise: Show the relation $\chi(V \oplus V') = \chi(V) \cup \chi(V')$ $\square$

Example 7.45. A vector field on $S^{2n} \times S^{2n}$ with non-degenerated zeros has at least 4 zeros. $\square$

7.5 Intersection numbers

We let M be a closed oriented manifold of dimension n. Then we have a Poincaré duality isomorphism

$$\mathcal{P}_M : H^k_{dR}(M) \stackrel{\cong}{\to} H^{n-k}_{dR}(M)^* .$$

It was given by

$$H^k_{dR}(M) \ni x \mapsto \left(H^{n-k}_{dR}(M) \ni y \mapsto (-1)^{(n+1)k} \int_M x \cup y \in \mathbb{R} \right) .$$

Let $P \subset M$ be an oriented closed submanifold of dimension p. Then we have the integration map

$$\int_P : H^p_{dR}(M) \to \mathbb{R} , \quad x \mapsto \int_P x_{|P} ,$$

i.e. an element $\int_P \in H^p_{dR}(M)^*$. By Poincaré duality we get a de Rham cohomology class

$$\{P\} := \mathcal{P}^{-1}_M(\int_P) \in H^{n-p}_{dR}(M)$$

called the **Poincaré dual of** P. Note that the degree of the Poincaré dual of P is the codimension of P.

Let now $Q \subset M$ be a second closed oriented submanifold of complementary dimension $q = n - p$. Then we can form the product $\{P\} \cup \{Q\} \in H^n_{dR}(M)$ and define the number

$$\int_M \{P\} \cup \{Q\} \in \mathbb{R} .$$

We assume that P and Q intersect transversely in M. Then $P \cap Q := P \times_M Q$ is a zero-dimensional submanifold of M. Since M is compact, this intersection is a finite set. It furthermore comes with an induced orientation, i.e. a function $s : P \cap Q \to \{1, -1\}$. Let us describe this orientation explicitly. Let $x \in P \cap Q$. Then we have an isomorphism of vector spaces $T_xQ \oplus T_xP \xrightarrow{\sim} T_xM$ induced by the inclusions. We define the sign $s(x) \in \{1, -1\}$ to be equal to 1 if this map preserves orientations, and as -1 else. Note the order of T_xP and T_xQ in this formula. We define the intersection number of P and Q in M as

$$\langle P \cap Q \rangle := \sum_{x \in P \cap Q} s(x) \ .$$

Theorem 7.46. *We have*

$$\langle P \cap Q \rangle = \int_M \{P\} \cup \{Q\} \ .$$

Proof. First we construct an explicit de Rham representative of the Poincaré dual $\{P\}$ of P. We choose a metric on M. It induces a metric on the normal bundle $\pi : N \to P$ which we identify with a tubular neighborhood of P, see Fact 3.3.

We can choose a form $\kappa \in \Omega^{q-1}(S(N))$ such that $\int_{S(N)/P} \kappa = 1$ and $dV = \pi^*\alpha$ for some form $\alpha \in \Omega^q(P)$. Indeed, if $\chi(S(N)) \neq 0$, then we can choose κ such that $d\kappa = f^*\alpha$ for a closed form $\alpha \in \Omega^p(P)$ representing the Euler class $\chi(S(N)) \in H^p_{dR}(P)$. Else we let κ be a representative of a Thom class $\mathtt{Th}(S(N)) \in H^{q-1}_{dR}(S(N))$ of $S(N)$.

We identify P with the zero section of N. Using the diffeomorphism

$$S(N) \times (0, \infty) \xrightarrow{\cong} N \setminus P \ , \quad (\xi, t) \mapsto t\xi$$

we extend the form κ to $N \setminus P$. Furthermore we consider a function $\chi \in C^\infty(0, \infty)$ such that $\chi(t) = 0$ for large t and $\chi(t) = 1$ for small t. The closed form

$$d(\chi\kappa) = d\chi \wedge \kappa + \chi \ \pi^*\alpha \in \Omega^p(N \setminus P)$$

extends smoothly to the zero section P and then by zero to M. We denote this extension by ω_P. We claim that this extension ω_P represents $(-1)^{(n+1)(n-p)+1}\{P\}$.

Let $x = [\beta] \in H^p_{dR}(M)$. Then we calculate using the projection formula

$$\begin{aligned}
\int_M [\omega_P] \cup x &= \int_N d(\chi\kappa) \wedge \beta \\
&= \int_{S(N)\times(0,\infty)} d(\chi\kappa) \wedge \beta \\
&= -\mathtt{lim}_{t\to 0} \int_{S(N)\times\{t\}} \kappa \wedge \beta \\
&= -\mathtt{lim}_{t\to 0} \int_{S(N)\times\{t\}} \kappa \wedge \pi^*\beta \\
&= -\int_P \beta \ .
\end{aligned}$$

On the other hand

$$\mathcal{P}_M([\omega_P])(x) = (-1)^{(n+1)(n-p)} \int_M [\omega_P] \cup x$$

We now calculate the quantity

$$\int_M \{P\}\cup\{Q\} = (-1)^{(n+1)(n-p)+1}(-1)^{(n+1)(n-q)}(-1)^{(n-p)(n-q)} \int_Q \omega_P = (-1)^{pq+1} \int_Q \omega_P \ .$$

Let $x \in P \cap Q$. Then we can assume after appropriate choice of the embedding $N \to M$ that $N_x \subset Q$. The contribution of the neighborhood of x to the integral $(-1)^{pq+1} \int_Q \omega_P$ is thus

$$-s(x) \int_{S(N_x)\times\mathbb{R}} d(\chi\kappa) = s(x)\, \mathtt{lim}_{t\to 0} \int_{S(N_x)\times\{t\}} \kappa = s(x) \ .$$

Note that the orientation of N_x is such that $T_xP \oplus N_x \xrightarrow{+} T_xM$ is orientation preserving. This is exactly the case of $(-1)^{pq} s(x) = 1$.

□

In particular, the number $\int_M \{P\} \cup \{Q\}$ is an integer. Note that the intersection number is only defined if P and Q are transverse. But the number $\int_M \{P\} \cup \{Q\}$ is always defined and can be taken as a definition of the intersection number in this case. Furthermore, in order to define the classes $\{P\}$ and $\{Q\}$ and therefore $\int_M \{P\} \cup \{Q\}$ it is not necessary to assume that $P \to M$ is an embedding. Just a smooth map suffices. Furthermore, theses classes, respective this number only depends on the homotopy class of these maps.

Proposition 7.47. *Let P and Q be two closed oriented submanifolds of M of dimensions p and q which intersect transversally. Then we have the relation*

$$\{P\}\cup\{Q\}=\{P\cap Q\}$$

in $H_{dR}^{n-p-q}(M)$.

Proof. We can assume by choosing the embedding $N\to M$ appropriately that $N\cap Q\to P\cap Q$ is the normal bundle of $P\cap Q$ in Q. Then we can take $(\omega_P)_{|Q}=\omega_{P\cap Q}$. We calculate for every $[\beta]\in H_{dR}^{n-p-q}(M)$ that

$$\begin{aligned}
\int_M \{P\}\cup\{Q\}\cup[\beta] &= (-1)^{(n+1)(2n-p-q)}\int_M \omega_P\wedge\omega_Q\wedge\beta\\
&= -(-1)^{(n+1)(2n-p-q)}\int_P \omega_Q\wedge\beta\\
&= (-1)^{(n+1)(2n-p-q)}\int_{P\cap Q}\beta\\
&= (-1)^{(n+1)(2n-p-q)+(n+1)(n-p-q)}\int_M \{P\cap Q\}\wedge[\beta]\\
&= \int_M \{P\cap Q\}\wedge[\beta]\ .
\end{aligned}$$

□

Example 7.48. Let $V\to B$ be a real oriented vector bundle of dimension k on a closed oriented manifold B of dimension n. We choose a section $s\in\Gamma(B,V)$ which is transverse to the zero section and let $Z=\{s=0\}$ be the smooth submanifold of zeros. We have $\dim(Z)=n-k$. We define the orientation of Z such that at $x\in Z$ the isomorphism $V_x\oplus T_xZ\to T_xM$ induced by $ds(x)$ preserves orientations. Note the order of the summands.

The following Lemma generalizes the Poincaré-Hopf theorem.

Lemma 7.49. *We have the equality $\{Z\}=\chi(V)$.*

Proof. We consider the bundle $V\oplus\mathbb{R}\to B$. Its sphere bundle $p:S(V\oplus\mathbb{R})\to B$ has a canonical section with image $N=\{(0_b,1)\mid b\in B\}$. Fibrewise stereographic projection gives a diffeomorphism $S(V\oplus\mathbb{R})\setminus N\stackrel{\cong}{\to}V$. We identify B with the zero section of V and therefore with a submanifold of $S(V\oplus\mathbb{R})$. We consider the class $\{B\}\in H^k_{dR}(S(V\oplus\mathbb{R}))$ which is represented by a form $(-1)^{(n+k+1)k+1}\omega_B$ constructed as above. The normal bundle at B is identified with p^*V, and its sphere bundle is $\pi : p^*S(V) \to B$ Therefore ω_B is the extension by zero of a form $d(\chi\kappa)$ with $d\kappa = \pi^*\alpha$ where α is closed and represents $\chi(V)$. We view the section s as a section of $S(V \oplus \mathbb{R})$. Then $s^*\omega_B$ is a choice for $(-1)^{kn}\omega_Z$, i.e. we get

$$s^*\{N\} = (-1)^{(n+1+k)k+1}(-1)^{(n+1)(n-k)+1}(-1)^{kn}\{Z\} = (-1)^{kn}\{Z\} \ .$$

The factor $(-1)^{kn}$ comes from our convention for the orientation of Z. Let z be the zero section of V. Then we have $z^*\omega_B = z^*d\kappa$ so that

$$z^*\{N\} = (-1)^{(n+k+1)k+1}\chi(S(V)) = (-1)^{kn}\chi(V) \ .$$

Note that s and z are homotopic by $(t, b) \mapsto ts(b)$. It follows that

$$\chi(V) = \{Z\} \ .$$

□

□

Example 7.50. Let $L^{taut,*} \to \mathbb{CP}^n$ be the dual of the tautological bundle. A hypersurface of degree d in $\mathbb{CP}^n$ is by definition a submanifold of the form $H = Z(s)$ for some holomorphic section $s \in \Gamma(\mathbb{CP}^n, (L^{taut,*})^d)$. By Lemma 7.41 we have that $\chi((L^{taut,*})^d_{|\mathbb{R}}) = c_1((L^{taut,*})^d) = dc_1$. Therefore, we have $\{H\} = dc_1$. Assume now that $n = 2$. Then we can consider the intersection number of two hypersurfaces H_1 and H_2 of degrees d_1 and d_2, respectively. We get

$$\langle H_1 \cap H_2 \rangle = \int_{\mathbb{CP}^2} d_1c_1 \cup d_2c_1 = d_1d_2$$

provided that the intersection is transverse. Since the H_i are complex submanifolds all signs are positive and we actually have

$$\langle H_1 \cap H_2 \rangle = \sharp H_1 \cap H_2 \ .$$

Exercise: Calculate the the number of points of $H_1 \cap \cdots \cap H_n$, where H_i is a hypersurface in $\mathbb{CP}^n$ of degree d_i for $i = 1, \ldots, n$. We assume that the intersection is transverse. □

Example 7.51. We consider the manifold $S^2 \times S^2$. Let $N \subset S^2$ be the north pole. Then we have submanifolds $S^2 \times \{N\}, \{N\} \times S^2$ of $S^2 \times S^2$. The middle cohomology of $S^2 \times S^2$ is generated by the Poincaré duals $[\mathrm{vol}_{S^2}] \times 1, 1 \times [\mathrm{vol}_{S^2}] \in H^2_{dR}(S^2 \times S^2)$ of the these submanifolds. They intersect transversally in the point (N, N) with index 1. This is compatible with $\int_{S^2 \times S^2} ([\mathrm{vol}_{S^2}] \times 1) \cup (1 \times [\mathrm{vol}_{S^2}]) = 1$. □

8 Interesting differential forms

8.1 G-manifolds and invariant forms

Let G be a connected Lie group which acts on a manifold M. For $g \in G$ let $a_g : M \to M$ be the action. If $M = G$, then we write L_g (resp. R_g) for the action by left (resp. right) multiplication. Note that $R_g(h) := hg^{-1}$, i.e. also R is a left-action. A form $\omega \in \Omega(M)$ is called G-invariant, if $a_g^* \omega = \omega$ holds for all $g \in G$.

For every $g \in G$ we have the equality $da_g^* = a_g^* d$. If ω is G-invariant, then so is $d\omega$. Since a_g^* preserves wedge products the wedge product of two G-invariant forms is again G-invariant.

We can thus define differential graded subalgebra

$$\Omega(M)^G := \{\omega \in \Omega(M) \mid (\forall g \in G \mid a_g^* \omega = \omega)\} \subseteq \Omega(M)$$

of G-invariant forms.

Lemma 8.1. *If G is compact, then there exists a right G-invariant volume form $\mathrm{vol}_G \in \Omega^{\dim(G)}(G)^G$ such that $\int_G \mathrm{vol}_G = 1$.*

Proof. We choose a non-trivial element $v \in \Lambda^{\dim(G)}(\mathrm{Lie}(G)^*)$. Then we define a form $\omega \in \Omega^{\dim(G)}(G)$ by

$$\omega(g) := dR^*_{g^{-1}} v \ , \quad g \in G \ .$$

This form is G-invariant since

$$(R_h^* \omega)(g) = dR^*_{h^{-1}} \omega(gh^{-1}) = dR^*_{h^{-1}} dR^*_{(h^{-1}g)^{-1}} v = (R_{g^{-1}h} R_{h^{-1}})^* v = R^*_{g^{-1}} v = \omega(g) \ .$$

We orient G using this form and define the normalized volume form by

$$\mathrm{vol}_G := \frac{1}{\int_G \omega} \, \omega \ .$$

□

We now make the technical assumption that M is closed and oriented in order to simplify the proof of the following Lemma.

Lemma 8.2. *If G is compact, then then inclusion $i : \Omega(M)^G \to \Omega(M)$ is a quasi-isomorphism.*

Proof. We define the averaging map

$$A : \Omega(M) \to \Omega(M)^G \ , \quad A(\omega) := \int_{g\in G} a_g \omega \, \mathrm{vol}_G(g) \ .$$

Indeed, A produces G-invariant forms since for every for $h \in G$ we have the chain of equalities

$$\begin{aligned} a_h^* A(\omega) &= \int_G a_h^* a_g^* \omega \, \mathrm{vol}_G(g) \\ &= \int_G a_{gh}^* \omega \, \mathrm{vol}_G(g) \\ &\overset{!}{=} \int_G a_g^* \omega \, (R_{h^{-1}}^* \mathrm{vol}_G)(g) \\ &= \int_G a_g^* \omega \, \mathrm{vol}_G(g) \\ &= A(\omega) \ , \end{aligned}$$

where at the marked equality we use the diffeomorphism $R_h^{-1} : G \to G$ in order to reparameterize the domain G of integration. Since G is compact and the integrand is smooth we can interchange differentiation and integration over G. The map A preserves the differential since

$$dA(\omega) = d \int_G a_g^* \omega \, \mathrm{vol}_G(g) = \int_G d a_g^* \omega \, \mathrm{vol}_G(g) = \int_G a_g^* d\omega \, \mathrm{vol}_G(g) = A(d\omega) \ .$$

If ω is G-invariant, then $A(\omega) = \omega$ by the normalization of vol_G, i.e. we have the equality $A \circ i = \mathtt{id}_{\Omega(M)^G}$. This shows that $i : H(\Omega(M)^G) \to H_{dR}(M)$ is injective.

We now show surjectivity. Surjectivity follows immediately from the claim that

$$[A\omega] = [\omega] \ ,$$

where $[\omega] \in H_{dR}^*(M)$ is the class of a closed form $\omega \in \Omega(M)$. In the proof of this claim we use the simplifying assumptions.

Remark 8.3. In order to drop the assumption that M is closed and oriented one can perform a similar argument using the pairing between simplicial homology and cohomology instead of the pairing with the cohomology in the complementary dimension. In the present course we have not developed this homology theory. $\square$

By Poincaré duality the equality $[A\omega] = [\omega]$ holds if and only if for every class $[\alpha] \in H^*_{dR}(G)$ we have the equality

$$\int_G \alpha \wedge \omega = \int_G \alpha \wedge A(\omega) \ . \tag{54}$$

We can express $A(\omega)$ as a uniform limit for $n \to \infty$ of a sequence $A_n(\omega)$, $n \in \mathbb{N}$, of Riemann sums of the form $\sum_i c_i a^*_{g_i}\omega$ such that $\sum_i c_i = 1$. Since G is connected, we have

$$[a^*_g\omega] = [\omega] \in H^*_{dR}(G) \ , \quad \forall g \in G$$

and therefore $[\sum_i c_i a^*_{g_i}\omega] = [\omega]$. We get

$$\int_G \alpha \wedge \omega = \int_G \alpha \wedge A_n(\omega)$$

for every n and thus (54) by taking the limit $n \to \infty$. This shows the claim. $\square$

We now apply this to the action of the group $G \times G$ on G by $(g,h)k := gkh^{-1}$.

Lemma 8.4. *For a compact connected Lie group G the evaluation at the identity provides an isomorphism*

$$(\Omega(G)^{G\times G}, d) \to (\Lambda(\mathrm{Lie}(G)^*)^G, 0) \ .$$

In particular we get an isomorphism

$$H_{dR}(G) \cong \Lambda(\mathrm{Lie}(G)^*)^G \ .$$

Proof. It is clear that the evaluation at the identity

$$\Omega(G)^{G\times\{1\}} \to \Lambda(\mathrm{Lie}(G)^*)$$

is an isomorphism of graded vector spaces since every element on the right-hand side can be uniquely extended to an invariant form. This extension is invariant under

$G \times G$ if and and only if it is invariant under the adjoint action of G on itself. Consequently, the evaluation at the identity is an isomorphism

$$\Omega(G)^{G\times G} \stackrel{\cong}{\to} (\Lambda(\mathrm{Lie}(G)^*)^G .$$

It remains to identify the differential on $(\Lambda(\mathrm{Lie}(G)^*)^G$ induced by the de Rham differential on $\Omega(G)^{G\times G}$. Let $\omega \in \Omega^n(G)^{G\times G}$. We insert $n+1$ left-invariant vector fields $X_1, \dots X_{n+1}$ into $d\omega$. Then the Cartan formula expresses $d\omega(X_1, \dots, X_{n+1})$ as a sum of terms indexed by i which vanish separately. First of all

$$\omega(X_1, \dots \check{X}_i, \dots, X_{n+1}) \in C^\infty(G)$$

is invariant and hence

$$X_i\omega(X_1, \dots \check{X}_i, \dots, X_{n+1}) = 0 .$$

Moreover, in the second group the terms are of the form (we write the case $i = 1$)

$$\begin{aligned} &\sum_{j=2}^{n+1}(-1)^j\omega([X_1, X_j], X_1, \dots, \check{X}_j, \dots, X_{n+1})(e) \\ &= (\mathrm{ad}(X_1)\omega(e))(X_1, \dots, \check{X}_j, \dots, X_{n+1}) \\ &= 0 \end{aligned}$$

□

Example 8.5. We can apply this to the torus T^n. We have

$$H^*_{dR}(T^n) \cong \Lambda^*\mathrm{Lie}(T^n)^* \cong \Lambda^*\mathbb{R}^n .$$

This reproduces the result of Example 2.35. □

Remark 8.6. The product $\mu : G \times G \to G$ turns the algebra $\Lambda(\mathrm{Lie}(G)^*)$ into a Hopf algebra over $\mathbb{R}$ with coproduct

$$\Delta = d\mu^* : \Lambda(\mathrm{Lie}(G)^*) \to \Lambda(\mathrm{Lie}(G)^*) \otimes \Lambda(\mathrm{Lie}(G)^*) .$$

Since conjugation by a fixed element is a group homomorphism the coproduct Δ restricts to a Hopf algebra structure

$$\Delta : \Lambda(\mathrm{Lie}(G)^*)^G \to \Lambda(\mathrm{Lie}(G)^*)^G \otimes \Lambda(\mathrm{Lie}(G)^*)^G .$$

We have the following structural result.

Proposition 8.7 (Milnor-Moore)**.** *A finite dimensional graded Hopf algebra over $\mathbb{R}$ is the graded commutative algebra $\mathbb{R}[(v_i)_{i\in A}]$ where v_i are primitive generators of odd degree and A is a finite set.*

We have already observed this structure for $G = U(n)$ (Example 6.20) and $G = SU(n)$ and (Example 6.17).

□

We consider $\mathtt{id}_{\mathrm{Lie}(G)}$ as an element of

$$\theta \in \Lambda^1(\mathrm{Lie}(G)^*) \otimes \mathrm{Lie}(G) \ .$$

This element in G-invariant under the tensor product action of G on $\Lambda^1(\mathrm{Lie}(G)^*) \otimes \mathrm{Lie}(G)$. It is called the canonical one-form.

Let $\rho : G \to \mathtt{Aut}(V)$ be a representation of G and set

$$\theta_\rho := (1 \otimes d\rho)(\theta) \in \Lambda^1(\mathrm{Lie}(G)^*) \otimes \mathtt{End}(V) \ .$$

This element is again G-invariant under the tensor product action of G. Finally, since $\mathtt{Tr} : \mathtt{End}(V) \to \mathbb{C}$ is G-invariant the form

$$\kappa_{\rho,p} := \mathtt{Tr}(\theta_\rho^{\wedge p}) \in \Lambda^p(\mathrm{Lie}(G)^*)^G$$

is G-invariant, too.

In this way we can construct cohomology classes on G.

We have $\Delta^*\theta = \theta \otimes 1 + 1 \otimes \theta$. This gives $\Delta^*\theta_\rho^p = \sum_{s+t=p} \binom{p}{s} \theta_\rho^s \otimes \theta_\rho^t$ and therefore

$$\Delta^*\kappa_{\rho,p} = \sum_{s+t=p} \binom{p}{s} \kappa_{\rho,s} \otimes \kappa_{\rho,t} \ .$$

For given representation ρ, if $p \geq 1$ is minimal such that $\kappa_{\rho,p} \neq 0$, then $\kappa_{\rho,p}$ is primitive.

Example 8.8. We shall make this explicit in the case $G = SU(2)$ which we identify for the present purpose with the unit quaternions. Then $\mathrm{Lie}(SU(2)) \cong \mathtt{Im}(\mathbb{H}) = \mathbb{R}\langle I, J, K\rangle$. We consider the representation ρ of $SU(2)$ on $\mathbb{H}$. We write

$$\theta_\rho = dx \otimes I + dy \otimes J + dz \otimes K \ ,$$

where x, y, z are coordinates on $\mathtt{Im}(\mathbb{H})$ dual to I, J, K. Then

$$\theta_\rho^3 = 6dx \wedge dy \wedge dz \otimes IJK = -6dx \wedge dy \wedge dz \otimes \mathtt{id}_{\mathbb{H}} \ .$$

Consequently

$$\kappa_{\rho,3} = \mathtt{Tr}(\theta_\rho^3) = -24 dx \wedge dy \wedge dz \ .$$

Under the identification $SU(2) \cong S^3$ the invariant extension of $dx \wedge dy \wedge dz$ becomes the Euclidean volume form. Since $\mathrm{vol}(S^3) = 2\pi^2$ we get

$$\kappa_{\rho,3} = -48\pi^2 [\mathrm{vol}_{SU(2)}] \ .$$

$\square$

Exercise: We consider the case $G = U(n)$ and the standard representation of $U(n)$ on $\mathbb{C}^n$. Determine $\lambda_p \in \mathbb{R}$ such that $\kappa_{\rho,p} = \lambda_p u_p$ for $p = 1, \dots, 2n-1$. $\square$

Example 8.9. We consider again a compact Lie group G. Then we can find an invariant scalar product $\langle ., . \rangle$ on the Lie algebra $\mathrm{Lie}(G)$. Indeed, we can start with an arbitrary scalar product and then average. We can now define the form $\omega \in (\Lambda^3 \mathrm{Lie}(G)^*)^G$ by

$$\omega(X, Y, Z) := \langle [X, Y], Z \rangle \ .$$

In order to see that it is alternating observe that

$$\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle = -\langle [X, Z], Y \rangle$$

by symmetry and invariance of the scalar product. If $\mathrm{Lie}(G)$ is not abelian, then $\omega \neq 0$.

Corollary 8.10. *If G is a compact Lie group with non-abelian Lie algebra, then $H^3_{dR}(G) \neq 0$.*

8.2 Chern forms

We consider a complex vector bundle $V \to B$. In this subsection we define differential forms representing the Chern classes $c_i(V) \in H^{2i}_{dR}(B)$ of V for $i \in \mathbb{N}$. If we choose a connection ∇ on V, then we can define the curvature form $R^\nabla \in \Omega^2(B, \mathtt{End}(V))$. We consider the inhomogeneus form

$$1 - \frac{1}{2\pi i} R^\nabla \in \Omega(B, \mathtt{End}(V))$$

with components in degree zero and two. We have a polynomial functor $\det : \mathtt{Vect}(B) \to \mathtt{Vect}(B)$ which maps a vector bundle to its maximal alternating power. Using the action of this functor on morphisms we can define the **total Chern form**

$$c(\nabla) := \det(1 - \frac{1}{2\pi i} R^\nabla) \in \Omega(B, \mathtt{End}(\det(V))) \cong \Omega(B, \mathbb{C}) \ .$$

Definition 8.11. *The homogeneous components of the total Chern form are denoted by $c_i(\nabla) \in \Omega^{2i}(B, \mathbb{C})$ and called the **Chern forms** of (V, ∇).*

We thus have

$$c(\nabla) = 1 + c_1(\nabla) + \cdots + c_{\dim(V)}(\nabla) \ .$$

Lemma 8.12. *The total Chern form $c(\nabla)$ is closed. Consequently the Chern forms $c_i(\nabla) \in \Omega^{2i}(B)$ are closed for all $i = 1, \dots, \dim(V)$.*

Proof. We calculate, using the identity $d \det(A) = \det(A)\mathtt{Tr}(A^{-1} dA)$ and the Bianchi identity $[\nabla, R^\nabla] = 0$,

$$\begin{aligned} dc(\nabla) &= d \det(1 - \frac{1}{2\pi i} R^\nabla) \\ &= [\nabla, \det(1 - \frac{1}{2\pi i} R^\nabla)] \\ &= \det(1 - \frac{1}{2\pi i} R^\nabla) \ \mathtt{Tr} \left(\frac{[\nabla, 1 - \frac{1}{2\pi i} R^\nabla]}{1 - \frac{1}{2\pi i} R^\nabla} \right) \\ &= 0 \ . \end{aligned}$$

□

Our goal is to show that the closed form $c(\nabla)$ represents the total Chern class of V. This will follows from the splitting principle and the corresponding assertions for line bundles. We prepare this argument by verifying some properties of these cohomology classes $c'(V) := [c(\nabla)]$. For the moment we use the superscript $'$ in order to distinguish these classes from the previously defined ones.

Lemma 8.13. *The Chern forms have the following properties:*

1. *The cohomology class of $c'(V) := [c(\nabla)] \in H_{dR}(B,\mathbb{C})$ does not depend on the choice of the connection.*
2. *For a smooth map $f : B' \to B$ we have $f^*c'(V) = c'(f^*V)$. In particular, $V \mapsto c'_i(V)$ is a characteristic class of degree $2i$ for complex vector bundles.*
3. *We have $c'(V \oplus V') = c'(V) \cup c'(V')$.*

Proof. Assume that ∇' is a second connection. Let $\mathtt{pr} : [0,1] \times B \to B$ be the projection. On $\mathtt{pr}^*V \to [0,1] \times B$ we consider the connection $\tilde{\nabla} := \mathtt{pr}^*\nabla + t\mathtt{pr}^*\alpha$, where $\alpha := \nabla' - \nabla \in \Omega^1(B, \mathtt{End}(V))$ and t is the coordinate of $[0,1]$. We define the **transgression Chern form**

$$\tilde{c}(\nabla', \nabla) := \int_{[0,1]\times B/B} c(\tilde{\nabla}) \in \Omega(B,\mathbb{C}) \ .$$

By Stoke's theorem we get

$$d\tilde{c}(\nabla', \nabla) = c(\nabla') - c(\nabla) \ .$$

This shows 1.

For 2. we use $f^*R^{\nabla} = R^{f^*\nabla}$ and $f^* \circ \det \cong \det \circ f^*$ in order to conclude that $f^*c(\nabla) = c(f^*\nabla)$. This implies the assertion.

Finally, for 3. we use the identity

$$\det(V \oplus V') \cong \det(V) \otimes \det(V')$$

which implies

$$\det(1 - \frac{1}{2\pi i} R^{\nabla\oplus\nabla'}) = \det(1 - \frac{1}{2\pi i}(R^{\nabla} \oplus R^{\nabla'})) = \det(1 - \frac{1}{2\pi i} R^{\nabla}) \wedge \det(1 - \frac{1}{2\pi i} R^{\nabla'}) \ .$$

□

Lemma 8.14. *If $\dim(V) = 1$, then we have $c'(V) = c(V)$.*

Proof. In this case

$$\det(1 - \frac{1}{2\pi i} R^{\nabla}) = 1 - \frac{1}{2\pi i} R^{\nabla}$$

under the natural identifications. The assertion now follows immediately from (42).

□

Lemma 8.15. *We have $c'(V) = c(V)$ in general.*

Proof. Let $p : \mathbb{F}(V) \to B$ be the bundle of total flags. Then we have

$$p^*c(V) = c(p^*V) \stackrel{(49)}{=} \prod_{i=1}^{k} c(L_i) \stackrel{Lemma\ 8.14}{=} \prod_{i=1}^{k} c'(L_i) = p^*c'(V) \ .$$

We finally use that p^* is injective. □

Example 8.16. If the complex vector bundle $V \to B$ admits a flat connection ∇, then $c_i(V) = 0$ for all $i \geq 1$. Indeed, if ∇ is flat, then we have $c_i(\nabla) = 0$.

Corollary 8.17. *If $V \to B$ is a complex vector bundle and $c(V) \neq 1$, then V does not admit a flat connection.*

□

8.3 The Chern character

We have seen in Example 6.33 that Chern classes do not behave well with respect to tensor products. In this subsection we introduce the Chern character which behaves well under sums and tensor products.

Let $V \to B$ be a vector bundle. We choose a connection ∇ and define the **Chern character form**

$$\mathbf{ch}(\nabla) = \mathtt{Tr}\, e^{-\frac{1}{2\pi i} R^\nabla} \ .$$

Here we consider the exponential function as a formal power series. We have

$$\mathbf{ch}(\nabla) = \dim(V) - \frac{1}{2\pi i}\mathtt{Tr}\ R^\nabla + \frac{1}{2!(2\pi i)^2}\mathtt{Tr}\ (R^\nabla)^2 - \frac{1}{3!(2\pi i)^3}\mathtt{Tr}\ (R^\nabla)^3 + \dots \ .$$

The sum is finite since $(R^\nabla)^k = 0$ for $2k > \dim(B)$.

Lemma 8.18.

1. *The form $\mathbf{ch}(\nabla)$ is closed.*
2. *The class $\mathbf{ch}(V) := [\mathbf{ch}(\nabla)] \in H_{dR}(B)$ is independent of the choice of the connection of V. It is called the **Chern character** of the bundle V.*
3. *For a map $f : B' \to B$ we have $f^*\mathbf{ch}(V) = \mathbf{ch}(f^*V)$.*

4. We have $\mathbf{ch}(V \oplus V') = \mathbf{ch}(V) + \mathbf{ch}(V')$, $\quad \mathbf{ch}(V \otimes V') = \mathbf{ch}(V) \cup \mathbf{ch}(V')$.

Proof. For 1. we calculate, using the Bianchi identity

$$\begin{aligned} d\mathbf{ch}(\nabla) &= d\mathtt{Tr}\, e^{-\frac{1}{2\pi i}R^\nabla} \\ &= \mathtt{Tr}[\nabla, e^{-\frac{1}{2\pi i}R^\nabla}] \\ &= 0 \end{aligned}$$

The arguments for 2. and 3. are the same as for corresponding assertions of Lemma 8.13. We have

$$\mathbf{ch}(\nabla \oplus \nabla') = \mathtt{Tr}\, e^{-\frac{1}{2\pi i}(R^\nabla \oplus R^{\nabla'})} = \mathtt{Tr}\, \left(e^{-\frac{1}{2\pi i}R^\nabla} \oplus e^{-\frac{1}{2\pi i}R^{\nabla'}}\right) = \mathbf{ch}(\nabla) + \mathbf{ch}(\nabla') \ .$$

Furthermore, we have

$$R^{\nabla\otimes 1 + 1 \otimes \nabla'} = R^\nabla \otimes 1 + 1 \otimes R^{\nabla'} \ .$$

This gives

$$e^{-\frac{1}{2\pi i}(R^\nabla \otimes 1 + 1 \otimes R^{\nabla'})} = e^{-\frac{1}{2\pi i}R^\nabla} \otimes e^{-\frac{1}{2\pi i}R^{\nabla'}} \ .$$

Consequently,

$$\mathbf{ch}(\nabla \otimes 1 + 1 \otimes \nabla') = \mathbf{ch}(\nabla) \wedge \mathbf{ch}(\nabla') \ .$$

This implies 4. $\square$

Let us fix $n \in \mathbb{N}$. We define the polynomials

$$s_k(x) := \sum_{i=1}^n x^k \in \mathbb{Z}[x_1, \dots, x_n] \ , k \in \mathbb{N} \ .$$

These polynomials are symmetric and therefore belong to the subring

$$\mathbb{Z}[\sigma_1(x), \dots, \sigma_n(x)] \subset \mathbb{Z}[x_1, \dots, x_n] \ .$$

There are unique polynomials

$$p_k(\sigma_1, \dots, \sigma_n) \in \mathbb{Z}[\sigma_1, \dots, \sigma_n]$$

such that

$$s_k(x) = p_k(\sigma_1(x), \dots, \sigma_n(x)) \ .$$

We have

$$\begin{aligned}
p_1(\sigma) &= \sigma_1 \\
p_2(\sigma) &= \sigma_1^2 - 2\sigma_2 \\
p_3(\sigma) &= \sigma_1^3 - 3\sigma_2\sigma_1 + 3\sigma_3 \\
&\vdots
\end{aligned}$$

Proposition 8.19. *For a vector bundle $V \to B$ we have*

$$\mathbf{ch}(V) = \dim(V) + \sum_{k=1}^{\infty} \frac{1}{k!} p_k(c_1(V), \dots, c_n(V)) \ .$$

Proof. Note that for $n = 1$ we have $p_k(c) = c^k$. Let $\dim(V) = 1$. Then we have by definition

$$\mathbf{ch}(V) = \sum_{k=0}^{\infty} \frac{1}{k!} c_1(V)^k = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} p_k(c_1(V)) \ .$$

We now consider the general case. We use the splitting principle. Let $p : \mathbb{F}(V) \to B$ be the bundle of full flags of V. Then we have

$$p^*\mathbf{ch}(V) = \mathbf{ch}(p^*V) = \mathbf{ch}(\bigoplus_{i=1}^{n} L_i) = \sum_{i=1}^{n} \mathbf{ch}(L_i) \ .$$

The right-hand side can be rewritten with $x_i = c_1(L_i)$ as

$$\sum_{i=1}^{n} \mathbf{ch}(L_i) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{1}{k!} x_i^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} x_i^k = \sum_{k=0}^{\infty} \frac{1}{k!} s_k(x) = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(c_1(V), \dots, c_n(V)) \ .$$

□

Vice versa one can express the Chern of a bundle classes in terms of the components of the Chern character. There are unique polynomials $h_i \in \mathbb{Q}[a_1, \dots, a_n]$ for $i = 1, \dots, n$ such that

$$\sigma_i(x) = h_i(s_1(x), \dots, s_n(x)) \ , \quad i = 1, \dots, n \ .$$

Then

$$c_i(V) = h_i(1!\mathbf{ch}_1(V), \dots, n!\mathbf{ch}_n(V)) \ .$$

We have

$$\begin{aligned}
h_1(a) &= a \\
h_2(a) &= \frac{1}{2}(a_1^2 - a_2) \\
h_3(a) &= \frac{1}{6}(a_1^3 - 3a_1a_2 + 2a_3) \\
&\vdots
\end{aligned}$$

For example,

$$c_3(V) = \frac{1}{6}(\mathbf{ch}_1(V)^3 - 6\mathbf{ch}_1(V)\mathbf{ch}_2(V) + 12\mathbf{ch}_3(V)) \ .$$

Example 8.20. We can continue Example 6.38. Note that $TGr(k, \mathbb{C}^n) \cong L^* \otimes \mathbb{C}^n/L$. Then

$$\mathbf{ch}(TGr(k, \mathbb{C}^n)) = \mathbf{ch}(L^*) \cup (n - \mathbf{ch}(L)) \ .$$

We now use that $\mathbf{ch}_i(L^*) = (-1)^i \mathbf{ch}(L)$. We get

$$\mathbf{ch}_k(TGr(k, \mathbb{C}^n)) = \sum_{j=1}^{n} (-1)^{n-j+1} \mathbf{ch}_{n-j}(L) \cup \mathbf{ch}_j(L) + (n-k)\mathbf{ch}_n(L) \ .$$

9 Exercises

1. Let $k, n \in \mathbb{N}$ and $0 \leq k \leq n$. We consider the set $V(k, \mathbb{R}^n)$ of k-tuples of linearly independent vectors in $\mathbb{R}^n$.

1. Equip $V(k, \mathbb{R}^n)$ with a smooth manifold structure by representing it as an open submanifold of $(\mathbb{R}^n)^k$.
2. Show that the linear action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ induces a smooth and transitive left action on $V(k, \mathbb{R}^n)$.
3. Show that the map $V(k, \mathbb{R}^n) \to Gr(k, \mathbb{R}^n)$, which maps the k-tuple of vectors to its span, is a locally trivial fibre bundle.
4. Show that $GL(k, \mathbb{R})$ acts freely from the right on $V(k, \mathbb{R}^n)$ preserving the fibres of π.
5. Show that π presents $Gr(k, \mathbb{R}^n)$ as the quotient $V(k, \mathbb{R}^n)/GL(k, \mathbb{R})$.

6. Give a presentation of $V(k,\mathbb{R}^n)$ as quotient $GL(n,\mathbb{R})/GL(n-k,\mathbb{R})$.

The manifold $V(k,\mathbb{R}^n)$ is called **Stiefel manifold**.

2. Consider integers $n,p,q \in \mathbb{N}$ such that $0 < p < q < n$. Let $F((p,q),\mathbb{R}^n)$ denote set of pairs of linear subspaces $V,W \subseteq \mathbb{R}^n$ such that $V \subseteq W$ and $\dim(V) = p$, $\dim(W) = q$. These tuples are called **flags of type** (p,q).

1. Equip $F((p,q),\mathbb{R}^n)$ with the structure of a smooth manifold by representing it as a submanifold of $Gr(p,\mathbb{R}^n) \times Gr(q,\mathbb{R}^n)$. The manifold $F(p,\mathbb{R}^n)$ is called **flag manifold** of type (p,q):
2. Show that the linear action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ induces a smooth action of $GL(n,\mathbb{R})$ on $F((p,q),\mathbb{R}^n)$.
3. Show that this action is transitive and describe the stabilizer of the standard flag $\mathbb{R}^p \subseteq \mathbb{R}^q$.
4. Show that map $F((p,q),\mathbb{R}^n) \to Gr(q,\mathbb{R}^n)$, $(V,W) \mapsto W$ is a locally trivial fibre bundle with fibre $Gr(p,\mathbb{R}^q)$.
5. Represent $F((p,q),\mathbb{R}^n)$ as a quotient of $GL(n,\mathbb{R})$.

The goal of the following two exercises is to practise explicit calculations with forms and the de Rham Lemma.

3. Let $n \in \mathbb{N}$, $n \geq 1$ and set $\mathbb{C}^{n+1,*} := \mathbb{C}^{n+1} \setminus \{0\}$. We have a projection $\pi : \mathbb{C}^{n+1,*} \to \mathbb{CP}^n$ which sends $x \in \mathbb{C}^{n+1,*}$ to the subspace spanned by x. Let $\alpha \in \Omega^2(\mathbb{C}^{n+1,*})$ be given by

$$\alpha := \frac{1}{2i}\frac{\sum_{i=0}^n dz^i \wedge d\bar{z}^i}{\|z\|^2} \quad .$$

1. Show that there exists a uniquely determined form $\omega \in \Omega^2(\mathbb{CP}^n)$ such that $\pi^*\omega = \alpha_{|\mathtt{ker}(d\pi)^\perp}$
2. Show that ω is real.
3. Show that ω is invariant under the natural action of $U(n+1)$ on $\mathbb{CP}^n$.
4. Show that $d\omega = 0$.
5. Show that ω^n is nowhere vanishing.

6. Let $\mathbb{CP}^1 \subset \mathbb{CP}^{n+1}$ be the submanifold of lines contained in $\mathbb{C}^2 \subseteq \mathbb{C}^{n+1}$. Show that $\omega_{|\mathbb{CP}^1}$ is nowhere vanishing. Orient $\mathbb{CP}^1$ using $\omega_{|\mathbb{CP}^1}$ and calculate $\int_{\mathbb{CP}^1} \omega$.
7. Show that ω^n is nowhere vanishing. Orient $\mathbb{CP}^n$ using ω^n and calculate $\int_{\mathbb{CP}^n} \omega^n$.

4. We consider the form $\mathrm{vol}_{\mathbb{R}^n} := dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$.

1. Show that $[\mathrm{vol}_{\mathbb{R}^n}] = 0$ in $H^n_{dR}(\mathbb{R}^n)$.
2. Show that there is a form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $d\alpha = \mathrm{vol}_{\mathbb{R}^n}$.
3. Determine a form α in 2. explicitly.
4. Show that such a form α is unique if one requires that it is $SO(n)$-invariant and determine this unique solution explicitly.
5. Show that there is no $GL(n,\mathbb{R})_0$- or $O(n)$-invariant form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $d\alpha = \mathrm{vol}_{\mathbb{R}^n}$. Here $GL(n,\mathbb{R})_0 := \{A \in GL(n,\mathbb{R}) \mid \det(A) > 0\}$.
6. Show that there is no bounded (i.e. the coefficients of α with respect to the standard basis $dx^1 \wedge \cdots \wedge dx^{n-1}, \dots$ are bounded) form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $d\alpha = \mathrm{vol}_{\mathbb{R}^n}$.

5. We fix positive numbers $n_1, \dots, n_r \in \mathbb{N}$ and consider the manifold

$$M := \prod_{i=1}^{r} S^{n_r} \ .$$

Construct an injective ring homomorphism

$$\mathbb{R}[x_1, \dots, x_r]/(x_1^2, \dots, x_r^2) \to H^*_{dR}(M) \ ,$$

where $|x_i| = n_i$.

6. Fix an integer n. Multiplication by n on $\mathbb{R}^n$ induces a map $f : T^n \to T^n$ given by $f([x]) := [nx]$. Show that $f^* : H^k_{dR}(T^n) \to H^k_{dR}(T^n)$ is multiplication by n^k (use without proof that $b_k(T^n) = \left(\begin{array}{c} n \\ k \end{array}\right)$).

7. We consider the exact sequence of abelian groups

$$0 \to (\mathbb{Z} \stackrel{9}{\to} \mathbb{Z}) \stackrel{(3,1)}{\to} (\mathbb{Z} \stackrel{3}{\to} \mathbb{Z}) \to (\mathbb{Z}/3\mathbb{Z} \to 0) \to 0 \ .$$

Calculate the long exact cohomology sequence explicitly.

8. Let M be a compact manifold with boundary $i : N \to M$.

1. Show that $i^* : \Omega(M) \to \Omega(N)$ is surjective.
2. Define $\Omega(M, N) := \mathtt{ker}(i^*)$. Furthermore, let $\Omega_c(M \setminus N)$ be the forms with compact support in $M \setminus N$. Show that $\Omega_c(M \setminus N) \subset \Omega(M, N)$.
3. Define $H^*_{dR}(M, N) := H^*(\Omega(M, N))$, and let $\partial : H^*_{dR}(N) \to H^*(\Omega(M, N))$ be the boundary operator for the sequence

$$0 \to \Omega(M, N) \to \Omega(M) \to \Omega(N) \to 0 \ .$$

   Show that for $x \in H^*_{dR}(N)$ the class ∂x has a representative in $\Omega_c(M \setminus N)$.

* Show that $\Omega_c(M \setminus N) \to \Omega(M, N)$ is a quasi-isomorphism.

9. Calculate the Betti numbers of the lens space $L(p, q)$ for coprime integers p, q.

10. For $0 \leq k < n$ we consider the usual embeddings $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$. Calculate the Betti numbers of $\mathbb{R}^n \setminus S^k$.

11. Let G be a connected Lie group and $H \subseteq G$ be a finite subgroup. Show that $\pi : G \to G/H$ induces an isomorphism $\pi^* : H^*_{dR}(G/H) \to H^*_{dR}(G)$.

12. Calculate the Betti numbers of a closed oriented surface Σ_g of genus g. Represent Σ_g as the sum of a 2-sphere with $2g$ discs removed and g copies of $[0, 1] \times S^1$.

13. Calculate all pages of the spectral sequence for the chain complex

$$\mathbb{Z} \stackrel{x \mapsto 2x}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$

with the filtration

$$\begin{pmatrix} \mathcal{F}^*C^0 \\ \mathcal{F}^*C^1 \\ \mathcal{F}^*C^2 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ \mathbb{Z}/2\mathbb{Z} \end{pmatrix} \subseteq \begin{pmatrix} 0 \\ 2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.$$

14. Let $(E_r^{*,*}, d_r)_{r\geq 0}$ be a bigraded spectral sequence consisting of finite-dimensional $\mathbb{R}$-vector spaces such that $\{(p,q) \in \mathbb{Z}^2 \mid E_1^{p,q} \neq 0\}$ is a finite set. For $r \geq 1$ we define

$$\chi_r := \sum_{(p,q)\in\mathbb{Z}^2} (-1)^{p+q} \dim E_r^{p,q} \in \mathbb{Z}$$

Show that $\chi_r = \chi_1$ for all $r \geq 1$.

15. We consider the manifold $M := \mathbb{R}^2 \setminus \mathbb{Z}^2$. Show that $H^1_{dR}(M)$ is infinite-dimensional.

16. For an open covering $\mathcal{U}$ of M one can consider the Čech complex $\check{C}^*(\mathcal{U}, \mathbb{C}^*)$ of $\mathbb{C}^*$-valued functions (the sums in the definition given in the course lecture are interpreted using the group structure of $\mathbb{C}^*$). Show that a cocycle $c \in \check{C}^1(\mathcal{U}, \mathbb{C}^*)$ is exactly the cocyle datum (as in Analysis IV) needed to define a one-dimensional complex vector bundle $L \to M$. Show that L is trivializable if and only if c is a boundary.

17. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of chain complexes. Consider the filtration of B given by $\mathcal{F}^0B := B$, $\mathcal{F}^1B := A$ and $\mathcal{F}^2B := 0$. Find the precise relation between the spectral sequence of $(B, \mathcal{F})$ and the long exact cohomology sequence associated to the short exact sequence above.

18. Let $\mathcal{U} := \{U, V\}$ be a covering of a manifold M by two open subsets. Find the precise relation between the Čech-de Rham spectral sequence and the Mayer-Vietoris sequence.

19. Show that the cohomology of the Stiefel manifolds $H^*_{dR}(V(k, \mathbb{R}^n))$ is finite-dimensional.

20. Let k be a ring. Let C, D be chain complexes of k-modules. We assume that $H^*(D)$, $H^*(C)$ and C^* are free. Show that

$$H^*(C) \otimes H^*(D) \cong H^*(C \otimes D) \ .$$

21. A knot is an embedding of S^1 as a closed submanifold $K \subset S^3$. Calculate the Betti numbers $b^i(S^3 \setminus K)$, $i \in \mathbb{Z}$, of the knot complement.

22. Let M and M' be compact manifolds with boundaries N and N', respectively. Furthermore, let $f : N \to N'$ be a diffeomorphism. Show that $\chi(M \cup_f M') = \chi(M) + \chi(M') - \chi(N)$.

23. Let M an M' be closed oriented manifolds of dimensions $4m$ and $4m'$. Show that $\mathtt{sign}(M \times M') = \mathtt{sign}(M)\mathtt{sign}(M')$.

Show further that this formula holds in general if we just assume that $\dim(M) + \dim(M') \equiv 0(4)$ and we define the signature of a manifold of dimension not divisible by 4 to be zero.

24. Let M be an oriented, connected, and non-compact manifold of dimension n. Show that $H^n_{dR}(M) \cong 0$.

25. We consider a vector bundle $V \to M$. Let $\mathtt{Tr} : \Omega(M, \mathtt{End}(V)) \to \Omega(M)$ be given on elementary tensors by $\mathtt{Tr}(\omega \otimes \Phi) := \omega \mathtt{Tr}(\Phi)$. Show that for a connection on V the form $\mathtt{Tr}(R^\nabla) \in \Omega^2(M)$ is closed.

26. We consider the standard inclusion $f : S^2 \to \mathbb{R}^3$. We have a trivial vector bundle $f^*T\mathbb{R}^3$ with a metric and a trivial connection. We consider TS^2 as a subbundle via $df : TS^2 \hookrightarrow f^*T\mathbb{R}^3$. Let $P : f^*T\mathbb{R}^3 \to TS^2$ be the orthogonal projection. Show that $\nabla := P\nabla^{triv}_{|\Gamma(S^2,TS^2)}$ is a connection on TS^2. Calculate $\int_{S^2} \mathtt{Tr}(R^\nabla)$.

27. We consider a closed oriented surface Σ_g of genus g. Describe explicitly a basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $H^1_{dR}(\Sigma_g)$ such that $\langle \alpha_i, \alpha_j \rangle = 0$, $\langle \beta_1, \beta_j \rangle = 0$ and $\langle \alpha_i, \beta_j \rangle = \delta_{i,j}$ for all $i, j \in \{1, \ldots, g\}$.

28. A link with k components is an embedding

$$\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{k\,\text{copies}} \to S^3 \ .$$

Let U be an open tubular neighbourhood of the link and $M := S^3 \setminus U$. Determine an explicit basis of $H_{dR}(M)$ and $H_{dR}(M, \partial M)$ and calculate the corresponding matrix of the pairing $H_{dR}(M) \otimes H_{dR}(M, \partial M) \to \mathbb{R}$.

29. Let $E \to B$ be a locally trivial fibre bundle and $\omega \in \Omega^p(E)$ be a closed form. Then we consider the section

$$[\omega]_{E/B} \in \Gamma(B, \mathcal{H}^p(E/B)) \ , \quad [\omega]_{E/B}(b) := [\omega_{|E_b}] \in H^p_{dR}(E_b) = \mathcal{H}^p(E/B)_b \ .$$

Show that $\nabla[\omega]_{E/B} = 0$, where ∇ is the Gauss-Manin connection of $\mathcal{H}^p(E/B)$.

30. Let $A \in SL(2, \mathbb{Z})$. We consider the trivial two-dimensional bundle $V := \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ with its trivial connection ∇^{triv}. We define the action of $\mathbb{Z}$ on this bundle by

$$n(t, v) := (t + n, Av) \ , \quad n \in \mathbb{Z} \ , \quad (t, v) \in \mathbb{R} \times \mathbb{R}^2 \ .$$

1. First show that this action preserves the connection.
2. Show that there exists a flat bundle $\mathbf{W} = (W, \nabla)$ on S^1 such that $\Omega(S^1, W) \cong \Omega(\mathbb{R}, V)^{\mathbb{Z}}$ (the space of $\mathbb{Z}$-invariant elements) so that ∇ is the restriction of ∇^{triv}.
3. Calculate $H^*_{dR}(S^1, \mathbf{W})$ explicitly.

   Hint: Complexify first and then use the Jordan decomposition of A)

31. Let M, N be manifolds and $\mathbf{V}$ be a flat vector bundle on M. Show the Künneth formula

$$H_{dR}(N \times M, \mathtt{pr}_M^* \mathbf{V}) \cong H_{dR}(N) \otimes H_{dR}(M, \mathbf{V})$$

under the condition that at least one of M or N is compact.

32. Let $\mathbf{V}$ and $\mathbf{W}$ be flat bundles on a manifold M and $V \to W$ be an injective bundle map which preserves connections. Show that the quotient bundle W/V has an induced flat connection and that there is a long exact sequence

$$\dots H^{p-1}_{dR}(M, \mathbf{W}/\mathbf{V}) \to H^p_{dR}(M, \mathbf{V}) \to H^p_{dR}(M, \mathbf{W}) \to H^p_{dR}(M, \mathbf{W}/\mathbf{V}) \to \dots \ .$$

33. Let $(\mathbf{V}, \nabla)$ be a flat vector bundle over a closed manifold E. Show that $H^q_{dR}(E, \mathbf{V})$ is finite dimensional for every $q \in \mathbb{Z}$. Let $f : E \to B$ be a surjective submersion. Show further, that

$$\mathcal{H}^q(E/B, \mathbf{V}) := \bigsqcup_{b \in B} H^q_{dR}(E_b, \mathbf{V}_{|E_b})$$

has a natural structure of a flat vector bundle over B.

34. Let $A \in SL(2, \mathbb{Z})$ and $f_A : T^2 \to T^2$ be the corresponding automorphism. Calculate the de Rham cohomology of the mapping torus T_{f_A}.

35. Let G be a compact Lie group with multiplication $\mu : G \times G \to G$. We consider the map

$$\Delta : H_{dR}(G) \stackrel{\mu^*}{\to} H_{dR}(G \times G) \stackrel{\text{Künneth}}{\cong} H_{dR}(G) \otimes H_{dR}(G) \; .$$

An element $x \in H_{dR}(G)$ is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$ and group-like if $\Delta(x) = x \otimes x$.

Determine the primitive and group-like elements in the de Rham cohomology of T^2 and $SU(2)$.

36. Let M be a manifold with boundary N and $\partial : H^*_{dR}(N) \to H^{*+1}_{dR}(M, N)$ be the boundary operator of the long exact sequence of the pair (M, N). Show the following identities for $x \in H_{dR}(N)$ and $y \in H_{dR}(M)$:

1. $\partial(x \cup y_{|N}) = \partial(x) \cup y$ for $x \in H_{dR}(N)$ and $y \in H_{dR}(M)$.
2. $\int_M \partial(x) \cup y = \int_N x \cup y_{|N}$ (if M is compact and oriented).

37. Let $E \to B$ be a trivial fibre bundle with compact fibres. Calculate the LSSS and relate it with the Künneth-formula.

38. Calculate the LSSS of the mapping torus $T_{f_A} \to S^1$, where $f_A : T^2 \to T^2$ is associated to $A \in SL(2, \mathbb{Z})$. Compare with Aufgabe 2., Blatt 9.

39. Let $E \to B$ be a fibre bundle with fibre S^1 and simply connected and connected base B. A normalized fibrewise volume form is an element $\omega \in \Omega^1(E)$ such that $\omega_{|E_b}$ is a volume form and $\int_{E_b} \omega = 1$ for all $b \in B$. Show:

1. There exists a normalized fibrewise volume form.
2. If there exists a normalized fibrewise volume form which is in addition closed, then the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ of the LSSS vanishes.

* Show the converse of 2.

40. Is S^3 diffeomorphic to a mapping torus T_f of an automorphism $f : \Sigma \to \Sigma$ for some closed surface Σ?

41. We consider the map

$$f : \mathbb{CP}^1 \to \mathbb{CP}^2 \ , \quad [z_0 : z_1] \mapsto [z_0^2 : z_0 z_1 : z_1^2] \ .$$

Calculate the number $\int_{\mathbb{CP}^1} f^* c_1$.

42. Use the Künneth formula in order to identify

$$H^*_{dR}(\mathbb{CP}^n \times \mathbb{CP}^m) \cong \mathbb{R}[a,b]/(a^{n+1}, b^{m+1}) \ .$$

Consider the map

$$p : \mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{(m+1)(n+1)-1} \ ,$$
$$([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \mapsto [x_1 y_1 : x_1 y_2 : \dots : x_1 y_m : x_2 y_1 : \dots : x_n y_m] \ .$$

Calculate $p^* c^k \in \mathbb{R}[a,b]/(a^{n+1}, b^{m+1})$ explicitly.

43. Let $V \to M$ be a real vector bundle. Show that $c_1(V \otimes \mathbb{C}) = 0$.

44. Show that there is no non-trivial characteristic class of degree 1 for complex vector bundles.

45. Calculate the de Rham cohomology of $SO(n)$ for $n = 2, 3, 4$ using the LSSS for the bundles $SO(n+1) \to S^n$ with fibre $SO(n)$. Discuss also the case $n = 5$, if possible.

46. We have found an isomorphism $H_{dR}(U(n)) \cong \mathbb{R}[u_1, \dots, u_{2n-1}]$, where u_{2k-1} primitive and of degree $2k-1$. Calculate the action of the inversion map $I : U(n) \to U(n)$, $I(g) := g^{-1}$ explicitly.

47. Calculate the first Chern class of $\Lambda^k_{\mathbb{C}} T^*\mathbb{CP}^n$. Note that we take the alternating power in the sense of complex vector bundles.

48. Let M be a manifold with a free action of $U(n)$. We consider the LSSS of the bundle $M \to M/G$ with fibre $U(n)$. Show that the element $u_{2k-1} \in E_2^{0,2k-1} \cong H^{2k-1}_{dR}(U(n))$ is $2k-1$-transgressive, i.e. belongs to the kernel of the differentials d_ℓ of the LSSS for all $\ell = 2, \dots, 2k-1$.

Hint: Use the sequence of sphere bundles

$$M \xrightarrow{S^1} M/U(1) \xrightarrow{S^2} \cdots \xrightarrow{S^{2n-2}} M/U(n-1) \xrightarrow{S^{2n-1}} M/U(n)$$

and the explicit description of u_{2k-1} given in the course.

49. Let $E \to B$ be a complex vector bundle. Show that $c_i(E^*) = (-1)^i c_i(E)$ for every $i \in \mathbb{N}$. Deduce that for a real vector bundle $V \to B$ we have $c_i(V \otimes \mathbb{C}) = 0$ for odd $i \in \mathbb{N}$.

50. Calculate $c(TGr(2, \mathbb{C}^4))$ explicitly as a polynomial in $c_1(L)$ and $c_2(L)$, where $L \to Gr(2, \mathbb{C}^4)$ is the tautological bundle.

51. Let $k, n \in \mathbb{N}$, $k \leq n$. Find for every $i \in \mathbb{N}$, $1 \leq i \leq k$ a closed oriented manifold M of dimension $2i$ and a map $f : M \to Gr(k, \mathbb{C}^n)$ such that $\int_M f^* c_i(L) \neq 0$.

52. Let $L \to Gr(3, \mathbb{C}^{20})$ be the tautological bundle which is considered as a subbundle of the n-dimensional trivial bundle. Let $L^\perp \to Gr(3, \mathbb{C}^{20})$ be its orthogonal complement. Calculate $c_3(L^\perp)$ explicitly as a polynomial in $c_1(L), c_2(L), c_3(L)$.

53. Let $f : M \to N$ be a map between closed connected oriented manifolds of the same dimension. Show that $f^* : H_{dR}(N) \to H_{dR}(M)$ is injective if and only if $\deg(f) \neq 0$.

54. Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial of degree k such that $p(0) = 1$. Calculate the degree of

$$f : \mathbb{CP}^n \to \mathbb{CP}^n \ , \quad [z_0 : \dots : z_n] \mapsto [z_0^k p(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) : z_1^k \dots : z_n^k] \ .$$

55. We consider an iterated bundle $E \to G \to B$ with closed fibres. Show that the choice of fibrewise orientations for two of the three bundles $E \to G$, $G \to B$ and $E \to B$ induces an orientation on the third such that

$$\int_{E/B} = \int_{G/B} \circ \int_{E/G}$$

holds.

56. We consider the manifold $\mathbb{F}(\mathbb{C}^n)$ of complete flags $(V_1 \subset \cdots \subset V_n)$ in $\mathbb{C}^n$. For $i = 1, \ldots, n$ let $x_i \in H^2_{dR}(\mathbb{F}(\mathbb{C}^n))$ be the first Chern class of the quotient of tautological bundles $V_i/V_{i-1} \to \mathbb{F}(\mathbb{C}^n)$, where we set $V_0 := 0$. Calculate the real number

$$\int_{\mathbb{F}(\mathbb{C}^n)} x_n^{n-1} \cup x_{n-1}^{n-2} \cup \cdots \cup x_3^2 \cup x_2 \ .$$