Duality of abelian groups stacks and T-duality

U. Bunke

September 6, 2006

String theory origin of T-duality

String theory :

Space with fields $\longrightarrow$ (susy) conformal field theory (CFT)

T-duality :

$$\begin{array}{ccc} \text{Space} & \xrightarrow{\text{Type IIA string theory}} & CFT \\ \big\downarrow {\scriptstyle ?} & & \big\downarrow {\scriptstyle \text{CFT-}T\text{-duality}} \\ \text{Space} & \xrightarrow{\text{Type IIB string theory}} & CFT \end{array}$$

? - space level T-duality

Math. Aspects : Mirror symmetry, Fourier-(Mukai) transform, **Pontrjagin-(Takai) duality**, Hitchin's generalized geometry

Topology of T-duality - history

$$\{\text{Space with fields}\} \overset{T\text{-duality}}{\leftrightarrow} \{\text{Space with fields}\}$$

$$\downarrow \text{forget geometry} \downarrow$$

$$\{\text{underlying top. space}\} \overset{\text{top. } T\text{-duality}}{\longleftrightarrow} \{\text{underlying top. space}\}$$

most studied for $\mathbb{T}^n$-principal bundles with B-field background

1. Bouwknegt, Evslin, Hannabuss, Mathai ($n = 1$) (2003)
2. Bunke, Schick ($n = 1$) (2004)
3. Mathai, Rosenberg ($n = 2$) (2004)
4. Bouwknegt, Evslin, Hannabuss, Mathai,... ($n \geq 1$) (2004-...)
5. Bunke, Rumpf, Schick ($n \geq 1$) (2005)
6. Bunke, Schick ($n = 1$, non-free actions of $\mathbb{T}$, orbifolds) (2004)

Basic objects over base B

pairs :

$$H \xrightarrow[\mathbb{T}\text{-gerbe}]{\mathcal{B}\mathbb{T}} E \xrightarrow[\mathbb{T}^n\text{-bundle}]{\mathbb{T}^n} B$$

Explanation of gerbe : topological background of B-field.
Alternative ways of realization :

- noncommutative geometry : bundle of algebras of compact operators (Mathai, Rosenberg)
- classical differential geometry : three form (Bouwknegt, Evslin, Hannabuss, Mathai, ...)
- homotopy theory : $E \to K(\mathbb{Z}, 3)$ (Bunke, Schick)
- topological stacks : map $H \to E$ of topological stacks with fibre $B\mathbb{T}$ (this talk)

The problem

Given (E, H).

What is a T-dual pair ? (Mathai,...):
The Buscher rules give the local transformation rules for the fields which are classical geometric objects. Topological T-duality is designed such that these Buscher can be realized globally.

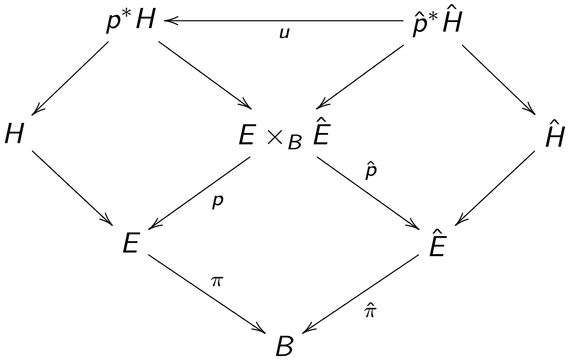
Does (E, H) admit a T-dual pair $(\hat{E}, \hat{H})$?
Yes, if $n = 1$. Under additional conditions, if $n \geq 2$.

Is the T-dual $(\hat{E}, \hat{H})$ unique?
Yes, if $n = 1$. In general no for $n \geq 2$.

Solution via T-duality triples

(Bunke, Schick)

$$((E, H), (\hat{E}, \hat{H}, u)$$



- (E, H) admits a T-dual iff it admits an extension to a T-duality triple.
- Classification of T-duality triples extending (E, H) leads to classification of T-dual pairs.

Solution via C^*-algebras

(Mathai-Rosenberg)

- Realize gerbe $H \to E$ as bundle of algebras of compact operators
- (E, H) admits T-dual if and only if $\mathbb{T}^n$-action on E amits lift to $\mathbb{R}^n$-action on H with trivial Mackey obstruction.
- Let $A := C(E, H)$, $\hat{A} := C(\hat{E}, \hat{H})$. Then

$$\hat{A} \cong \mathbb{R}^n \ltimes C(E, H)$$

- different $\mathbb{R}^n$-actions correspond to different T-duals

Connection with T-duality triples : (A. Schneider (Göttingen))

Solution via duality of abelian group stacks

proposed by T Pantev

worked out in detail by : Bunke, Schick, Spitzweck, Thom (2006)

Abelian groups stacks

- Site $\mathbf{S}$: category of compactly generated locally contractible spaces, open coverings (e.g. topological submanifolds of $\mathbb{R}^\infty$)
- Abelian group stack : stack on $\mathbf{S}$ with abelian group structure (Precise notion : Strict Picard stack (Deligne, SGA 4 XVIII)),
$$\mathtt{PIC}(\mathbf{S})$$
- isom. classes of objects and automorphisms of $P \in \mathtt{PIC}(\mathbf{S})$:
$$H^0(P), H^{-1}(P) \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}$$

Classification : $A, B \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}$

- $\mathtt{Ext}^2_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(A, B) \cong \{P \in \mathtt{PIC}(\mathbf{S}) \mid H^0(P) \cong A,\ H^{-1}(P) \cong B\}/\sim$

Pontrjagin duality for locally compact abelian groups

Topological abelian group G gives sheaf $\underline{G} \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}$:

$$\mathbf{S} \ni U \mapsto C(U, G) \ .$$

For $G, H \in \mathbf{S}$: $\mathtt{Hom}_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(\underline{G}, \underline{H}) \cong \underline{\mathtt{Hom}(G, H)}$

Dual sheaf: $D(F) := \underline{\mathtt{Hom}}_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}})$, $F \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}$

Pontrjagin duality : (for $G \in \mathbf{S}$)

$$\underline{G} \xrightarrow{\sim} D(D(\underline{G}))$$

The dual of an abelian group stack

Define $\mathcal{B}\underline{\mathbb{T}} \in \mathtt{PIC}(\mathbf{S})$ such that

$$H^0(\mathcal{B}\underline{\mathbb{T}}) \cong 0 \ , \quad H^{-1}(\mathcal{B}\underline{\mathbb{T}}) \cong \underline{\mathbb{T}}$$

For $P, Q \in \mathtt{PIC}(\mathbf{S})$ we have

$$\underline{\mathtt{HOM}}_{\mathtt{PIC}(\mathbf{S})}(P, Q) \in \mathtt{PIC}(\mathbf{S})$$

Definition: Dual group stack:

$$D(P) := \underline{\mathtt{HOM}}_{\mathtt{PIC}(\mathbf{S})}(P, \mathcal{B}\underline{\mathbb{T}})$$

Pontrjagin duality for abelian group stacks

Theorem : Assume that $P \in \mathtt{PIC}(\mathbf{S})$, $H^i(P) \cong \underline{\mathbb{T}}^{n_i} \times \underline{\mathbb{R}}^{n_i} \times \underline{F}_i$, F_i - finitely generated

1. $H^0(D(P)) \cong D(H^{-1}(P))$, $\quad H^{-1}(D(P)) \cong D(H^0(P))$
2. $P \xrightarrow{\sim} D(D(P))$
3. $\mathcal{D} : \mathtt{Ext}^2_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(B, A) \to \mathtt{Ext}^2_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(D(A), D(B))$

$$[D(P)] = \mathcal{D}([P])$$

No counter example with $H^i(P) \cong \underline{G}_i$ with $G_i \in \mathbf{S}$ locally compact!
Main technical result:

$$\underline{\mathtt{Ext}}^k_{\mathtt{Sh}_{\mathtt{Ab}}\mathbf{S}}(H^i(P), \underline{\mathbb{T}}) = 0 \ , \quad k = 1, 2$$

Application to T-duality : Pairs and group stacks

$B \in \mathbf{S}$

$$\text{principal } \mathbb{T}^n\text{-bundle } E \to B$$

$$\downarrow \text{sheaf of sections}$$

$$\text{sheaf of } \underline{\mathbb{T}}_{|B}\text{-torsors}$$

$$\uparrow \text{ preimage of } 1$$

$$0 \to \underline{\mathbb{T}}^n_{|B} \to \mathcal{E} \to \underline{\mathbb{Z}}_{|B} \to 0$$

$P \in \mathtt{PIC}(\mathbf{S}/B)$ with $H^0(P) \cong \mathcal{E}$, $H^{-1}(P) \cong \underline{\mathbb{T}}_{|B}$ defines pair

$$\begin{array}{ccc} H & \longrightarrow & P \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \underline{B \times \{1\}} & \longrightarrow & \underline{\mathbb{Z}}_{|B} \end{array}$$

We say that P extends E.

T-duality via abelian group stacks

Theorem: There is a bijection between the sets

$$\{\text{Extensions of } E \text{ to abelian group stacks}\}$$

$$\downarrow \text{construction}$$

$$\{\text{Extensions of } E \text{ to } T\text{-duality triples}\}$$

Construction

P extending $(E, H) \rightsquigarrow$ triple $((E, H), (\hat{E}, \hat{H}), u)$

$$\begin{array}{ccccccc}
\hat{H}^{op} & \longrightarrow & & & \tilde{\hat{H}} & \longrightarrow & D(P) \\
\downarrow & & & & \downarrow & & \downarrow {\scriptstyle \underline{\mathbb{T}}-gerbe} \\
\hat{E} & \xrightarrow{\texttt{can}} & R^{\hat{E}}_{\mathbb{Z}^n} & \xrightarrow{\cong} & R & \longrightarrow & \overline{D(P)} \\
 & \searrow & \downarrow & & \downarrow {\scriptstyle \underline{\mathbb{Z}^n}-gerbe} & & \downarrow \\
 & & B & = & \underline{B \times \{1\}} & \longrightarrow & \underline{\mathbb{Z}}_{|B}
\end{array} .$$

$R^{\hat{E}}_{\mathbb{R}^n} \longrightarrow B$: gerbe of $\mathbb{R}^n$-reductions of $\hat{E}$

$\mathtt{ev} : P \times D(P) \longrightarrow \mathcal{B}\underline{\mathbb{T}}_{|B}$ induces $u : \hat{p}^* \hat{H} \to p^* H$.