NONCOMMUTATIVE HOMOTOPY THEORY II

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2.1 Basic Definitions

2.1.1 G-C^*-algebras

G - a group

- BG category with one object $*$ and automorphisms G

Definition 2.1. *We define the category of G-C^*-algebras as $GC^*\mathbf{Alg}^{\mathrm{nu}} := \mathbf{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$.*

explicitly:

- objects: C^*-algebras A with action $\alpha : G \to \mathtt{Aut}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A)$

– write (A, α)

– $g \mapsto \alpha_g$

– $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all g, h in G

- morphisms: $f : (A, \alpha) \to (B, \beta)$

– $f : A \to B$ - morphism of C^* algebras

– condition: $f(\alpha_g a) = \beta_g f(a)$ for all g in G

this is good for discrete groups

- for topological group G: use topological enrichment to put continuity requirement

- BG is topologically enriched

– $\mathtt{Hom}_{BG}(*, *) \cong G$

- $C^*\mathbf{Alg}^{\mathrm{nu}}$ is topologically enriched

– $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B)$ has point-norm topology

- write $\mathbf{Fun}_c$ for functors in the enriched sense: continuous on topological mapping spaces

Definition 2.2. *For a topological group we define the category of G-C^*-algebras as $GC^*\mathbf{Alg}^{\mathrm{nu}} := \mathbf{Fun}_c(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$.*

explicitly:

- additional requirement: $G \ni g \mapsto \alpha_g(a) \in A$ is continuous for every a in A

note: $\alpha : G \to \mathtt{Aut}(A)$ is not necessarily continuous for the norm topology

2.1.2 First examples

trivial action:

- A in $C^*\mathbf{Alg}^{\mathrm{nu}}$

- set $\alpha_g := \mathtt{id}_A$ for all g in G

- get (A, α) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- often denoted by $\underline{A}$

X locally compact space

- $\rho : G \times X \to X$ continuous G-action

- $\alpha_g : C_0(X) \to C_0(X)$

- $(\alpha_g f)(x) := f(\rho_{g^{-1}}(x))$

- is continuous

– get $(C_0(X), \alpha)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

even better: Gelfand duality is topologically enriched

$\mathtt{Aut}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(C_0(X)) \cong \mathtt{Aut}_{\mathbf{Top}}(X)$

- compact open topology on $\mathtt{Aut}_{\mathbf{Top}}(X)$

- point-norm topology in $\mathtt{Aut}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(C_0(X))$

some warnings:

note: in general G does not act continuously on $C_b(X)$

Problem 2.3. *Show that the action of* $\mathbb{R}$ *on* $C_b(\mathbb{R})$ *is not continuous.*

- $G \to \mathtt{Aut}(C_0(X))$ is not norm continuous

Problem 2.4. *Let* T_u *be the translation by* u *in* $U(1)$*. Show that* $\|T_u - \mathtt{id}\| = 2$ *if* $u \neq 1$*.*

recall multiplier algebra $M(A)$ of A

- hast strict topology:

- $m_i \to m$ if $m_i a \to ma$ in norm for all a in A

$\rho : G \to U(M(A))$ homomorphism

- continuous for the strict topology

- define $\alpha : G \to \mathtt{Aut}(A)$

– $\alpha_g a := \rho_g a \rho_{g^{-1}}$

– $g \mapsto \alpha_g$ is continuous

- get (A, α) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

$\rho : G \to U(H)$ unitary representation of G on Hilbert space

- assume ρ is strongly continuous (will always be assumed)

– means: $(g, h) \mapsto \rho_g h$ is norm continuous for all h in H

Problem 2.5. *Recall that $B(H) = M(K(H))$. Show that the strict and the strong topology on $U(B(H))$ coincide.*

- hence ρ is strictly continuous

- for any G-invariant (under conjugation) subalgebra A of $K(H)$

- (A, α) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

– $\alpha_g a := \rho_g a \rho_{g^{-1}}$

Example 2.6. it is not natural to require that ρ is norm continuous

- $G \times X \to X$ continuous on locally compact space

– $L_g : X \to X$ action of g in G

- μ a G-invariant Radon measure

– recall Radon measure:

— finite on compact sets

— $\mu(C) = \inf_{C\subseteq U} \mu(U)$ (outer regular)

— $\mu(U) = \sup_{K\subseteq U} \mu(K)$ (inner regular on opens)

– means: $L_{g,*}\mu = \mu$ for all g in G

- $L^2(X,\mu)$ has unitary G-action

– $(\rho_g f)(h) := f(g^{-1}h)$

– unitary: $\int_G |f(g^{-1}h|^2\mu(h) = \int_G |f(h)|^2 L_{g,*}\mu(g) = \int_G |f(h)|^2\mu(g)$

– also notation: $L_{g,*}\mu(h) = \mu(gh)$

- $\rho : G \to U(L^2(X,\mu))$ is strongly continuous, but in general not norm continuous

Problem 2.7. *Show these assertions.*

□

2.1.3 Categorical properties of $GC^*\mathbf{Alg}^{\mathrm{nu}}$

recall: $C^*\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete

have forgetful functor $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$

Corollary 2.8. *The forgetful functor $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is conservative.*

Corollary 2.9. *For a discrete group G the category $GC^*\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete and $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ preserves limits and colimits.*

for a diagram $A : I \to C^*\mathbf{Alg}^{\mathrm{nu}}$

- limit or colimit is formed in $C^*\mathbf{Alg}^{\mathrm{nu}}$

- gets induced G-action

for topological group:

- $\mathtt{colim}_I\, A$ has induced G-action

- it is again continuous

Problem 2.10. *Show that the induced G-action on a colimit of G-C^*-algebras is continuous.*

Lemma 2.11. *For a topological group the category $GC^*\mathbf{Alg}^{\mathrm{nu}}$ is cocomplete and $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ preserves colimits.*

- $\mathtt{lim}_I\, A$ also has an induced G-action

– this is not always continuous

Example 2.12. $U(1)$ is a topological group

- $C(U(1))$ has actions α_n given by $(\alpha_{n,u} f)(v) := f(u^n v)$

- action on $\prod_{n\in\mathbb{N}}(C(S^1), \alpha_n)$ is not continuous

Problem 2.13. *Show this assertion.*

□

but finite limits are ok

Lemma 2.14. *$GC^*\mathbf{Alg}^{\mathrm{nu}}$ is finitely complete and $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ preserves limits.*

Problem 2.15. *Show Lemma 2.14.*

Proposition 2.16. *$GC^*\mathbf{Alg}^{\mathrm{nu}}$ has all products.*

Proof. $((A_i, \alpha_i))_{i\in I}$ family in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- form $\prod_{i\in I} A_i$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$

- get induced G-action α

- $\alpha_g := \prod_{i\in I} \alpha_{i,g}$

- $g \mapsto \alpha_g f$ is not continuous in general

- call f continuos if this is the case

$(\prod_{i\in I} A_i)^c$ subset of continuous elements

- observe: is G-invariant closed $*$-subalgbera

Problem 2.17. *Show this assertion.*

α_g^c - restriction of α_g to continuous elements

claim: $((\prod_{i\in I} A_i)^c, \alpha^c)$ represents products

check universal property:

$(f_i : (T,\beta) \to (A_i, \alpha_i))$ given

- induced map $f : T \to \prod_{i\in I} A_i$ is G-equivariant such that $\mathtt{pr}_i \circ f = f_i$

- takes values in continuous elements

- $\|\alpha_g f(t) - f(t)\| = \sup_{i\in I} \|\alpha_{i,g} f_i(t) - f_i(t)\| = \sup_{i\in I} \|f_i(\beta_g t - t)\| \leq \|\beta_g t - t\|$

– use that f_i is contractive for every i

□

Corollary 2.18. *For every topological group the category $GC^*\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete.*

G -topological

- G^δ - G with discrete topology

- (A, α) in $G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}$

define $A^c := \{f \in A \mid G \ni g \mapsto \alpha_g f \text{ is continuous}\}$

Lemma 2.19. *A^c is a sub-C^*-algebra and $\alpha_{|A^c}$ is continuous.*

Proof. f, f' in A^c implies that $f + \lambda f'$, ff' and $f*$ belong to A^c

- since operations of A are continuous

- α_g preserves A^c by associativity

A^c is closed $a_i \to a$, $a_i \in A^c$ implies $a \in A^c$

- $\|\alpha_g a - a\| \leq \|\alpha_g(a - a_i)\| + \|\alpha_g a_i - a_i\| + \|a_i - a\|$

– first chose i to make $\|a_i - a\|$ small

– then also $\|\alpha_g(a - a_i)\|$ is small independently of g

– then choose g to make $\|\alpha_g a_i - a_i\|$ small

□

(A, α)

Proposition 2.20. *Show that there is a right Bousfield localization*

$$\mathrm{Res}^G_{G^\delta} : GC^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows G^\delta C^*\mathbf{Alg}^{\mathrm{nu}} : (-)^c \ .$$

Proof. $\mathtt{Hom}_{GC^*\mathbf{Alg}^{\mathrm{nu}}}(A, B^c) \cong \mathtt{Hom}_{G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{Res}^G_{G^\delta} A, B)$

it is clear that $\mathtt{Hom}_{GC^*\mathbf{Alg}^{\mathrm{nu}}}(A, B^c) \subseteq \mathtt{Hom}_{G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{Res}^G_{G^\delta} A, B)$

given $f \in \mathtt{Hom}_{G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{Res}^G_{G^\delta} A, B)$

- claim f takes values in B^c

- $\alpha_g f(a) = f(\beta_g a)$

– use $g \mapsto \beta_g a$ is continuous □

the following are egeneral facts following from the Bousfield localization

Corollary 2.21. $GC^*\mathbf{Alg}^{\mathrm{nu}}$ *is complete and cocomplete. Colimits are calculated in* $G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}$ *and limits are given by the composition* $(\mathtt{lim}^{G^\delta C^*\mathbf{Alg}^{\mathrm{nu}}} \mathrm{Res}^G_{G^\delta}(-))^c$.

2.1.4 Two-categorical structure

$C^*\mathbf{Alg}^{\mathrm{nu}}$ has some two categorical structure

- $f, g : A \to B$

- could be conjugated by u in $M(B)$: $f = ugu^*$

- turns $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B)$ into a category $\mathbf{Fun}(A, B)$

- composition of 2-morphism u with 1-morphism h is only partially defined: $h \circ u := M(h)(u)$

– needs h to be essential

$(A, \alpha), (B, \beta)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- G acts on $\mathbf{Fun}(A, B)$ by conjugation

- $g^* f := \beta_g^{-1} \circ f \circ \alpha_g$

$f : (A, \alpha) \to (B, \beta)$

- f can be equivariant

- $f \in \mathbf{Fun}(A, B)^G$ - one-categorical invariants

- $g^* f = f$

– $f \circ \alpha_g = \beta_g \circ f$

could also require $f \in \mathbf{Fun}(A, B)^{hG}$ - two categorical invariants

– f is weakly equivariant:

– f extends to pair (f, ρ)

— $\rho : G \to U(M(B))$ strictly continuous

— cocylcle relation: $\beta_h(\rho_g)\rho_h = \rho_{hg}$

— $g^* f = \rho_g \cdot f \cdot \rho_g^*$ for all g in G

– $\rho_g : f \xrightarrow{\cong} g \cdot f$

2.1.5 Tensor products

consider ? in $\{\min, \max\}$

$- \otimes_? - : C^*\mathbf{Alg}^{\mathrm{nu}} \times C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is enriched bifunctor

- get induced tensor product $- \otimes_? - : GC^*\mathbf{Alg}^{\mathrm{nu}} \times GC^*\mathbf{Alg}^{\mathrm{nu}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}$

Corollary 2.22. *$\otimes_?$ equips $GC^*\mathbf{Alg}^{\mathrm{nu}}$ with a symmetric monoidal structure.*

the tensor products inhertis the exactenss properties from the non-equivariant case

- $\otimes_{\max}$ preserves exact sequences

- $\otimes_{\min}$ preserves inclusions

2.2 Induction and Restriction

additional richnesss of equivariant theory comes from change of group functors

2.2.1 Restriction

$\phi : H \to G$ continuous homomorphism

get restriction functor

- $\phi^* : GC^*\mathbf{Alg}^{\mathrm{nu}} \to HC^*\mathbf{Alg}^{\mathrm{nu}}$

- $\phi^*(A, \alpha) := (A, \alpha \circ \phi)$

write often $\mathrm{Res}^G_H := \phi^*$ - in particular if ϕ is inclusion of a subgroup

forgetful functor $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is special case

2.2.2 Induction

assume:

- G locally compact

- $H \to G$ inclusion of closed subgroup

– G/H - locally compact space

A in $HC^*\mathbf{Alg}^{\mathrm{nu}}$ with H-action α

- consider space of bounded continuous functions $f : G \to A$ such that:

– $f(gh) = \alpha_{h^{-1}} f(g)$ for all h in H

– $\mathtt{pr}_{G/H}(\mathtt{supp}(f))$ is compact

- form closure wr.t. norm $\|f\| := \sup_{g\in G} \|f(g)\|$ in $C_b(G, A)$

- denote resulting C^*-algebra by $\mathtt{Ind}_H^G(A)$

- has continuous G-action $(\rho_g f)(g') := f(g^{-1}g')$

continuity not completely obvious: $\mathtt{supp}(f)$ is not compact on G in general

Problem 2.23. *Show continuity of G-action*

extend $\mathtt{Ind}_H^G$ to morphisms:

$\phi : A \to A'$

- define $\mathtt{Ind}_H^G(f) : \mathtt{Ind}_H^G(A) \to \mathtt{Ind}_H^G(A')$

– $\mathtt{Ind}(\phi)(f) := \phi \circ f$

Definition 2.24. *The functor $\mathtt{Ind}_H^G : HC^*\mathbf{Alg}^{\mathrm{nu}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}$ is called the induction functor.*

Example 2.25.

$C_0(G) \cong \mathtt{Ind}_1^G(\mathbb{C})$

$C_0(G/H) \cong \mathtt{Ind}_H^G(\underline{\mathbb{C}})$ □

H can be open and closed

- the connected component of G

- any subgroup if G discrete

- a clopen subgroup if G totally disconnected, e.g. $\mathbb{Z}_p$

have natural transformation

$b : \mathtt{id} \to \mathrm{Res}_H^G \circ \mathtt{Ind}_H^G$

- $b_A : A \to \mathrm{Res}_H^G(\mathtt{Ind}_H^G(A))$

- $b_A(a)(g) := \begin{cases} \alpha_{h^{-1}} a & h \in H \\ 0 & else \end{cases}$

looks like unit of adjunction, no obvious counit $\mathtt{Ind}_H^G \circ \mathrm{Res}_H^G(A)) \to A$

2.2.3 Coinduction

assume: G/H is compact or G discrete

consider again subspace $C_b(G, A)^H := \{f \in C_b(G, A) \mid (\forall h \in H \mid \alpha_h f(gh) = f(g))\}$

- has G-action by left-regular representation

- $\mathrm{Coind}_H^G(A) := (C_b(G, A)^H)^c$ - continuous vectors

- $\phi : A \to B$ homomorphism

– induces $\mathrm{Coind}_H^G(\phi) : \mathrm{Coind}_H^G(A) \to \mathrm{Coind}_H^G(A)$, $f \mapsto \phi \circ f$

get coinduction functor $\mathrm{Coind}_H^G : HC^*\mathbf{Alg}^{\mathrm{nu}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}$

- if G/H is compact, then $\mathtt{Ind}_H^G = \mathrm{Coind}_H^G(A)$

- have natural transformation

- $c : \mathrm{Res}_H^G \circ \mathrm{Coind}_H^G \to \mathtt{id}$

– $c_A(\mathrm{Res}_H^G(\mathrm{Coind}_H^G(A)) \to A$, $f \mapsto f(e)$

looks like counit of an adjunction

- indeed have unit $e : \mathrm{Coind}_H^G \circ \mathrm{Res}_H^G \to \mathtt{id}$

– $e_A : A \to \mathrm{Coind}_H^G(\mathrm{Res}(A))$

– $e_A(a)(g) := \alpha_{g^{-1}} a$

Proposition 2.26. *We have an adjunction*

$$\mathrm{Res}_H^G : GC^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows HC^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{Coind}_H^G \ .$$

Problem 2.27. *Show Proposition 2.26*

2.2.4 multiplicative induction

Z - finite G-set

- can define $A^{\otimes Z} := \bigotimes_Z A$

- get G-action by permutations of tensor factors

- $A^{\otimes Z} \in GC^*\mathbf{Alg}^{\mathrm{nu}}$

for unital A can assume Z infinite

- for finite subset F of Z consider $\bigotimes_F A$

- for $F \to F'$ inclusion

– use unit to define $\bigotimes_F A \to \bigotimes_{F'} A$

– $\otimes_{f\in F} a_f \mapsto \otimes_{f\in F} a_f \otimes \otimes_{x\in F'\setminus F} 1_A$

– $\bigotimes_Z A := \mathtt{colim}_{F\subseteq Z\ ,|F<\infty|} \bigotimes_F A$

– get G-action by permutation of tensor factors

$\bigotimes_{\mathbb{Z}} \mathtt{Mat}_2(\mathbb{C})$ - spin chain

2.3 Crossed products

2.3.1 Haar measures

X - locally compact space

- μ - Radon measure

– properties:

— finite on compact sets

— $\mu(C) = \inf_{C\subseteq U} \mu(U)$ (outer regular), U runs over open subsets

— $\mu(U) = \sup_{K \subseteq U} \mu(K)$ (inner regular on opens), K runs over compact subsets

- μ determined by the functional $C_c(X) \to \mathbb{C}$

– $f \mapsto \int_X f(x)\mu(x)$

$\phi : X \to X'$ proper map

- $\phi^* : C_c(X') \to C_c(X)$

- ϕ_* - push-forward of measures

– defining relation: $\int_{X'} f(x')(\phi_*\mu)(x) = \int_X f(\phi(x))\mu(x)$

G - locally compact group

- μ - Radon measure on G

$L_{g,*}\mu$

- say μ is left invariant if $L_{g,*}\mu = \mu$

– means for all f in $C_c(G)$ and g in G

$$\int_G f(g^{-1}h)\mu(h) = \int_G f(h)\mu(h)$$

Definition 2.28. *A non-zero left invariant Radon measure on G is called a Haar measure.*

Theorem 2.29. *On G there is a unique (up normalization) Haar measure on G.*

Remark 2.30. have natural normalization in some cases:

- for compact G: $\int_G \mu(g) = 1$

- for infinite discrete groups: $\mu(\{e\}) = 1$ □

Example 2.31.

G discrete: counting measure: $\sum_{g \in G} \delta_g$ is a Haar measure

$\mathbb{R}^n$ - Lebesgue measure is a Haar measure

G - a Lie group

- choose $\mathrm{vol} \in \Lambda^{\max}\mathfrak{g}^*$

– extends uniquely to left invariant volume form $(L^*_{g^{-1}}\mathrm{vol})(g) := \mathrm{vol}$

- defines Haar measure by $\int_G f(g)\mu(g) = \int_{G,or} f(g)\mathrm{vol}(g)$

□

μ - Haar measure

- in general μ is not right invariant

- $\int_G f(h)R_{g,*}\mu(h) = \int_G f(hg)\mu(h)$

- $R_{g,*}\mu$ is left invariant, Radon

– by uniqueness of Haar measure: there exists $\Delta(g)$ in $\mathbb{R}^+$ such that $R_{g,*}\mu = \Delta(g)\mu$

Proposition 2.32. $\Delta : G \to \mathbb{R}^*_+$ *is a continuous homomorphism.*

Example 2.33.

G is called unimodular if $\Delta = 1$

- compact groups

- discrete groups

- abelian groups

- for a Lie group: if $\det \mathrm{Ad} : G \to \mathtt{Aut}(\mathfrak{g}) \to \mathbb{R}^*$ is constant 1 □

Example 2.34. Consider $ax+b$-group $\mathbb{R} \rtimes \mathbb{R}^*$

- determine Haar measure and Δ explicitly

$I : G \to G$ - inversion

- $I_*\mu = \Delta^{-1}\mu$

– $\int_G f(g^{-1})\mu(g) = \int_G f(g)\Delta(g)^{-1}\mu(g)$

– $I_*\mu$, $\Delta^{-1}\mu$ are right invariant

– conclude: $I_*\mu = c\Delta^{-1}\mu$ for some constant c

– apply I_* again:

– get $\mu = c^2 \Delta\Delta^{-1}\mu = c^2\mu$

– conclude $c = 1$

2.3.2 The maximal crossed product

- A in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- consider $C_c(G, A)$ with convolution product

- $(f * f')(g) := \int_G f(h)\alpha_h(f'(h^{-1}g))\mu(h)$

Problem 2.35. *Check associativity*

$$
\begin{aligned}
(f'' * (f * f'))(g) &= \int_G f''(h)\alpha_h(\int_G f(h')\alpha_h(f'(h'^{,-1}h^{-1}g))\mu(h'))\mu(h) \\
&= \int_G \int_G f''(h)\alpha_h(f(h'))\alpha_{hh'}(f'(h'^{,-1}h^{-1}g))\mu(h'))\mu(h) \\
&= \int_G \int_G f''(h)\alpha_h(h^{-1}l)\alpha_l(f'(l^{-1}g))\mu(l)\mu(h) \\
&= \int_G (\int_G f''(h)\alpha_h(h^{-1}l)\mu(h))\alpha_l(f'(l^{-1}g))\mu(l) \\
&= ((f'' * f) * f')(g)
\end{aligned}
$$

define $*$-operation: $f^*(g) := \alpha_g(f(g^{-1})^*)\Delta(g)^{-1}$

Problem 2.36. *Check* $(f^*)^* = f$ *and* $(f' * f)^* = f^* * f'^{,*}$.

Proof. $(f^*)^*(g) = \alpha_g(f^*(g^{-1}))\Delta(g)^{-1} = \alpha_g(\alpha_{g^{-1}}(f(g)))\Delta(g)^{-1}\Delta(g^{-1})^{-1} = f(g)$

$$
\begin{aligned}
(f' * f)^*(g) &= \alpha_g(\int_G f'(h)\alpha_h(f(h^{-1}g^{-1}))\mu(h))^*\Delta(g)^{-1} \\
&= \int_G \alpha_{gh}(f(h^{-1}g^{-1}))\alpha_g(f'(h))^*\mu(h)\Delta(g)^{-1} \\
&= \int_G \alpha_l(f(l^{-1}))\alpha_g(f'(g^{-1}l))^*\mu(l)\Delta(g)^{-1} \\
&= \int_G \alpha_l(f(l^{-1}))\Delta(l)^{-1}\alpha_l\alpha_{l^{-1}g}(f'((l^{-1}g)^{-1})^*)\Delta(l^{-1}g)^{-1}\mu(l) \\
&= f^* * f'^{,*}
\end{aligned}
$$

□

G acts by multipliers on $C_c(G, A)$

- $(h * f)(g) := \alpha_h f(g^{-1}h)$

- $(f' * h)(g) := f'(gh)$

- $h^* = h^{-1}$

A acts by multipliers

- $(a * f)(g) := af(g)$

- $(f * a)(g) := f(g)\alpha_{g^{-1}}(a)$

Problem 2.37. *Check $f' * (h * f) = (f' * h) * f$ and $(f' * a) * f = f' * (a * f)$.*

Check: $h * a * h^{-1} = \alpha_h(a)$ in multipliers

Proposition 2.38. *$C_c(G, A)$ with the convolution product and the involution as indicated is a pre-C^*-algebra.*

Proof. Exercise for discrete groups.

For non-discrete groups

- consider non-degenerated representation $\phi : C_c(G, A) \to B$

– means: $C_c(G, A)B \subseteq B$ dense

- get homomorphism $\rho : G \to U(M(B))$

- get homomorphism $\psi : A \to M(B)$

– have equality $\phi(f) = \int_G \psi(f(g))\rho_g \mu(g)$

- get bound: $\|\phi(f)\| \le \|f\|_{L^1(G,A)}$

$\square$

Definition 2.39. *We define the maximal crossed product* $A \rtimes G := \mathrm{compl}(C_c(G,A))$.

Proposition 2.40. *We have a functor* $- \rtimes G : GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$.

Proof. $A \mapsto C_c(G,A)$ is functor $GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$

- $\phi : A \to B$ maps to $f \mapsto (g \mapsto \phi \circ f)$ $\square$

Remark 2.41. $- \rtimes G$ is functorial for weakly equivariant maps

$(\phi, \rho) : A \to B$ weakly equivariant $A \to B$

- define $f \mapsto (g \mapsto \rho_g \phi(f(g)))$ $\square$

2.3.3 Covariant representations

(A, α) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

Definition 2.42. *A covariant representation of* A *is a pair* (ϕ, ρ) *of a unitary representation* $\rho : G \to U(H)$ *and a homomorphism* $\phi : A \to B(H)$ *such that* $\phi(\alpha_g a) = \rho_g \phi(a) \rho_g^*$ *for all* g *in* G *and* a *in* A.

note that conjugation action on $B(H)$ is not continuous in general

- can therefore not say that ϕ is just morphism in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- get map $\bar{\phi}_c : C_c(G,A) \to B(H)$

- $\bar{\phi}_c(f) := \int_G \phi(f(g))\rho_g \mu(g)$

Problem 2.43. *Show that this is a* $*$*-homomorphism.*

$\bar{\phi}_c$ is called the integrated form of (ρ, ϕ)

- extends to $\bar{\phi} : A \rtimes G \to B(H)$

Definition 2.44. *(ϕ, ρ) is non-degenerated if $\phi(A)H$ is dense in H.*

Proposition 2.45. *There is a bijection between the sets the non-degenerated covariant representation (ϕ, ρ) of (A, G) and non-degenerated representations $\bar{\phi} : A \rtimes G \to B(H)$*

Proof. given (ϕ, ρ) construct $\bar{\phi}_c$ and finally $\bar{\phi}$

A and G act as multipliers on $A \rtimes G$

given $\bar{\phi}$ - construct $\phi : A \to B(H)$ and $\rho : G \to U(H)$ as above

□

Remark 2.46. if (ϕ, ρ) is not non-generated, then lose the information about ρ on $(\phi(A)H)^{\perp}$

2.3.4 The reduced crossed product

choose an injective representation $\psi : A \to B(H)$

- consider $\rho : G \to U(B(L^2(G, H))$ given by $(\rho_h v)(g) = v(h^{-1}g)$

- define representation $\phi : A \to B(L^2(G, H))$ by $(\phi(a)v)(g) := \psi(\alpha_{g^{-1}}a)v(g)$

– check: (ϕ, ρ) is covariant

$$\begin{aligned}
(\rho_h \phi(a) \rho_{h^{-1}} v)(g) &= (\phi(a)\rho_{h^{-1}} v)(h^{-1}g) \\
&= \psi(\alpha_{g^{-1}h} a)(\rho_{h^{-1}} v)(h^{-1}g) \\
&= \psi(\alpha_{g^{-1}h} a)v(g) \\
&= \phi(\alpha_h a)v(g)
\end{aligned}$$

the covariant representation induces $C_c(G, A) \to B(L^2(G, H))$

- get norm $\| - \|_r$ in $C_c(G, A)$ - called the reduced norm

Definition 2.47. *We define the reduced crossed product $A \rtimes_r G := \overline{C_c(G, A)}^{\| - \|_r}$.*

get functor $- \rtimes_r G : GC^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$

Problem 2.48. *Show that $\| - \|_r$ is independent of choice of ψ.*

Problem 2.49. *Show that $A \rtimes_r G$ extends naturally to a functor which preserves injections.*

have canonical morphism $A \rtimes G \to A \rtimes_r G$

2.3.5 Further aspects and examples

Example 2.50.

$C^*(G) := \underline{\mathbb{C}} \rtimes G$ -maximal group C^*-algebra

$C^*_r(G) := \underline{\mathbb{C}} \rtimes_r G$ - reduced group C^*-algebra □

Remark 2.51 (Fourier transformation)**.**

G abelian

- $\hat{G}$ - dual group of characters

- Fourier transformation

- $f \mapsto \hat{f}$

– $\hat{f}(\xi) = \int_G \xi^{-1}(g) f(g) \mu(g)$

- dual Fourier transformation

– $\check{h}(g) := \int_{\hat{G}} h(\xi) \hat{\mu}(\xi)$

- normalize $\hat{\mu}$ on $\hat{G}$ such that

- $\check{\hat{f}} = f$ □

Example 2.52.

$\hat{\mathbb{Z}} \cong U(1)$

$\hat{U}(1) \cong \mathbb{Z}$

$\widehat{\text{discrete group}} = \text{compact group}$

counting measure corresponds to normalized Haar measure

$\hat{\mathbb{R}} \cong \mathbb{R}$

$\widehat{|-|} = \frac{1}{2\pi}|-|$ (Lebesguemeasure)

Lemma 2.53. *The Fourier transformation induces an isomorphism* $C^*(G) \cong C_0(\hat{G})$

Example 2.54 (dual group action). $\hat{G}$ acts on $A \rtimes G$

- $(\xi, f) \mapsto (g \mapsto \xi(g)f(g)$

- $(\xi f) * (\xi f') = \int_G \xi(h)f(h)\alpha_h(\xi(h^{-1}g)f'(h^{-1}h))d\mu(h) = \xi(g)\int_G f(h)\alpha_h(f'(h^{-1}g))\mu(h) = (\xi(f * f'))(g)$

- $(A \rtimes G) \rtimes \hat{G} \cong K(L^2(G)) \otimes A$ (Takai duality) □

Example 2.55 (G-graded algebras). G finite

Definition 2.56. *A G-graded algebra is a C^*-algebra with a decomposition $A \cong \bigoplus_{g \in G} A_g$ such that $A_g A_{g'} \subseteq A_{gg'}$ for all g, g' in G and $A_g^* \subseteq A_{g^{-1}}$.*

$A \rtimes G$ is G-graded

- $A \rtimes G \cong \bigoplus_{g \in G} A$

- write elements as (g, A)

- $(g, a) * (g', a') = (gg', \alpha_g(a)a'$

G-grading is same information as action of $\hat{G}$ (for G abelian)

- $(A \rtimes G)_g$ is image of action of projection $p : \int_{\hat{G}} \xi(g)^{-1}\hat{\alpha}_\xi \hat{\mu}(\xi)$ □

Example 2.57 (finite groups).

G finite

- $L^2(G) \cong \bigoplus_{\pi \in \hat{G}} V_\pi \otimes V_\pi^*$ - Peter-Weil

- $C^*(G)$ generated by $L_g = \oplus_{\pi \in \hat{G}} \pi(g) \otimes \mathtt{id}_{V_\pi}$

- projection to factor $V_\pi \otimes V_\pi^*$ is in $C^*(G)$

– given by $\int_G \chi_\pi(g)^{-1} L_g \mu(g)$ (where χ_π is the character)

- hence $\pi(g) \otimes \mathtt{id}_{V_\pi}$ is in $C^*(G)$

- Schur Lemma: $\mathtt{End}(V_\pi) \otimes \mathtt{id}_{V_\pi^*}$ is in $C^*(G)$

- $C^*(G) \cong \bigoplus_{\pi\in\hat{G}} \mathtt{End}(V_\pi)$ - sum of matrix algebras

- $K_*(C^*(G)) \cong \mathbb{Z}[\hat{G}]$ representation "ring" □

3 KK^G

3.1 Homotopy invariance

3.1.1 The localization

start with $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- category is topologically enriched

- write $\underline{\mathtt{Hom}}_G(A, B)$ for the topological mapping space

- $\underline{\mathtt{Hom}}_G(A, B) = \underline{\mathtt{Hom}}(A, B)^G$ - G-fixed points with conjugation action

– $\mathtt{Hom}_{\mathbf{Top}}(X, \underline{\mathtt{Hom}}(A, B)) = \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, C(X) \otimes B)$ for all compact spaces X

get notion of homotopy equivalence

Definition 3.1. *We define the Dwyer-Kan localization $L_h : GC^*\mathbf{Alg}^{\mathrm{nu}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}_h$ at the homotopy equivalences.*

the following are proved the same way as in the non-equivariant case

Proposition 3.2.

1. $\mathtt{Map}_{GC^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \simeq \ell\underline{\mathtt{Hom}}_G(A, B)$.

2. *L_h is symmetric mononidal for $\otimes_?$ with $?$ in $\{\max, \min\}$.*

3. *L_h sends Schochet fibrant squares to pull-back squares.*

4. *$GC^*\mathbf{Alg}^{\mathrm{nu}}_h$ is left-exact.*

5. *The bifunctor $\otimes_?$ on $GC^*\mathbf{Alg}^{\mathrm{nu}}_h$ is bi-left-exact.*

6. $GC^*\mathbf{Alg}^{\mathrm{nu}}_h$ *has all coproducts and* L_h *preserves them.*

$L_h^* : \mathbf{Fun}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{W_h}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$L_h^* : \mathbf{Fun}^{\mathrm{lex}}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{h,Sch}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$L_h^* : \mathbf{Fun}^{\otimes}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{\otimes,W_h}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$L_h^* : \mathbf{Fun}^{\otimes,\mathrm{lex}}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{\otimes,Sch}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$\Omega \circ L_h \simeq L_h \circ S$ loops and suspension

Puppe sequence for $f : A \to B$

$$\cdots \to L_h(S(C(f))) \xrightarrow{\Omega(i_f)} L_h(S(A)) \xrightarrow{S(f)} L_h(S(B)) \xrightarrow{\partial_f} L_h(C(f)) \xrightarrow{i_f} L_h(A) \xrightarrow{L_h(f)} L_h(B)$$

each segment is fibre sequence

the verifications are completely analogous as in the non-equivariant case

3.1.2 Descend of functors

$H \to G$

$G \subseteq L$

consider functors: Res^G_H, $\mathtt{Ind}^L_G$, Coind^L_G, $- \rtimes G$, $- \rtimes_r G$

Lemma 3.3. *The functor* Res^G_H, $\mathtt{Ind}^L_G$, $- \rtimes G$, $- \rtimes_r G$ *functors refine to topologically enriched functors.*

for Coind^L_G is is only true if L/G is compact

- this case is then covered by $\mathtt{Ind}^L_G$

use: $F : GC^*\mathbf{Alg}^{\mathrm{nu}} \to G'C^*\mathbf{Alg}^{\mathrm{nu}}$ a functor

Proposition 3.4. *If there is a natural transformation* $F(A \otimes B) \cong F(A) \otimes B$ *for all commutative algebras* B *such that* $F(A) \cong F(A \otimes \mathbb{C}) \cong F(A) \otimes \mathbb{C} \cong F(A)$ *is the identity, then* F *is topologically enriched.*

Proof.

$$\begin{array}{rcl} \mathtt{Hom}_{\mathbf{Top}}(X, \underline{\mathtt{Hom}}_G(A,B)) & \cong & \underline{\mathtt{Hom}}_G(A, B\otimes C(X)) \\ & \to & \underline{\mathtt{Hom}}_{G'}(F(A), F(B\otimes C(X))) \\ & \cong & \underline{\mathtt{Hom}}_{G'}(F(A), F(B)\otimes C(X)) \\ & \cong & \mathtt{Hom}_{\mathbf{Top}}(X, \underline{\mathtt{Hom}}_{G'}(F(A), F(B))) \end{array}$$

use additional property to check that this map is the correct one on underlying sets □

Lemma 3.5. *We have for any C^*-algebra B and choice of tensor product that*

$$\mathrm{Res}^G_H(A\otimes B) \cong \mathrm{Res}^G_H(A)\otimes B \ .$$

Proof. obvious □

$H \subseteq G$

Lemma 3.6. *For B in $C^*\mathbf{Alg}^{\mathrm{nu}}$, A in $GC^*\mathbf{Alg}^{\mathrm{nu}}$ and $? \in \{\min, \max\}$ we have*

$$\mathtt{Ind}^G_H(A)\otimes_? B \cong \mathtt{Ind}^G_H(A\otimes_? B) \ .$$

Proof. - not completely obvious

- $\iota : C_b(G,A)\otimes_? B \to C_b(G, A\otimes_? B)$ is a map

- but not an isomorphism in general

- similarly $\iota : \mathtt{Ind}^G_H(A)\otimes_? B \to \mathtt{Ind}^G_H(A\otimes_? B)$

for surjectivity:

$f \in \mathtt{Ind}^G_H(A\otimes_? B)$

- choose function χ on G with proper support over G/H such that $\int_G \chi(gh)\mu(h) = 1$

- $\chi f \in C_0(G, A\otimes_? B)$

- $f(g) = \int_G (\alpha_h \otimes \mathtt{id}_B)(\chi(gh) f(gh))\mu(h)$

- find approximation $\chi f = \sum_i^{\text{finite}} f_i \otimes b_i + r$ with r as small as we want

- can assume: $\tilde\chi f_i = f_i$, $\tilde\chi r = r$ for some function with proper support over G/H

- $f(g) = \sum_i^{\text{finite}} \int_H \alpha_h f_i(gh)\otimes b_i \mu(h) + \int_H \alpha_h r(gh)\mu(h)$

- $\int_H \alpha_h r(gh)\mu(h) = \int_H \alpha_h r(gh)\tilde{\chi}(gh)\mu(h)$

- this is small if r is small

for injectivity:

$\mathtt{Ind}_H^G(A) \otimes_? B \to \mathtt{Ind}_H^G(A \otimes_? B) \xrightarrow{\chi} C_0(G, A \otimes_? B)$ is injective

since it is also $\mathtt{Ind}_H^G(A) \otimes_? B \xrightarrow{\chi \otimes \mathtt{id}_B} C_0(G, A) \otimes_? B \to C_0(G, A \otimes_? B)$

□

Corollary 3.7. *The functor* $\mathtt{Ind}_G^L$ *descends to the homotopy localization.*

$f \mapsto \mathrm{Coind}_G^L(f)$ in general not continuous

- only if G/L is compact

- the following exercise shows where the problem is

Problem 3.8. *Show that the functor* $A \mapsto C_b(A)$ *on* $C^*\mathbf{Alg}^{\mathrm{nu}}$ *is not continuous.*

Lemma 3.9. *We have* $B \otimes_{!!} (A \rtimes_! G) \cong (B \otimes_{!!} A) \rtimes_! G$.

Proof. have map $B \otimes_{!!} (A \rtimes_! G) \to (B \otimes_{!!} A) \rtimes_! G$

- [Wil07, Thm. 2.75] for maximal products

- [Ech10, Lem. 4.1] for minimal/reduced □

Corollary 3.10. *The functors* $- \rtimes G$ *and* $- \rtimes_r G$ *descend to the homotopy localization.*

Lemma 3.11. *If* G *is closed in* L *and* L/G *is compact, then we have an adjunction*

$$\mathrm{Res}_G^L : LC^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows GC^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{Coind}_G^L \ .$$

Proof. adjunctions descend if the functors do □

3.2 G-stability

3.2.1 The localization

general principle

$\mathbf{C}$ - ∞-category

- $F : \mathbf{C} \to \mathbf{C}$ endofunctor

- W_F - morphisms that are sent to equivalences by F

– called F-equivalences

- want to understand $\ell : \mathbf{C} \to \mathbf{C}[W_F^{-1}]$

assume: zig-zag $\eta : \mathtt{id} \rightsquigarrow F$

- assume: $\rightsquigarrow \in W_F$

– more precisely: have sequence of natural transformations

$$\mathtt{id} \to F_1 \leftarrow F_2 \to \cdots \leftarrow F_n = F$$

– all components of all these transformations are in W_F

let $F\mathbf{C}$ - full subcategory of $\mathbf{C}$ on image of F

- we say that η preserves $F\mathbf{C}$ if $F_i(F\mathbf{C}) \subseteq F\mathbf{C}$ and the components of $F_i \to F_{i\pm 1}$ are equivalences for all objects in $F\mathbf{C}$

notation:

$i : F\mathbf{C} \to \mathbf{C}$ inclusion

$L : \mathbf{C} \to F\mathbf{C}$ - corestriction of F

Lemma 3.12. *If η preserves $F\mathbf{C}$, then the functor $L : \mathbf{C} \to F\mathbf{C}$ presents its target as the Dwyer-Kan localization of $\mathbf{C}$ at W_F.*

Proof. must show:

$L^* : \mathbf{Fun}(F\mathbf{C}, \mathbf{D}) \overset{\simeq}{\to} \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D})$

- $\Phi : F\mathbf{C} \to \mathbf{D}$

- $L^*\Phi := \Phi \circ F$ obviously inverts W_F

– so functor takes values in target as indicated

claim: $i^* : \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D}) \to \mathbf{Fun}(F\mathbf{C}, \mathbf{D})$ is inverse

consider $L^* \circ i^* : \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D}) \to \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D})$

- this is $\Phi \mapsto \Phi \circ F$

- $\eta : \mathtt{id} \rightsquigarrow F$ induces $\alpha_\Phi := \Phi(\eta) : \Phi \rightsquigarrow \Phi \circ F$

– since Φ inverts W_F we know that $\Phi(\eta)$ is equivalence

– get equivalence $\alpha : \mathtt{id} \to L^* \circ i^* : \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D}) \to \mathbf{Fun}^{W_F}(\mathbf{C}, \mathbf{D})$

— components α_Φ

consider $i^* \circ L^* : \mathbf{Fun}(F\mathbf{C}, \mathbf{D}) \to \mathbf{Fun}(F\mathbf{C}, \mathbf{D})$

- this is functor $\Psi \mapsto \Psi \circ F_{|F\mathbf{C}}$

- have transformation $\eta_{|F\mathbf{C}} : \mathtt{id}_{F\mathbf{C}} \rightsquigarrow F_{|F\mathbf{C}} : F\mathbf{C} \to F\mathbf{C}$

– this is equivalence

– get equivalence $\beta_\Psi := \Psi(\eta_{|F\mathbf{C}}) : \Psi \simeq \Psi \circ F_{|F\mathbf{C}}$

– get equivalence $\beta : \mathtt{id} \to i^* \circ L^* : \mathbf{Fun}(F\mathbf{C}, \mathbf{D}) \to \mathbf{Fun}(F\mathbf{C}, \mathbf{D})$

— with components β_Ψ □

Lemma 3.13. *If F is left-exact, then the localization $\ell : \mathbf{C} \to \mathbf{C}[W_F^{-1}]$ is left-exact*

Proof. W_F is closed under

- pull-backs

- 2-out-of-3

□

Lemma 3.14. *If* $\mathbf{C}$ *is symmetric monoidal with bi-left exact* $\otimes$*, and* $F = -\otimes D$ *for some object* D*, then* $\ell : \mathbf{C} \to \mathbf{C}[W_F^{-1}]$ *is left-exact symmetric monoidal.*

Proof.

f in W_F

- C any object

– $D \otimes (C \otimes f) \simeq C \otimes (D \otimes f)$

– $(D \otimes f)$ is equivalence since $f \in W_F$

– hence $D \otimes (C \otimes f)$ is equivalence

– hence $C \otimes f \in W_F$

conclude: ℓ is symmetric monoidal

in $\mathbf{C}[W^{-1}]$

- show: $E \otimes -$ is left-exact:

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

- use model $F\mathbf{C}$

- all objects in $F\mathbf{C}$

– extend to pull-back in $\mathbf{C}$

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

- since $F = -\otimes D$ is left-exact have $P \in F\mathbf{C}$

- square is pull-back in $F\mathbf{C}$ (since latter is full subcategory)

$$\begin{array}{ccc} E \otimes P & \longrightarrow & E \otimes A \\ \downarrow & & \downarrow \\ E \otimes B & \longrightarrow & E \otimes C \end{array}$$

is also pull-back in $F\mathbf{C}$

□

G -locally compact, second countable

$L^2(G)$ - has left-regular representation

- is separable if G is second countable

- define $K_G := K(L^2(G) \otimes \ell^2)$ with conjugation action

Definition 3.15. *A morphism $f : A \to B$ in $GC^*\mathbf{Alg}_h^{\mathrm{nu}}$ is called a K_G-equivalence if $f \otimes K_G : A \otimes K_G \to B \otimes K_G$ is an equivalence.*

V - Hilbert space with unitary G-action

- $K(V)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$ - compact operators with G-action by conjugation

- $V \to V'$ unitary embedding - induces morphism $K(V) \to K(V')$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

Lemma 3.16. *If V is non-zero and V' is separable, then $K(V) \to K(V')$ is a K_G-equivalence.*

Proof.

$K_G \cong K(L^2(G)) \otimes K(\ell^2)$ - is K-stable

$V \to V'$ unitary embedding of separable Hilbert spaces (no G-action)

- will show: $K(V) \to K(V')$ is K_G-equivalence

– use $K(V) \otimes K \to K(V') \otimes K$ is isomorphic to left upper corner

– $K(V) \otimes K \otimes K \to K(V') \otimes K \otimes K$ is homotopy equivalence

– use $K_G \cong K_G \otimes K \otimes K$

(V, ρ) - separable Hilbert space with G-action

- $V \otimes L^2(G) \cong L^2(G, V)$ mit action $(g \cdot f)(h) = \rho_g f(g^{-1}h)$

- construct equivariant unitary: $\phi : V \otimes L^2(G) \cong \mathrm{Res}_1^G(V) \otimes L^2(G)$

– $\phi : f \mapsto (h \mapsto \rho_{h^{-1}} f(h))$

– write action on target as $g \circ f$ for the moment: $(g \circ f)(h) = f(g^{-1}h)$

- check: $(g \circ \phi(f))(h) = \rho_{h^{-1}g} f(g^{-1}h) = \phi(g \cdot f)(h)$

- conclusion:

$$K(V) \otimes K_G \cong \mathrm{Res}_1^G K(V) \otimes K_G$$

for unitary embedding $V \to V'$ of unitary representations on separable Hilbert spaces

- $K(V) \otimes K_G \to K(V') \otimes K_G$ is isomorphic to $\mathrm{Res}_1^G K(V) \otimes K_G \to \mathrm{Res}_1^G K(V') \otimes K_G$

- is equivalence

□

$F : GC^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{D}$ - functor

Definition 3.17. *The functor F is called G-stable if for every equivariant unitary embedding $V \to V'$ of separable Hilbert spaces the induced map $F(A \otimes K(V)) \to F(A \otimes K(V'))$ is a equivalence.*

write $\mathbf{Fun}^{Gs}(\dots, \dots)$ for G -stable functors

define $\hat{K}_G := K((\mathbb{C} \oplus L^2(G)) \otimes \ell^2)$

- $\mathbb{C} \to \mathbb{C} \otimes \ell^2 \to (\mathbb{C} \oplus L^2(G)) \otimes \ell^2 \leftarrow L^2(G) \otimes \ell^2$ induce

- $\mathbb{C} \to K \to \hat{K}_G \leftarrow K_G$

- $F := - \otimes K_G$

- $\hat{F} := - \otimes \hat{K}_G$

- get zig-zag

$$\eta : \mathtt{id} \to \hat{F} \leftarrow F$$

Lemma 3.18. *$F(\eta)$ is an equivalence*

Proof. Lemma 3.16

□

Definition 3.19. *We define the Dwyer-Kan localization*

$$L_{K_G} : GC^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}$$

at the K_G-equivalences.

set $L_{h,K_G} := L_{K_G} \circ L_h : GC^*\mathbf{Alg}^{\mathrm{nu}} \to L_{K_G} C^*\mathbf{Alg}^{\mathrm{nu}}_h$

Corollary 3.20. *Assume that G is second countable.*

1. $\mathtt{Map}_{L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \simeq \ell\underline{\mathtt{Hom}}_G(K_G \otimes A, K_G \otimes B)$

2. *L_{K_G} is left exact.*

3. *L_{K_G} is symmetric monoidal and induced tensor product on $L_{K_G} C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is bi-left-exact*

4. *For every stable infty category $\mathbf{D}$ we have an equivalence*

$$L^*_{h,K_G} : \mathbf{Fun}(L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Gs}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$$

Proof.

1. Lemma 3.12

2. Lemma 3.13

3. Lemma 3.14

4.

any functor which inverts K_G-equivalence is G-stable:

- use $A \otimes K(V) \to A \otimes K(V')$ is a K_G-equivalence

- L_{h,K_G} is G-stable

any homotopy invariant G-stable functor F inverts K_G-equivalences

$f : A \to B$ - K_G-equivalence

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes \hat{K}_G & \longleftarrow & A \otimes K_G \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \simeq} \\ B & \longrightarrow & B \otimes \hat{K}_G & \longleftarrow & B \otimes K_G \end{array}$$

- F inverts horizontal arrows

- hence F inverts left vertical arrow f $\square$

$L^*_{h,K_G} : \mathbf{Fun}^{\mathrm{lex}}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Gs,Sch}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$L^*_{h,K_G} : \mathbf{Fun}^{\otimes}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\otimes,h,Gs}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

$L^*_{h,K_G} : \mathbf{Fun}^{\otimes,\mathrm{lex}}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\otimes,h,Gs,Sch}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$

Proposition 3.21. *$L_{K_G}C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is semi-additive*

Proof. same proof as for non-equivariant case

$\square$

Lemma 3.22. *$L_{K_G}C^*\mathbf{Alg}^{\mathrm{nu}}_h$ has and L_{h,K_G} preserves all countable coproducts.*

Proof. L_K is Bousfield localization

- preserves all coproducts

for i countable:

- $L_K(\coprod_{i \in I} A_i) \simeq L_K(\bigoplus_{i \in I} A_i)$

- $K_G \otimes \bigoplus_{i \in I} A_i \cong \bigoplus_{i \in I} K_G \otimes A_i$

$$
\begin{aligned}
\ell\underline{\mathtt{Hom}}_G(K_G \otimes \bigoplus_{i\in I} A_i, K_G \otimes B) &\simeq \ell\underline{\mathtt{Hom}}_G(K \otimes \bigoplus_{i\in I} K_G \otimes A_i, K_G \otimes B) \\
&\simeq \ell\underline{\mathtt{Hom}}_G(K \otimes \bigsqcup_{i\in I} K_G \otimes A_i, K \otimes K_G \otimes B) \\
&= \prod_{i\in I} \ell\underline{\mathtt{Hom}}_G(K \otimes K_G \otimes A_i, K \otimes K_G \otimes B) \\
&= \prod_{i\in I} \ell\underline{\mathtt{Hom}}_G(K_G \otimes A_i, K_G \otimes B)
\end{aligned}
$$

□

if G is compact

- have $\mathbb{C} \to L^2(G) \otimes \ell^2$

– $1 \mapsto \mathtt{const} \otimes e_0$

– get $\epsilon : \mathbb{C} \to K_G$

Proposition 3.23. *(K_G, ϵ) is tensor idempotent in $GC^*\mathbf{Alg}_h^{\mathrm{nu}}$*

Proof. $\mathbb{C}^\perp$ - complement of $\mathbb{C}$ in $L^2(G) \otimes \ell^2$

$$
\begin{aligned}
(L^2(G) \otimes \ell^2) \otimes (L^2(G) \otimes \ell^2) &\cong L^2(G) \otimes \ell^2 \oplus \mathbb{C}^\perp \otimes (L^2(G) \otimes \ell^2) \\
&\cong L^2(G) \otimes \ell^2 \oplus L^2(G) \otimes \ell^2
\end{aligned}
$$

$$
\begin{array}{ccc}
L^2(G) \otimes \ell^2 & \longrightarrow & L^2(G)) \otimes \ell^2 \oplus L^2(G)) \otimes \ell^2 \\
\downarrow v & & \downarrow w \\
L^2(G) \otimes L^2((-\infty, 0]) & \longrightarrow & L^2(G) \otimes L^2((-\infty, 1])
\end{array}
$$

find family of isometries $U_t : L^2((-\infty, 0]) \to L^2((-\infty, 1])$ interpolating from the inclusion to unitary

$\phi_t := w^* U_t v(-) v^* U_t^* w : K_G \to K_G \otimes K_G$

$\phi_0 = \epsilon_G$

ϕ_1 is isomorphism

□

Corollary 3.24. *If G is compact, then $L_{K_G} : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}}$ is a left Bousfield localization.*

Corollary 3.25. *$L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}}$ has all coproducts and L_{h,K_G} preserves coproducts.*

3.2.2 Descend of functors

all groups second countable

restriction:

- $H \to G$

- $\mathrm{Res}^G_H : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to HC^*\mathbf{Alg}_h^{\mathrm{nu}}$

Lemma 3.26. Res^G_H *descends to* $\mathrm{Res}^G_H : L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_{K_H}HC^*\mathbf{Alg}_h^{\mathrm{nu}}$.

Proof. - want to show: $L_{K_H} \circ \mathrm{Res}^G_H$ sends K_G-equivalences to equivalences

- equivalently: this functor is G-stable

- $V \to V'$ - embedding of G-Hilbert spaces

- $i : K(V) \to K(V')$

- $A \otimes i : A \otimes K(V) \to A \otimes K(V')$ induced map

- $\mathrm{Res}^G_H(A \otimes i) \simeq \mathrm{Res}^G_H(A) \otimes \mathrm{Res}^G_H(i)$

- $\mathrm{Res}^G_H(i)$ is $K(\mathrm{Res}^G_H(V)) \to K(\mathrm{Res}^G_H(V'))$

– is induced by $\mathrm{Res}^G_H(V) \to \mathrm{Res}^G_H(V')$ - isometric inclusion of H-Hilbert spaces

- hence $L_{K_H} \circ \mathrm{Res}^G_H(A \otimes i)$ is an equivalence

□

induction

- G a closed subgroup of L

- generalize Lemma 3.6

Lemma 3.27. *For A in $GC^*\mathbf{Alg}^{\mathrm{nu}}$ and B in $LC^*\mathbf{Alg}^{\mathrm{nu}}$ and $? \in \{\min, \max\}$ we have an isomorphism*

$$\mathtt{Ind}_G^L(A) \otimes_? B \cong \mathtt{Ind}_G^L(A \otimes_? \mathrm{Res}_G^L(B)) \ .$$

Proof. same as Lemma 3.6

- have canonical map $\mathtt{Ind}_G^L(A) \otimes B \to \mathtt{Ind}_G^L(A \otimes \mathrm{Res}_G^L(B))$

– must show injectivity and surjectivity

- use $f \mapsto (L \ni l \mapsto (\mathtt{id}_A \otimes \beta_l)f(l) \in A \otimes B)$ in order to identify

- $C_b(G, A \otimes \mathrm{Res}_G^L(B))^G \cong C_b(G, A \otimes \mathrm{Res}_1^L(B))^G$

- this preserves supports

– restricts to: $\mathtt{Ind}_G^L(A \otimes \mathrm{Res}_G^L(B)) \cong \mathtt{Ind}(A \otimes \mathrm{Res}_1^L(B))$

- then apply Lemma 3.6

□

Lemma 3.28. *Assume that L is second countable. The functor $\mathtt{Ind}_G^L : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to LC^*\mathbf{Alg}_h^{\mathrm{nu}}$ descends to a functor $\mathtt{Ind}_G^L : L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_{K_L}LC^*\mathbf{Alg}_h^{\mathrm{nu}}$.*

Proof. want to show: $L_{K_L} \circ \mathtt{Ind}_G^L$ sends K_G-equivalences to equivalences

abbreviate $F := L_{K_L} \circ \mathtt{Ind}_G^L : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_{K_L}LC^*\mathbf{Alg}_h^{\mathrm{nu}}$

- $\hat{F} := F(- \otimes \mathrm{Res}_G^L(\hat{K}_L))$

– $\hat{F} \simeq (- \otimes \hat{K}_L) \circ F$

- $\tilde{F} := F(- \otimes \mathrm{Res}_G^L(K_L))$

– $\tilde{F} \simeq (- \otimes K_L) \circ F$

- have zig-zag $F \to \hat{F} \leftarrow \tilde{F}$

– by Lemma 3.27 is equivalent to $F \to (- \otimes \hat{K}_L) \circ F \leftarrow (- \otimes K_L) \circ F$

– these maps are equivalences

now use $\mathrm{Res}^L_G(K_L) \cong K_G$ - see below

- $\tilde{F}$ obviously sends K_G-equivalences to equivalences

- $\mathrm{Res}^L_G(L^2(L)) \cong L^2(G) \otimes \ell^2$

– $L \to L/G$ has measurable section s

— here we need that L and L/G are polish spaces

—- this is true since separable locally compact Hausdorff spaces are polish

— then apply the measurable section theorem to the image of the map $L \to L/G \times L$, $l \mapsto (eG, l)$ and the projection $L/G \times L \to L/G$

— this image is universally measurable

measurable G- isomorphism

- $G \times L/G \to L$, $(g, lG) \mapsto gs(lG)$

– induced measure $\mu \otimes \nu$ for Haar measure μ on G and some measure on L/G

– $L^2(L) \cong L^2(G) \otimes L^2(G/L, \nu) \cong L^2(G) \otimes \ell^2$

□

crossed products

$? \in \{-, r\}$

Lemma 3.29. *If A is in $GC^*\mathbf{Alg}^{\mathrm{nu}}$ and (V, ρ) is a G-Hilbert space, then we have an isomorphism*

$$A \rtimes_? G \otimes \mathrm{Res}^G_1(K(V)) \cong (A \otimes K(V)) \rtimes_? G \ .$$

Proof. since $K(V)$ is nuclear do not have to specify $\otimes$

for $? = -$

- use $\otimes_{max}$

$C_c(G, A \otimes K(V)) \stackrel{\cong}{\to} C_c(G, A \otimes \mathrm{Res}^G_1(K(V)))$

- $f \mapsto (g \mapsto f(g)(\mathtt{id} \otimes \rho_g))$

– isomorphism of $*$-algebras

– inverse: $f \mapsto (g \mapsto f(g)(\mathtt{id} \otimes \rho_{g^{-1}}))$

– use then [Wil07, Lem. 2.75] or Lemma 3.9

for $* = r$

- use $\otimes_{min}$

– use same isomorphism of $*$-algebras as above

– apply Lemma 3.9

– $\phi : A \to B(H)$ injective to define $\psi : A \rtimes_r G \to B(L^2(G, H))$

– use $\psi : C_r(G, A) \to B(L^2(H))$ and $K(V) \to B(V)$ to define minimal tensor product

– $\phi \otimes \mathtt{id} : A \otimes \mathrm{Res}_1^G(K(V)) \to B(H \otimes V)$

- use this to define $(A \otimes \mathrm{Res}_1^G(K(V))) \rtimes_r G$ via rep on $L^2(G, H \otimes V)$

- use $L^2(G, H \otimes V) \cong L^2(G, H) \otimes V$

- conclude isomorphism above is isometric

□

Example 3.30. Assume: $\sigma : G \to U(M(B))$ representation

- $\beta_g := \sigma_g - \sigma_{g^{-1}}$

- makes $B \in GC^*\mathbf{Alg}^{\mathrm{nu}}$

Lemma 3.31. *For A in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and $(?, !) \in \{(-, \max), (r, \min)\}$ we have an isomorphism $(B \otimes_! A) \rtimes_? G \cong \mathrm{Res}^G(B) \otimes_! (A \rtimes_? G)$*

Proof. $C_c(G, A) \otimes B \to C_c(G, A \otimes B)$

- $f \otimes b \mapsto (g \mapsto (\mathtt{id}_A \otimes \sigma_{g^{-1}})(f \otimes b))$

- induces isomorphism

□

Lemma 3.32. *The functor* $- \rtimes_? G : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to C^*\mathbf{Alg}_h^{\mathrm{nu}}$ *descends to a functor* $- \rtimes_? G : L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$.

Proof. abbreviate $F := L_K \circ (- \rtimes_? G) : GC^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$

- consider isometric embedding of separable G-Hiilbert spaces $V \to V'$

- must show $F(A \otimes K(V)) \to F(A \otimes K(V'))$ is an equivalence

use Lemma 3.29

- $F(A \otimes K(V)) \to F(A) \otimes \mathrm{Res}_1^G(K(V))$ is equivalent to

– $F(A) \otimes \mathrm{Res}_1^G(K(V)) \to F(A) \otimes \mathrm{Res}_1^G(K(V'))$

– this is equivalence by stability

□

Lemma 3.33. *If H is closed in G and G/H is compact, then we have an adjunction*

$$\mathrm{Res}_H^G : L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows L_{K_H}HC^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{Coind}_H^G \ .$$

Proof. adjunctions descend if functors do □

Lemma 3.34. *If H is open in G, then we have an adjunction*

$$\mathtt{Ind}_H^G : L_{K_H}HC^*\mathbf{Alg}_h^{\mathrm{nu}} \leftrightarrows L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}} : \mathrm{Res}_H^G \ .$$

Proof.

start with description of unit and counit

$\epsilon : \mathtt{id} \to \mathrm{Res}_H^G \circ \mathtt{Ind}_H^G$

- $\epsilon_A : A \to \mathrm{Res}_H^G \circ \mathtt{Ind}_H^G(A)$

- $\epsilon_A(a) = \chi_H(g)\alpha_{g^{-1}}a = \begin{cases} \alpha_{g^{-1}}a & g \in H \\ 0 & else \end{cases}$

- $\eta : \mathtt{Ind}_H^G \circ \mathrm{Res}_H^G \to \mathtt{id}$

- $\eta_B : \mathtt{Ind}_H^G(\mathrm{Res}_H^G(B)) \to B$

– $\mathtt{Ind}_H^G(\mathrm{Res}_H^G(B)) \subseteq C_b(G,B)^H$

— invariance condition $f(gh) = \beta_{h^{-1}} f(g)$

— G-action by $(g' \cdot f)(g) = f(g',^{-1} g)$

— $C_b(G,B)^H \stackrel{\cong}{\to} C_b(G/H,B)$

— $f \mapsto (gH \mapsto \beta_g f(g)$

— restricts to $\mathtt{Ind}_H^G(\mathrm{Res}_H^G(B)) \cong C_0(G/H,B) \cong C_0(G/H) \otimes B$

— G-action diagonally

— $C_0(G/H) \otimes B \to K(L^2(G/H)) \otimes B$

— functions act by multiplication operator

— multiplication operators by C_0-functions are compact by discreteness of G/H

— $\eta_B : \mathtt{Ind}_H^G(\mathrm{Res}_H^G(B)) \cong C_0(G/H) \otimes B \to K(L^2(G/H)) \otimes B \simeq B$

check triangle equalities

$$\mathrm{Res}_H^G(B) \stackrel{\epsilon_{\mathrm{Res}_H^G(B)}}{\to} \mathrm{Res}_H^G(\mathtt{Ind}_H^G(\mathrm{Res}_H^G(B))) \stackrel{\mathrm{Res}(\eta_B)}{\to} \mathrm{Res}_H^G(B)$$

$$\begin{array}{rcl} b & \mapsto & (g \mapsto \chi_H(g)\beta_{g^{-1}} b) \\ & \mapsto & (g \mapsto \chi_H(g)\beta_g \beta_{g^{-1}} b) \\ & = & (g \mapsto \chi_H(g) b) \\ & \mapsto & \chi_H \otimes b \in \mathrm{Res}_H^G(K(L^2(G/H)) \otimes B) \\ & \stackrel{\simeq}{\leftarrow} & b \in \mathrm{Res}_H^G(B) \end{array}$$

- the last map is left upper corner inclusion

- it follows that $\mathrm{Res}_H^G(\eta_B) \circ \epsilon_{\mathrm{Res}_H^G(B)} \simeq \mathtt{id}_{\mathrm{Res}_H^G(B)}$

$$\mathtt{Ind}_H^G(A) \stackrel{\mathtt{Ind}_H^G(\epsilon_A)}{\to} \mathtt{Ind}_H^G(\mathrm{Res}_H^G(\mathtt{Ind}_H^G(A))) \stackrel{\eta_{\mathtt{Ind}_H^G(A)}}{\to} \mathtt{Ind}_H^G(A)$$

$$
\begin{aligned}
((g \mapsto f(g)) \in \mathtt{Ind}_H^G(A)) &\mapsto (g \mapsto (l \mapsto \chi_H(l)\alpha_{l^{-1}} f(g))) \in \mathtt{Ind}_H^G(\mathrm{Res}_H^G(\mathtt{Ind}_H^G(A))) \\
&\mapsto (g \mapsto (l \mapsto \chi_H(g^{-1}l)\alpha_{(g^{-1}l)^{-1}} f(g))) \in C_0(G/H) \otimes \mathtt{Ind}_H^G(A) \\
&= (g \mapsto (l \mapsto \chi_H(g^{-1}l) f(l))) \in C_0(G/H) \otimes \mathtt{Ind}_H^G(A) \\
&= \sum_{k \in G/H} \chi_{kH} \otimes \chi_{kH} f \in K(L^2(G/H)) \otimes \mathtt{Ind}_H^G(A)
\end{aligned}
$$

must still compose with

$$
K(L^2(G/H)) \otimes \mathtt{Ind}_H^G(A) \xrightarrow{\simeq} K(\mathbb{C} \oplus L^2(G/H)) \otimes \mathtt{Ind}_H^G(A) \xleftarrow{\simeq} \mathtt{Ind}_H^G(A)
$$

- denote embedding $i : K(L^2(G/H)) \to K(\mathbb{C} \oplus L^2(G/H))$

– p in $K(\mathbb{C} \oplus L^2(G/H)$ projection onto summand $\mathbb{C}$

– $i(\chi_{kH}) \in K(\mathbb{C} \oplus L^2(G/H))$ - one-dimensional projection

– choose $u \in K(\mathbb{C} \oplus L^2(G/H))$ one-dimensional partial isometry such that $upu^* = i(\chi_H)$

– define $u_k := ku$ for all k in G/H

– $u_k p u_k^* = i(\chi_{kH})$

– family of g-equivariant homomorphisms $A \mapsto K(L^2(G/H)) \otimes \mathtt{Ind}_H^G(A)$

$$
f \mapsto \sum_{k \in G/H} (\cos(\frac{\pi}{2}t)^2 i(\chi_{kH}) + \sin(\frac{\pi}{2}t)^2 p + \cos(\frac{\pi}{2}t)) \sin(\frac{\pi}{2}t)(u_k + u_k^*)) \otimes \chi_{kH} f
$$

– $t = 0$: get $\sum_{k \in G/H} \chi_{kH} \otimes \chi_{kH} f$

– $t = 1$: get $f \mapsto p \otimes f$

conclude:

$$
\eta_{\mathtt{Ind}_H^G(A)} \circ \mathtt{Ind}_H^G(\epsilon_A) \simeq \mathtt{id}_{\mathtt{Ind}_H^G(A)}
$$

□

note: this argument needs homotopy and stabilization

3.2.3 Murray von Neumann equivalence and weakly equivariant maps, Thomsen stability

$f : A \to B$ - a morphism in $C^*\mathbf{Alg}^{\mathrm{nu}}$

- consider v in $M(B)$

– assume: u is partial isometry

– $f(-)vv^* = f(-)$

– then get new homomorphism $v^*f(-)v : A \to B$

– call this the conjugated homomorphism

$f, g : A \to B$

Definition 3.35. *We say that f and g are Murray-von Neumann (MvN) equivalent if there exists a partial isometry v in $M(B)$ such that $fvv^* = f$ and $v^*f(-)v = g(-) : A \to B$.*

Lemma 3.36. *If f and g are MvN-equivalent, then we have an equivalence*

$$L_{h,K}(f) \simeq L_{h,K}(g) \ .$$

Proof.

$B \xrightarrow{b \mapsto (b,0)} \mathtt{Mat}_2(B)$ is equivalence in $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$

- consider compositions:

– $f \oplus 0 : A \xrightarrow{f} B \xrightarrow{b \mapsto (b,0)} \mathtt{Mat}_2(B)$

– $g \oplus 0 : A \xrightarrow{g} B \xrightarrow{b \mapsto (b,0)} \mathtt{Mat}_2(B)$

– suffices to show $f \oplus 0 \simeq g \oplus 0$

consider $u := \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}$ in $\mathtt{Mat}_2(M(B))$

- is unitary

- $u^*(f \oplus 0)u = (g \oplus 0)$

- i is homotopic to $1_{\mathtt{Mat}_2(M(B))}$

– here is a homotopy

— $\begin{pmatrix} \cos(\frac{\pi}{2}t)v & 1-(1-\sin(\frac{\pi}{2}t))vv^* \\ (1-\sin(\frac{\pi}{2}t))v^*v-1 & \cos(\frac{\pi}{2}t)v^* \end{pmatrix}$ is homotopy from u to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

— this can further be connected with $1_{\mathtt{Mat}_2(M(B))}$

□

(A, α), (B, β) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- usually write A, B

$f : A \to B$ morphism in $C^*\mathbf{Alg}^{\mathrm{nu}}$

- $g \cdot f := \beta_g \circ f \circ \alpha_{g^{-1}}$

- conjugation action on $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B)$

$f : A \to B$ morphism in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- means f is equivariant $g \cdot f = f$

Definition 3.37. *A cocycle on B is a continuous map $G \to U(M(B))$ (strict topology on the target) such that $\beta_h(\sigma_g)\sigma_h = \sigma_{hg}$ for all h, g in G.*

$$\begin{aligned} (hg) \cdot f &= \sigma_{hg} f \sigma_{hg}^* \\ h \cdot (g \cdot f)) &= h \cdot (\sigma_g f \sigma_g^*) \\ &= \beta_h(\sigma_g)\sigma_h f \sigma_h^* \beta_h(\sigma_g^*) \end{aligned}$$

if $\beta = \mathtt{id}$, then σ is an action of G^{op}

Definition 3.38. *A cocycle σ on B extends f to a weakly equivariant map if $g \cdot f(-) = \sigma_g f(-)\sigma_g^*$ for all g in G.*

(A, α), (B, β) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- $f : A \to B$ equivariant

- v isometry in $M(B)$

– $v^*v = 1_{M(B)}$

– $p := vv^*$

– $\beta_g(p) = p$ for all g in G

– $fp = f$

Lemma 3.39. *$v^*f(-)v$ extends to a weakly equivariant map with cocycle*

$$g \mapsto \sigma_g := \beta_g(v^*)v \ . \tag{3.1}$$

Proof.

unitaryness

– $\sigma_g^*\sigma_g = v^*\beta_g(v)\beta_g(v^*)v = v^*\beta_g(p)v = v^*pv = 1_{M(B)}$

– cocycle

– $\beta_h(\beta_g(v^*)v)\beta_h(v^*)v = \beta_{hg}(v^*)pv = \beta_{hg}(v^*)v$

$(v^*f(-)v, \sigma)$ is weakly equivariant morphism

- $\beta_g(v^*f(\alpha_{g^{-1}}a)v) = \beta_g(v^*\beta_{g^{-1}}(f(a))v) = \beta_g(v^*)vv^*f(a)vv^*\beta_g(v)) = \sigma_g v^*f(a)v\sigma_{g^*}$

□

Lemma 3.40. *A weakly equivariant map $(f, \sigma) : A \to B$ functorially induces an equivariant homomorphism $A \otimes K_G \to B \otimes K_G$.*

functorial means: as long as composition is defined

Proof.

suffices to construct morphisms $A \otimes K(L^2(G)) \to B \otimes K(L^2(G))$

- identify $B \otimes K(L^2(G))$ with B-valued convolution kernels $b(g, g')$ on G

- $(bb')(g,g'') = \int_G b(g,g')b'(g',g'')\mu(g')$

- G-action: $(hb(g,g') = \beta_h b(h^{-1}g, h^{-1}g')$

similarly with $A \otimes K(L^2(G))$

define map $A \otimes K(L^2(G)) \to B \otimes K(L^2(G))$ by

- $a(g,g') \mapsto \sigma_g f(a(g,g'))\sigma^*_{g'}$

– is homomorphism

– $\alpha_h(a(h^{-1}g, h^{-1}g'))$ goes to $\sigma_g f(\alpha_{h^{-1}}(a(h^{-1}g, h^{-1}g')))\sigma^*_{g'}$

$$\begin{aligned}
\sigma_g f(\alpha_h(a(h^{-1}g, h^{-1}g')))\sigma^*_{g'} &= \sigma_g \beta_h(\beta_{h^{-1}} f(\alpha_h(a(h^{-1}g, h^{-1}g'))))\sigma^*_{g'} \\
&= \sigma_g \beta_h(\sigma_{h^{-1}} f(a(h^{-1}g, h^{-1}g'))\sigma^*_{h^{-1}})\sigma^*_{g'} \\
&= \beta_h(\sigma_{h^{-1}g} f(a(h^{-1}g, h^{-1}g'))\sigma^*_{h^{-1}g}))
\end{aligned}$$

- conclude: $A \otimes K(L^2(G)) \to B \otimes K(L^2(G))$ is equivariant homomorphism

this is compatible with the partially defined composition

in order to see that we land in $B \otimes K(L^2(G))$

- consider image of kernels $a \otimes \chi_K(g)\chi_{K'}(g')$

– K compact in G

- goes to $(g,g') \mapsto \sigma_g a \sigma^*_{g'} \chi_K(g)\chi_{K'}(g') \in B$

- approximate $\sigma_g a \sigma^*_{g'}$ on K uniformly by locally constant functions

- the resulting kernel is obviously in $B \otimes K(L^2(G))$

□

(A,α), (A,α') in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

Definition 3.41. *We say that A and A' are exterior equivalent if $\mathtt{id}_A$ extends to a weakly equivariant map.*

Corollary 3.42. *If A ane A' are exterior equivalent, then we have an equivalence $L_{h,K_G}(A) \simeq L_{h,K_G}(A')$ in $L_{K_G}C^*\mathbf{Alg}_h^{\mathrm{nu}}$*

note: the equivalence in the corollary above might depend on the choice of the cocycle extending $\mathtt{id}_A$

consider $A = (A, \alpha)$

- consider G-action $\tilde{\alpha}$ on $A \otimes K$

Definition 3.43 (Thomsen)**.** *We say that $\tilde{\alpha}$ is compatible with α if there exists an equivariant map $A \to A \otimes K$, $a \mapsto a \otimes e$, for a minimal projection e.*

Proposition 3.44. *If $\tilde{\alpha}$ is compatible with α, then $\tilde{\alpha}$ is exterior equivalent to $\alpha \otimes \mathtt{id}_K$ by a cocycle σ with $\sigma_g(\alpha_g \otimes \mathtt{id})\sigma_g^* = \tilde{\alpha}_g$ and $\sigma_g(a \otimes e)\sigma_g^* = a \otimes e$ for all a in A.*

Proof.

define $\sigma_g := \sum_i \tilde{\alpha}_g(1 \otimes e_{i,1})(1 \otimes e_{1,i})$

$$\begin{aligned}
\sigma_g^*\sigma_g &= \sum_j (1 \otimes e_{j,1})\tilde{\alpha}_g(1 \otimes e_{1,j}) \sum_i \tilde{\alpha}_g(1 \otimes e_{i,1})(1 \otimes e_{1,i}) \\
&= \sum_j (1 \otimes e_{j,1})\tilde{\alpha}_g(1 \otimes e_{1,1})(1 \otimes e_{1,j}) \\
&= \sum_j (1 \otimes e_{j,1})(1 \otimes e_{1,1})(1 \otimes e_{1,j}) \\
&= 1
\end{aligned}$$

- $\sigma_{hg} = \sum_i \tilde{\alpha}_{hg}(1 \otimes e_{i,1})(1 \otimes e_{1,i})$

$$\begin{aligned}
\tilde{\alpha}_h(\sigma_g)\sigma_h &= \tilde{\alpha}_h(\sum_i \tilde{\alpha}_g(1 \otimes e_{i,1})(1 \otimes e_{1,i})) \sum_j \tilde{\alpha}_h(1 \otimes e_{j,1})(1 \otimes e_{1,j}) \\
&= \sum_i \tilde{\alpha}_{hg}(1 \otimes e_{i,1})\tilde{\alpha}(1 \otimes e_{1,1}))(1 \otimes e_{1,i}) \\
&= \sum_i \tilde{\alpha}_{hg}(1 \otimes e_{i,1})\tilde{\alpha}(1 \otimes e_{1,i})
\end{aligned}$$

$$
\begin{aligned}
\sigma_g(\alpha_g(a) \otimes e_{kl})\sigma_g^* &= \sum_i \tilde{\alpha}_g(1 \otimes e_{i,1})(1 \otimes e_{1,i})(\alpha_g(a) \otimes e_{kl}) \sum_j (1 \otimes e_{j,1})\tilde{\alpha}_g(1 \otimes e_{1,j}) \\
&= \tilde{\alpha}_g(1 \otimes e_{k,1})(1 \otimes e_{1,k})(\alpha_g(a) \otimes e_{kl})(1 \otimes e_{l,1})\tilde{\alpha}_g(1 \otimes e_{1,l}) \\
&= \tilde{\alpha}_g(1 \otimes e_{k,1})(\alpha_g(a) \otimes e_{11})\tilde{\alpha}_g(1 \otimes e_{1,l} \\
&= \tilde{\alpha}_g(1 \otimes e_{k,1})\tilde{\alpha}_g(a \otimes e_{11})\tilde{\alpha}_g(1 \otimes e_{1,l}) \\
&= \tilde{\alpha}_g(a \otimes e_{k,l})
\end{aligned}
$$

□

Corollary 3.45. *If $\tilde{\alpha}$ is compatible with α, then the map $(A, \alpha) \to (A \otimes K, \tilde{\alpha})$ is a K_G-equivalence.*

Proof.

$$
A \otimes K_G \xrightarrow{(a \mapsto a \otimes e) \otimes \mathtt{id}_{K_G}} (A \otimes K \otimes K_G, \tilde{\alpha} \otimes \ell) \cong (A \otimes K \otimes K_G, \alpha \otimes \mathtt{id}_K \otimes \ell)
$$

- second isomorphism induced by exterior equivalence $(A \otimes K, \tilde{\alpha}) \to (A \otimes K, \alpha \otimes \mathtt{id}_K)$ obtained from Proposition 3.44

- this equivalence preserves $a \otimes e$

- whole composition is left upper corner inclusion tensored with K_G

– hence a homotopy equivalence by stability of K_G

conclude: first map is homotopy equivalence

□

$F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{M}$

- F homotopy invariant

Definition 3.46 (Thomsen [Tho98]). *F is called Thomsen stable if it sends $F(A, \alpha) \to F(A \otimes K, \tilde{\alpha})$ to equivalences provided α and $\tilde{\alpha}$ are compatible*

Lemma 3.47. *G-stability is equivalent to Thomsen stability.*

Proof.

- by Corollary 3.45: a G-stable functor is stable in the sense of Thomsen

show: stable functor in the sense of Thomsen is K_G-stable

– $A \to A \otimes \hat{K}_G$ is Thomsen equivalence

– $A \otimes K_G \to A \otimes \hat{K}_G$ is Thomsen equivalence

$$\hat{K}_G \cong \begin{pmatrix} K_G & K(\ell^2, L^2(G) \otimes \ell^2) \\ K(L^2(G) \otimes \ell^2, \ell^2) & K(\ell^2, \ell^2) \end{pmatrix} \cong \begin{pmatrix} K_G \otimes e & eK_G \otimes Ke^{\perp} \\ e^{\perp}Ke & e^{\perp}K_G \otimes Ke^{\perp} \end{pmatrix} \cong K_G \otimes K$$

- e - one-dimensional in K

– some action preserving this structure

– use here some identification $K_G \otimes K$ with K (no action)

— write $A \otimes K_G = (A', \alpha')$

— $A \otimes \hat{K}_G = (A' \otimes K, \tilde{\alpha}')$

— get Thomsen equivalence

$f : A \to B$ - K_G -equivalence

- use diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes \hat{K}_G & \longleftarrow & A \otimes K_G \\ \downarrow f & & \downarrow & & \downarrow \simeq \\ B & \longrightarrow & B \otimes \hat{K}_G & \longleftarrow & B \otimes K_G \end{array}$$

- F sends horizontal arrows to equivalences since they are Thomsen equivalences

- F sends right vertical map to equivalence since it is homotopy equivalence

– hence: F sends left vertical map to equivalence

□

consider (A, α) in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- p in $M(A)^G$ - invariant projection

- $(B, p\alpha p)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- $i : B \to A$ invariant inclusion

Definition 3.48. *B is called a corner of A.*

Definition 3.49. *It is called full if $ApA = A$.*

Recall: A separable implies A has strictly positive element

Proposition 3.50. *If A admits a strictly positive element, then there exists a weakly equivariant isomorphism $v : B \otimes K \to A \otimes K$. Furthermore $L_{h,K_G}(v) \simeq L_{h,K_G}(i)$.*

Proof.

apply [Bro77, Cor. 2.6]

– $(B \otimes K) = (p \otimes 1)A \otimes K)(p \otimes 1)$

- find isometry v in $M(A \otimes K)$ with $v^*v = p \otimes 1$

- $v^* - v : B \otimes K \stackrel{\cong}{\to} A \otimes K$

apply Lemma 3.39

- get canonical extension by cocycle to weakly equivariant map

i and v are Murray von Neumann equivalent

- $i \oplus 0$ and $v \oplus 0$ are conjugate by unitary u

- u is homotopic to 1

- can extend whole homotopy from $i \oplus 0$ to $v \oplus 0$ to homotopy of weakly equivariant maps (use explicit formula for cocycle (3.1))

- get homotopy of equivariant maps $\mathtt{Mat}_2(A) \otimes K_G \to \mathtt{Mat}_2(B) \otimes K_G$

□

Corollary 3.51. *If A is separable, then a full corner inclusion $B \to A$ induces an equivalence $L_{h,K_G}(B) \to L_{h,K_G}(A)$.*

3.2.4 Hilbert C^*-modules and bimodules

B - C^*-algebra

- E - $\mathbb{C}$- vector space

– consider the following additional structures:

— B-right module structure

— B-valued scalar product: $\langle -, - \rangle : E \otimes_{\mathbb{C}} E \to B$

—- $\langle be, e'b' \rangle = b^* \langle e, e' \rangle b'$ for all b, b' in B, e, e' in E

—- $\langle e, e' \rangle = \langle e', e \rangle^*$

—- $\langle e, e \rangle \geq 0$

– define seminorm: $\|e\| := \|\langle e, e \rangle\|^{1/2}$

— check: semi-norm properties (exercise)

- so far: $(E, \langle -, - \rangle)$ - a pre Hilbert B-module

Definition 3.52. $(E, \langle -, - \rangle)$ *is a Hilbert B-module if $(B, \| - \|)$ is a Banach space.*

set $I := \overline{\langle E, E \rangle}$

- is ideal in B

Lemma 3.53. *$EI \subseteq E$ is dense*

Proof. $\langle e - ei, e - ei \rangle = \langle e, e \rangle - \langle e, e \rangle i - i^* \langle e, e \rangle + i^* \langle e, e \rangle i$

- can make this as small as we want

- take i in approximate unit of I □

$A : E \to E$ a map

Definition 3.54. *A is adjointable if there exists $A^* : E \to E$ such that $\langle Ae, e' \rangle = \langle e, A^* e \rangle$ for all e, e' in E*

Lemma 3.55. *If A is adjointable, then A is linear, B-linear and bounded (in the sense of Banach spaces) and A^* is uniquely determined by A.*

Proof. uniqueness: exercise

- linearity: exercise

- boundedness: use closed graph theorem □

$B(E)$ - adjointable operators on E

Lemma 3.56. *$B(E)$ is a C^*-algebra.*

Proof. $B(E)$ is closed in bounded operators on E

- $*$ is involutive, isometric

- $\|A^*A\| = \|A\|^2$

– Chauchy-Schwarz: $\|\langle e, f \rangle\|^2 \leq \|e\|^2 \|f\|^2$ (exercise)

– implies $\|\langle Ae, Ae \rangle\|^2 \leq \|A^*A\|^2 \leq \|A\|^4$ for unit vectors e

– $\|A\|^2 \leq \|A^*A\| \leq \|A\|^2$ - hence equality

□

consider e, e' in E

- define $\mathbb{C}$-linear map $\Theta_{e,e'} : E \to E$

– $\Theta_{e,e'}(e'') := e\langle e', e'' \rangle$

– is B linear: $\Theta_{e,e'}(e''b) = e\langle e', e''b \rangle = e\langle e', e'' \rangle b = \Theta_{e,e'}(e'')b$

– is adjointable:

$$\begin{aligned}
\langle \Theta_{e,e'}(e''), e''' \rangle &= \langle e\langle e', e''\rangle, e''' \rangle \\
&= \langle e', e''\rangle^* \langle e, e''' \rangle \\
&= \langle e'', e'\rangle \langle e, e''' \rangle \\
&= \langle e'', e'\langle e, e''' \rangle\rangle \\
&= \langle e'', \Theta_{e',e}(e''')\rangle
\end{aligned}$$

$\Theta_{e,e'}$ is called elementary compact

Definition 3.57. *We define $K(E)$ as the C^*-subalgebra of $B(E)$ generated by the elementary compact operators.*

Lemma 3.58. *$K(E)$ is an ideal in $B(E)$ and $B(E) \cong M(K(E))$.*

Proof. ideal: exercise

multiplier: see [Bla98, 13.4.1] □

Example 3.59. Example: $B = \mathbb{C}$

- Hilbert $\mathbb{C}$-modules are Hilbert spaces, $B(E)$ and $K(E)$ have the usual meaning

-note: the elements of $K(E)$ are in general not compact in the sense of bounded operators on a Banach space □

Example 3.60. B is Hilbert B-module

- $\langle b, b' \rangle := b^* b'$

- $B(B) = M(B)$ and $K(B) = B$

can form orthogonal sum of Hilbert B-modules

$B^n := \bigoplus_{i=1}^n B$ as Hilbert B-modules

$K(B^n) \cong \mathtt{Mat}_n(B)$

$B(B^n) \cong \mathtt{Mat}_n(M(B))$ □

Example 3.61. can for direct sum of Hilbert B-modules

$E \oplus F$

- scalar product $\langle e \oplus f, e' \oplus f' \rangle := \langle e, e' \rangle + \langle f, f' \rangle$

Example 3.62. have maps $B^n \to B^{n+1}$

- form $H_B^\circ := \mathtt{colim}_{n\in\mathbb{N}} B^n$ in right B-modules

- get scalar product

- $H_B :=$ completion of H_B°

elements: $(b_i)_{i\in\mathbb{N}}$ with $\sum_{i\in\mathbb{N}} b_i^* b_i$ converges in B

- norm: $\|(b_i)_{i\in\mathbb{N}}\|^2 = \|\sum_{i\in\mathbb{N}} b_i^* b_i\|$

note: $\|\sum_{i\in\mathbb{N}} b_i^* b_i\| \le \|\sum_{i\in\mathbb{N}} \|b_i\|^2$ but in general not equal □

Example 3.63. X -locally compact space

(V, h) - euclidean vector bundle

$\Gamma_0(X, V)$ is right $C_0(X)$-module

- $\langle v, v'\rangle(x) := h(v(x), v'(x))$ is scalar product

- $B(\Gamma_0(X, V)) = \Gamma_b(X, \mathtt{End}(V))$

- $K(\Gamma_0(X, V)) = \Gamma_0(X, \mathtt{End}(V))$

- $\mathtt{id}_V$ is compact if and only if X is compact

□

Example 3.64. can talk about adjointable operators $A : E \to E'$

- equivalently: $\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} : E \oplus E' \to E \oplus E'$ is adjointable

here is an example of a non-adjointable bounded B-linear map

$B := B(\ell^2)$ is B-Hilbert C^*-module

- $K := K(\ell^2)$ is submodule

- $A : K \to B$ is isometric inclusion of right B-modules

Claim: A is not adjointable.

□

everything has an equivariant version

G - action on E

- $\sigma : G \to U(B(E))$ homomorphism

- strongly continuous: $g \mapsto \sigma_g(e)$ continuous

Lemma 3.65. *The action $G \to \mathtt{Aut}(K(E))$ (by conjugation) is continuous.*

Proof. Exercise! □

Definition 3.66. *A Hilbert-B-module is called full, if $\langle E, E\rangle$ is dense in B.*

Example 3.67. E - equivariant Hilbert B-module

- I - ideal in B generated by $\langle E, E\rangle$

- is invariant

E is full equivariant I Hilbert B-module

Lemma 3.68. $B(E) \cong B(E_{|I})$

Proof. (u_i) approximate unit of I

- A in $B(E_{|I})$

- for all e, e' in E, b in B

$$\begin{aligned}\langle e, A(e'b) - A(e')b\rangle &= \lim_i \langle e, A(e'b) - A(e')b\rangle u_i \\ &= \lim_i \langle e, A(e'bu_i) - A(e')bu_i\rangle \\ &= 0\end{aligned}$$

- shows: $A(e'b) = A(e')b$ □

□

Example 3.69. can consider left Hilbert A-modules in analogy

- start with Hilbert B-module E

- is left $K(E)$-module

- define $K(E)$-valued scalar product $(e,e') := \Theta_{e,e'}$:

- check $(\Theta_{e''',e''}e,e') = \Theta_{e'''\langle e'',e\rangle,e'} = \Theta_{e''',e''}\Theta_{e,e'} = \Theta_{e''',e''}(e,e')$

– $(e,e) = \Theta_{e,e}$ is positive (exercise ?)

– show $\|\theta_{e,e} - t\| \leq t$

– $\|(e,e)\| = \|\Theta_{e,e}\| = \|e\|^2$ (exercise ?)

conclude: E is left Hilbert $K(E)$-module

- compatible scalar products:

$$(e,e')e'' = \Theta_{e,e'}(e'') = e\langle e',e''\rangle$$

- full by construction □

Construction 3.70. follow [BGR77]

A, B - G-C^*-algebras

- X - (right) B-Hilbert module and (left) A-Hilbert module

- compatible scalar products $\langle x,x'\rangle_A x'' = x\langle x',x''\rangle_B$

- define X^* - (B,A) - bimodule

– underlying vector space same as X with conjugated complex structure:

– operations: $(x,a) \mapsto a^*x$, $(b,x) \mapsto xb^*$

– conjugated scalar product

- define linking algebra $C^0 := \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$

– product: $\begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + \langle x,y'\rangle_A & ax' + xb' \\ ya' + by' & bb' + \langle y,x'\rangle_B \end{pmatrix}$

$$\left(\left(\begin{array}{cc} a & x \\ y & b \end{array}\right)\left(\begin{array}{cc} a' & x' \\ y' & b' \end{array}\right)\right)\left(\begin{array}{cc} a'' & x'' \\ y'' & b'' \end{array}\right)$$

$$= \left(\begin{array}{cc} aa' + \langle x, y'\rangle_A & ax' + xb' \\ ya' + by' & bb' + \langle y, x'\rangle_B \end{array}\right)\left(\begin{array}{cc} a'' & x'' \\ y'' & b'' \end{array}\right)$$

$$= \left(\begin{array}{cc} (aa' + \langle x, y'\rangle_A)a'' + \langle ax' + xb', y''\rangle_A & (ax' + xb')b'' + (aa' + \langle x, y'\rangle_A)y'' \\ (ya' + by')a'' + (bb' + \langle y, x'\rangle_B)y'' & (bb' + \langle y, x'\rangle_B)b'' + \langle ya' + by', x''\rangle_B \end{array}\right)$$

$$\left(\begin{array}{cc} a & x \\ y & b \end{array}\right)\left(\left(\begin{array}{cc} a' & x' \\ y' & b' \end{array}\right)\left(\begin{array}{cc} a'' & x'' \\ y'' & b'' \end{array}\right)\right)$$

$$= \left(\begin{array}{cc} a & x \\ y & b \end{array}\right)\left(\begin{array}{cc} a'a'' + \langle x', y''\rangle_A & x'b'' + a'y'' \\ y'a'' + b'y'' & b'b'' + \langle y', x''\rangle_B \end{array}\right)$$

$$= \left(\begin{array}{cc} a(a'a'' + \langle x', y''\rangle_A) + \langle x, y'a'' + b'y''\rangle_A & a(x'b'' + a'y'') + x(b'b'' + \langle y', x''\rangle_B) \\ \cdots & \cdots \end{array}\right)$$

look at right upper corner: here need compatibility of scalar products for associativity

involution:

$$\left(\begin{array}{cc} a & x \\ y & b \end{array}\right)^* = \left(\begin{array}{cc} a^* & y \\ x & b^* \end{array}\right)$$

- consider representation of C^0 on $X \oplus B$ by matrix multiplication

- induces seminorm

- define C as closure

clear: $B \cong \left(\begin{array}{cc} 0 & 0 \\ 0 & B \end{array}\right) \subseteq C$ as corner

full: $C\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)C = C$?

these are the elements of the form $\left(\begin{array}{cc} \langle x, y''\rangle_A & xb'' \\ by'' & b \end{array}\right)$

- need: A-valued scalar product is full

- $XB \subseteq X$ is dense, Lemma 3.53

assume: A, B - separable, X separable

- then C separable

- $A \cong \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \to C$ is homomorphism (not necessarily injective)

Proposition 3.71. *If X is a (A, B)-Hilbert bimodule such that*

1. *X is full as left A-Hilbert module*

2. *A, B, X are separable.*

Then we get a morphism $L_{h,K_G}(A) \to L_{h,K_G}(C) \stackrel{\simeq}{\leftarrow} L_{h,K_G}(B)$

□

Definition 3.72. *An equivariant separable (A, B)-Hilbert bimodule is called an equivariant Morita bimodule if it is full as right B-module and as left A-module.*

Corollary 3.73. *An (A, B)- Morita bimodule induces an equivalence in $L_{h,K_G}(A) \simeq L_{h,K_G}(B)$.*

E - a separable right B-Hilbert module

- then it is also $(K(E), B)$-Hilbert bimodule

- is full as $K(E)$-module

- is full as a I-rightmodule for $I := \overline{\langle E, E\rangle}$

- by Proposition 3.50

Proposition 3.74. *If E is a separable (A, B)-Hilbert bimodule such that: $A \to K(E)$, then we get a morphism*

$$E_* : L_{h,K_G}(A) \to L_{h,K_G}(K(E)) \to L_{h,K_G}(X) \stackrel{\simeq}{\leftarrow} L_{h,K_G}(I) \to L_{h,K_G}(B) \ .$$

Construction 3.75.

E - (A, B) - Hilbert bi-module

F - (B, C)-Hilbert bimodule

define $E \otimes_B F$

- $E \otimes_B^{\mathrm{alg}} F$ as vector space

- left action by a: $a(e \otimes f) := ae \otimes f$

- right action by C: $(e \otimes f)c := e \otimes fc$

- C-valued scalar product $\langle e \otimes f, e' \otimes f' \rangle := \langle f, \langle e, e' \rangle f' \rangle$

- form completion $E \otimes_B F$ with respect to induced semi-norm

- show: operations extend by continuity

Lemma 3.76. $K(E) \xrightarrow{k \mapsto k \otimes \mathtt{id}} K(E \otimes_B F)$

Proof. exercise* □

E - (A, B) - Hilbert bi-module

F - (B, C)-Hilbert bimodule

Lemma 3.77. *We have* $L(F) \circ L(E) \simeq L(F \otimes_B E) : L_{h,K_G}(A) \to L_{h,K_G}(B)$.

Proof. need a good argument! □

Example 3.78. in this example translate two-morphisms into homotopies

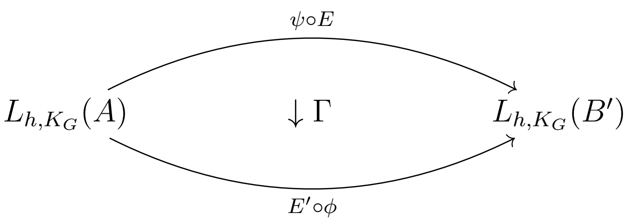
$\phi : A \to A'$, $\psi : B \to B'$ - algebra homomorphisms

$E : A \to A'$, $E' : B \to B'$ - bi-modules

- can form new bimodules:

– $A \xrightarrow{\phi} A' \xrightarrow{E'} B'$ - gives $E' \circ \phi : A \to B'$

– $A \xrightarrow{E} B \xrightarrow{\psi} B'$ (by $E \otimes_B B'$) - gives $\psi \circ E : A \to B'$



- $\Gamma : E \to E'$ structure preserving iso in obvious sense

- induces homotopy $E \otimes_B B' \to E' \circ \phi$

– form mapping cone $C([0,1], E') \circ \phi \oplus_{0,\Gamma} \psi \circ E$

– is $(A, C([0,1], B'))$-bimodule

– evaluation at 0 is $\psi \circ E$

– evaluation at 1 is $E' \circ \phi$

$\square$

Example 3.79. $(A, \alpha), (A, \mathtt{id}_A)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

- $\sigma : G \to U(M(A))$ homomorphism

- assume: $(\mathtt{id}_A, \sigma) : (A, \alpha) \to (A, \mathtt{id}_A)$ weakly equivariant map

- consider vector space $\mathcal{A} := A$ with:

– G-action: $a \mapsto \sigma_g a$

– $\mathcal{A}$ is right $(A, 1)$-Hilbert C^*-module

— action aa' is product in A

— scalar product $\langle a, a \rangle := a^* a'$

– $(A, \alpha) \to K(\mathcal{A})$ equivariant $a \mapsto (a' \mapsto aa')$

— equivariance $\sigma_g a \sigma_{g^{-1}} = \alpha_g(a)$ by assumptions

— is isomorphism

$\mathcal{A}$ is $(A, \alpha), (A, \mathtt{id})$-Morita bimodule

Lemma 3.80. $L(\mathcal{A}) \simeq L(\mathtt{id}_A, \sigma)$

$\square$

3.2.5 Imprimitivity and some adjunctions

$H \subset G$ - closed subgroup

Theorem 3.81 (Green's imprimitivity theorem). *For $? \in \{r, -\}$ there is an equivalence of functors*

$$- \rtimes_? H \to \mathtt{Ind}_H^G(-) \rtimes_? G$$

from $L_{K_H} HC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$.

Proof. A in $HC^*\mathbf{Alg}^{\mathrm{nu}}$

- define Morita $(\mathtt{Ind}_H^G(A) \rtimes_r G, A \rtimes_r H)$-bimodule $X(A)$

- $X_c(A) := C_c(G, A)$

– left action: $(bx)(s) = \int_G b(t,s)x(t^{-1}s)\Delta_G(t)^{1/2}\mu_G(t), \quad b(t,s) \in C_c(G, \mathtt{Ind}_H^G(A))$

– right action $(xa)(s) = \int_G \alpha_h(x(sh)a(h^{-1}))\Delta_H(h)^{-1/2}\mu_H(h), \quad a \in C_c(G, A)$

– ${}_{\mathtt{Ind}_H^G(A) \rtimes_? G}\langle x, y\rangle(s,t) := \Delta_G(s)^{-1/2} \int_H \alpha_h(x(th)y(s^{-1}th)^*)\mu_H(h)$

– $\langle x, y, \rangle_{A \rtimes_? H}(h) = \Delta_H(h)^{-1/2} \int_G x(t^{-1})^* \alpha_h(y(t^{-1}h))\mu_G(t)$

form closure with respect to induced norm

- continuous extension of actions and scalar products

- show Morita property

for history and references see discussion in [Ech10] □

Theorem 3.82 (Green-Julg theorem). *If G is compact, then we have an adjunction*

$$\mathrm{Res}_G : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \leftrightarrows L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} : - \rtimes G \ .$$

Proof.

unit: $\epsilon_A : A \to \mathrm{Res}_G(A) \rtimes_r G$

- $a \mapsto \mathtt{const}_a$ in $C(G, A) \subseteq C^*(G, A)$

– use that Haar measure is normalized to see that this is homomorphism

description of the unit as bimodule

- more general:

– B in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

– E a equivariant (right) Hilbert B-module

– action map γ

– form $\hat{E}$ - a $B \rtimes G$-Hilbert module

— right action: $eb := \int_G \gamma_s(ef(s^{-1}))\mu(s)$

— $B \rtimes G$-valued scalar product: $\langle e, e' \rangle(s) = \langle e, \gamma_s(e') \rangle$

apply to A with trivial action

- A becomes right $A \rtimes G$-module $\hat{A}$

– $\hat{A}$ induces morphism $\epsilon_A : L_{h,K}(A) \to \mathrm{Res}_G(L_{h,K}A) \rtimes_r G$

argument that this is the case

- $\langle \hat{A}, \hat{A} \rangle =: I$ - constant functions in $A \rtimes G$

– is ideal in $A \rtimes G$

– linking algebra C for (A, I) is $\mathtt{Mat}_2(A)$

– $A \to C$ left upper corner

– $I \to C$ right lower corner

– induces $A \to I$ (identity on A)

– $\hat{A}$ thus induces $A \to A \rtimes G$ given by inclusion of I

– this is precisely the unit

counit:

- $L^2(G, B)$ becomes equivariant $(B \rtimes G, B)$-bimodule

– B-valued scalar product: $\langle h, h' \rangle := \int_G \beta_s(h(s^{-1})^* h'(s))\mu(s)$

– right B-action: $(hb)(t) = h(t)\beta_t(b)$

– left $B \rtimes G$-action: $(fh)(t) = \int_G f(s)\beta_s(h(s^{-1}t))\mu(s)$

— check: goes to $K(L^2(G,B))$

– G-action $\sigma_s(h)(t) = f(ts)$

- $\mathrm{Res}_G(B \rtimes G) \to K(L^2(G,B))$

– left convolution commutes with right translation

$L^2(G,B)$ induces counit map $\eta_B : \mathrm{Res}_G(B \rtimes G) \to B$ in $L_{K_G}GC^*\mathbf{Alg}_h^{\mathrm{nu}}$

check triple identities

$\mathrm{Res}_G(A) \xrightarrow{\mathrm{Res}_G(\epsilon_A)} \mathrm{Res}_G(\mathrm{Res}_G(A) \rtimes_r G) \xrightarrow{\eta_{\mathrm{Res}_G(A)}} \mathrm{Res}_G(A)$

- $a \mapsto \mathtt{const}_a \to \mathtt{const}_a$ (convolution) in $K(L^2(G,A)) \cong A \otimes K(L^2(G))$

– this is left upper corner inclusion with projection onto the G-invariants

$B \rtimes G \xrightarrow{\epsilon_{B \rtimes G}} \mathrm{Res}_G(B \rtimes G) \rtimes G \xrightarrow{\eta_B \rtimes G} B \rtimes G$

- write this as tensor products of bimodules

$\eta_{\mathrm{Res}_G(B \rtimes G)} \rtimes G \circ \epsilon_{B \rtimes G}$ is given by

$\widehat{\mathrm{Res}_G(B \rtimes G)} \otimes_{\mathrm{Res}_G(B \rtimes G) \rtimes G} (L^2(G,B) \rtimes G) \cong ...$

this represents identity

□

Theorem 3.83. *If G is discrete, then we have an adjunction*

$$- \rtimes_{\max} : L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \leftrightarrows L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} : \mathrm{Res}_G$$

Proof. unit: $\epsilon_A : A \to \mathrm{Res}_G(A \rtimes_{\max} G)$

- $a \mapsto a\delta_e$

- weakly equivariant with cocycle: $\sigma_g := \delta_g$

- $\delta_g(a\delta_e)\delta_{g^{-1}} = \delta_g(a\delta_{g^{-1}}) = \alpha_g(a)\delta_e$

– get map $\epsilon_A : L_{h,K_G}(A) \to L_{h,K_G}(\mathrm{Res}_G(A \rtimes_{\max} G))$

can be more explicit: is useful for calculations

- $g \mapsto \delta_g$ is homomorphism $G \to U(M(A \rtimes_{\max} G))$

– get $(A \rtimes_{\max} G, \delta)$ in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

— $A \to (A \rtimes_{\max} G, \delta)$ is equivariant

— $\epsilon_A : L_{h,K_G}(A) \xrightarrow{a \mapsto a\delta_e} L_{h,K_G}(A \rtimes_{\max} G, \delta) \xrightarrow{L(E)} L_{h,K_G}(\mathrm{Res}_G(A \rtimes_{\max} G))$

– E is $(A \rtimes_{\max} G, \delta), \mathrm{Res}_G(A \rtimes_{\max} G))$ -bimodule as in Example 3.79

– get bimodule $\mathrm{Res}_G(A \rtimes_{\max} G)$

counit: $\eta_B : \mathrm{Res}_G(B) \rtimes_{\max} G \to B$

- trivial G-action and left multiplication on B extends to $B \rtimes_{\max} G$-action on B

- get $\hat{B}$ - a $(\mathrm{Res}_G(B) \rtimes_{\max} G, B)$-bimodule

- induces a map $\mathrm{Res}_G(B) \rtimes_{\max} G \to B$

– $f \mapsto \sum_{s \in G} f(s)$

check triple identities:

$\mathrm{Res}_G(B) \xrightarrow{\epsilon_{\mathrm{Res}_G(B)}} \mathrm{Res}_G(\mathrm{Res}_G(B) \rtimes_{\max} G) \xrightarrow{\mathrm{Res}_G(\eta_B)} \mathrm{Res}_G(B)$

- $b \mapsto \sum_{s \in G} (b\delta_e)(s) = b$

- this is obviously the identity

$A \rtimes_{\max} G \xrightarrow{\epsilon_A \rtimes_{\max} G} \mathrm{Res}_G(A \rtimes_{\max} G) \rtimes_{\max} G \xrightarrow{\eta_{A \rtimes_{\max} G}} A \rtimes_{\max} G$

see e.g. [Par15, Sec. 3]

$$
\begin{array}{ccccccc}
A \rtimes_{\max} G & \xrightarrow{\epsilon_A \rtimes_{\max} G} & (A \rtimes_{\max} G, \delta) \rtimes_{\max} G & \xrightarrow{E \rtimes_{\max} G} & \mathrm{Res}_G(A \rtimes_{\max} G) \rtimes_{\max} G & \xrightarrow{\eta_{A \rtimes_{\max} G}} & A \rtimes_{\max} G \\
& \searrow^{\phi} & \downarrow \Psi & & \| & \nearrow_{\eta_{A \rtimes_{\max} G}} & \\
& & \mathrm{Res}_G(A \rtimes_{\max} G) \rtimes_{\max} G & \xrightarrow{E' \rtimes_{\max} G} & \mathrm{Res}_G(A \rtimes_{\max} G) \rtimes_{\max} G & &
\end{array}
$$

Ψ is given by Lemma 3.31

- $E^{'}$ is like E but for trivial action

- the same map as in Lemma 3.31 also induces a two-morphism from $E \rtimes_{\max} G$ to $E' \rtimes_{\max} G \circ \Psi$ making the diagram commute

– use Example 3.78 to produce homotopy

- $\phi(f)(g,h) = (\delta_h \cdot (f(h)\delta_e))(g)\delta_e = f(h)\delta_h(g)$

- $\eta_{A \rtimes_{\max} G}(\phi(f)(g,h)) = \sum_{h \in G} \phi(f)(g,h) = f(g)$

□

3.3 Forcing exactness and Bott

3.3.1 The localization $L_!$

$! \in \{\mathrm{ex}, \mathrm{se}, \mathrm{splt}\}$

want a left exact localization

$$L_! : L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

- such that

$$L_{h,K_G,!} : GC^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{L_k} GC^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_{K_G}} L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_!} L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

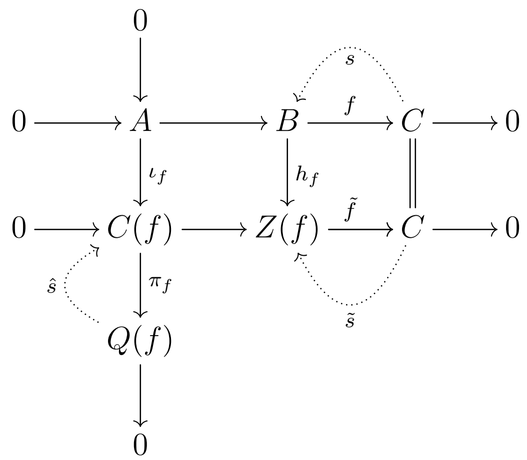
sends !-exact sequences of C^*-algebras to fibre sequences

- in case $! = \mathrm{se}, \mathrm{splt}$: require the corresponding splits equivariant

consider !-split exact sequence of G-C^*-algebras

$$0 \to A \to B \xrightarrow{f} C \to 0$$

form diagram:



(3.2)

$\hat{W}_!$ - set of morphisms $L_{h,K_G}(\iota_f)$ for all !-exact sequences as above with C contractible

- $W_!$ - closure of $\hat{W}_!$ under 2-out-of 3 and pull-backs

Definition 3.84.

$$L_! : L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

is the Dwyer Kan localization at $W_!$.

Proposition 3.85.

1. $L_!$ *is left exact.*

2. $L_!$ *symmetric monoidal.*

3. $\otimes$ *on* $L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ *is bi-left exact.*

4. $L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ *is semi-additive and* $L_!$ *preserves finite coproducts.*

Proof. same as non-equivariant case □

universal properties:

- for any left exact ∞-category $\mathbf{D}$:

$$L^*_{h,K_G,!} : \mathbf{Fun}^{\mathrm{lex}}(GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{h,Gs,Sch+!}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$$

- for any symmetric monoidal left exact ∞-category $\mathbf{D}$:

$$L^*_{h,K_G,!} : \mathbf{Fun}^{\otimes,\mathrm{lex}}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \stackrel{\simeq}{\to} \mathbf{Fun}^{\otimes,h,Gs,Sch+!}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$$

there is a separable version of all that

Remark 3.86 (Descend of functors)**.**

the functors Res^L_G, $\mathtt{Ind}^G_H$ and $- \rtimes_? G$ preserve suitable exact sequences but:

- it is not clear that they preserve Schochet fibrations

- therefore not clear that the descends to $L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_h$ are left-exact

- they perserve $\hat{W}_!$

- but not clear that they preserve $W_!$

– so do not expect that these functors descend to $L_{K_G}GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$

- fortunatlely this is intermediate step □

3.3.2 Bott periodicity and $\mathrm{KK}^G_{\mathrm{sep}}$ and $\mathrm{E}^G_{\mathrm{sep}}$

have Toeplitz extension

$$0 \to K \to \mathcal{T} \to C(S^1) \to 0$$

- no G-action

- reduced Toeplitz extension

$$0 \to K \to \mathcal{T}_0 \to S(\mathbb{C}) \to 0$$

Lemma 3.87. *If $F : GC^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{M}$ is homotopy invariant, G-stable, split-exact and takes values in groups, then $F(\mathcal{T}_0) \simeq 0$.*

Proof. same as in non-equivariant case ☐

! in $\{\mathrm{ex}, \mathrm{se}\}$

- reduced Toeplitz extension is semisplit

- get $\beta_{\mathbb{C},!} : \Omega^2(L_{h,K_G,!}(\mathbb{C})) \simeq \Omega(L_{h,K_G,!}(S(\mathbb{C}))) \to L_{h,K_G,!}(K)) \simeq L_{h,K_G,!}(\mathbb{C})$

- $\beta_{A,!} := \beta_{\mathbb{C},!} \otimes A$

for A in $GC^*\mathbf{Alg}^{\mathrm{nu}}$:

Corollary 3.88. *If $E : L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to \mathbf{M}$ is left exact and takes values in groups, then the boundary map $E(\beta_{A,!}) : E(\Omega^2_! A) \to E(A)$ is an equivalence.*

Proof. - consider $F := E(- \otimes A)$

- $F(\beta_{\mathbb{C},!}) = E(\beta_{A,!})$

- F of reduced Toeplitz sequence is E of $0 \to K \otimes A \to \mathcal{T}_0 \otimes A \to S(A) \to 0$

- is fibre sequence

- F annihilates middle term

☐

Corollary 3.89. *If A is a group in $L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$, then $\beta_{A,!} : \Omega^2_!(A) \to A$ in $L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ is an equivalence.*

Corollary 3.90. *We have a Bousfield localization*

$$\mathrm{incl} : (L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!})^{\mathrm{group}} \leftrightarrows L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} : \Omega^2_!$$

with counit $\beta : \Omega^2_! \to \mathtt{id}$.

have separable version

Definition 3.91. *We define the ∞-category*

$$\mathrm{KK}^G_{\mathrm{sep},!} := (L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{h,!})^{\mathrm{group}}$$

and

$$\mathrm{kk}_{\mathrm{sep},!} : GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{L_{\mathrm{sep},h}} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \xrightarrow{L_{K_G}} L_{\mathrm{sep},K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \xrightarrow{L_{\mathrm{sep},!}} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \xrightarrow{\Omega^2_{\mathrm{sep},!}} \mathrm{KK}^G_{\mathrm{sep},!}$$

Lemma 3.92. *If $F : GC^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{M}$ is a homotopy invariant and semi-exact functor, then it is Schochet exact.*

Proof.

note: Schochet exact means: F sends Schochet fibrant pull-back squares

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle Schochet} \\ C & \longrightarrow & D \end{array}$$

to pull-back squares

- by stability of $\mathbf{M}$: it suffices to consider case with $C = 0$, i.e. Schochet exact sequences

assume: $0 \to A \to B \to C \to 0$ is Schochet exact

- have diagram

$$\begin{array}{ccccc} F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) \\ \downarrow{\scriptstyle F(\iota_f)} & & \downarrow{\scriptstyle F(h_f)} & & \| \\ F(C(f)) & \longrightarrow & F(Z(f)) & \longrightarrow & F(C) \\ \downarrow & & & & \\ F(Q(f)) & & & & \end{array} .$$

- lower sequence is fibre sequence since mapping cone sequence is semi-exact and F is semiexact

L_h sends both sequences to fibre sequences by Schochet exactness

- $L_h(h_f)$ is equivalence

- $L_h(\iota_f)$ is equivalence

- hence $F(\iota_f)$ is equivalence by homotopy invariance of F

the horizontal sequence in the diagram above are equivalent

- upper sequence is fibre sequence □

consider

- $\otimes_?$ in connection with localization $! \in \{\mathrm{se}, \mathrm{ex}\}$

- allowed combinations:

! \ ?	min	max
se	yes	yes
ex	no	yes

Theorem 3.93.

1. $\mathrm{KK}^G_{\mathrm{sep},!}$ *is a stable* ∞*-category.*

2. $\mathrm{kk}^G_{\mathrm{sep},!}$ *is symmetric monoidal and* $\otimes_?$ *is bi-exact.*

3. $\mathbf{Fun}^{ex}(\mathrm{KK}^G_{\mathrm{sep},!}, \mathbf{D}) \overset{\mathrm{kk}^{G,*}_{\mathrm{sep},!}}{\simeq} \mathbf{Fun}^{h,Gs,!}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$ *for any stable* ∞*-category* $\mathbf{D}$.

4. $\mathbf{Fun}^{\otimes,ex}_{(\mathrm{lax})}(\mathrm{KK}^G_{\mathrm{sep},!}, \mathbf{D}) \overset{\mathrm{kk}^{G,*}_{\mathrm{sep},!}}{\simeq} \mathbf{Fun}^{\otimes,h,Gs,!}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$ *for any symmetric monoidal stable* ∞*-category* $\mathbf{D}$.

standard notation

$$\mathrm{KK}^G_{\mathrm{sep}} := \mathrm{KK}^G_{\mathrm{sep,se}}\ , \quad \mathrm{kk}^G_{\mathrm{sep}} := \mathrm{kk}^G_{\mathrm{sep,se}}$$
$$\mathrm{E}^G_{\mathrm{sep}} := \mathrm{KK}^G_{\mathrm{sep,ex}}\ , \quad \mathrm{e}^G_{\mathrm{sep}} := \mathrm{kk}^G_{\mathrm{sep,ex}}$$

3.3.3 Descend of functors

$L^G := \Omega^2_{\mathrm{sep},!} \circ L_{\mathrm{sep},!} : GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to \mathrm{KK}^G_{\mathrm{sep},!}$

by construction: for any stable ∞-category $\mathbf{D}$

$$L^* : \mathbf{Fun}^{ex}(\mathrm{KK}^G_{\mathrm{sep},!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex},!}(L_{K_G}C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \simeq \mathbf{Fun}^{!}(L_{K_G}C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}$$

use Lemma 3.92

- $\mathbf{Fun}^!$- which send (images of) !-exact sequences to fibre sequences

$G \to L$ - homomorphism

$$\begin{array}{ccc} L_{K_L} LC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} & \xrightarrow{\mathrm{Res}^L_G} & L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \\ \downarrow{\scriptstyle \mathrm{kk}^L_{\mathrm{sep},!}} & & \downarrow{\scriptstyle \mathrm{kk}^G_{\mathrm{sep},!}} \\ \mathrm{KK}^L_{\mathrm{sep},!} & \xrightarrow{\mathrm{Res}^L_G} & \mathrm{KK}^G_{\mathrm{sep},!} \end{array}$$

- $\mathrm{Res}^L_G : L_{K_L} LC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ preserves !-exact sequences

- $L^G \circ \mathrm{Res}^L_G \in \mathbf{Fun}^!(L_{K_L} LC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D})$ sends !-exact sequences to fibre sequences

Corollary 3.94. *We have a left-exact descended functor*

$$\mathrm{Res}^L_G : \mathrm{KK}^L_{\mathrm{sep},!} \to \mathrm{KK}^G_{\mathrm{sep},!}$$

$H \subseteq G$ closed subgroup

$$\begin{array}{ccc} L_{K_H} HC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} & \xrightarrow{\mathtt{Ind}^G_H} & L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \\ \downarrow{\scriptstyle \mathrm{kk}^G_{\mathrm{sep},!}} & & \downarrow{\scriptstyle \mathrm{kk}^G_{\mathrm{sep},!}} \\ \mathrm{KK}^H_{\mathrm{sep},!} & \xrightarrow{\mathtt{Ind}^G_H} & \mathrm{KK}^G_{\mathrm{sep},!} \end{array}$$

Lemma 3.95. $\mathtt{Ind}^G_H$ *preserves* !*-exact sequences.*

Proof. construct for any A natural retract:

$$\mathtt{Ind}^G_H(A) \xrightarrow{\alpha} C_0(\mathtt{supp}(\chi)) \otimes A \xrightarrow{\beta} \mathtt{Ind}^G_H(A)$$

- consider function $\chi \in C(G)$

– $\int_H \chi(gh)\mu(h) = 1$

– require that for every g in G there exists a open U of G and compact K in H such that $\chi(g'h) = 0$ for $g' \in U$, $h \notin K$

- define maps:

– $\alpha : f \mapsto (g \mapsto \chi(g)f(g))$

– $\beta : f \mapsto (g \mapsto \int_H \alpha_h f(gh)\mu(h))$

— check H-equivariance: $gh' \mapsto \int_H \alpha_h f(gh'h)\mu(h) = \alpha_{h',-1} \int_H \alpha_h f(gh)\mu(h)$

— check retract: $\beta(\alpha(f)) = f$

—- $\int_H \alpha(h)\chi(gh)f(gh)\mu(h) = \int_H \chi(g)f(g)\mu(h) = f(g)$

$C_0(\mathtt{supp}(\chi)) \otimes -$ is preserves !-exact sequences

- a retract of a !-exact sequence is again one

□

? in $\{\max, r\}$

Corollary 3.96. *We have a left-exact descended functor* $\mathtt{Ind}_H^G : \mathrm{KK}^H_{\mathrm{sep},!} \to \mathrm{KK}^G_{\mathrm{sep},!}$.

$\rtimes_? G$ preserves contractibility and zero

- use $(A \otimes C(X)) \rtimes_? G \cong (A \rtimes G) \otimes C(X)$

- it preserves contractible algebras

– use $\mathtt{Ind}_H^G(A \otimes C(X)) \cong \mathtt{Ind}_H^G(A) \otimes C(X)$

– $\mathtt{Ind}_H^G(0) \cong 0$

consider

$$\begin{array}{ccc} L_{K_G} GC^*\mathbf{Alg}^{\mathrm{nu}}_h & \xrightarrow{\rtimes_? G} & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \\ \downarrow{\scriptstyle \mathrm{kk}^G_{\mathrm{sep},!}} & & \downarrow{\scriptstyle \mathrm{kk}_{\mathrm{sep},!}} \\ \mathrm{KK}^H_{\mathrm{sep},!} & \xrightarrow{-\rtimes_?} & \mathrm{KK}^G_{\mathrm{sep},!} \end{array}$$

- $\rtimes_?$ in connection with localization $! \in \{\mathrm{se}, \mathrm{ex}\}$

- allowed combinations:

! \ ?	r	max
se	yes	yes
ex	no	yes

Lemma 3.97. $- \rtimes_? G$ *preserves* !*-exact sequences.*

Proof. for ex and max:

$0 \to I \to A \to Q \to 0$

$0 \to I \rtimes_{\max} G \to A \rtimes_{\max} G \to Q \rtimes_{\max} G \to 0$

$C_c(G, -)$ preserves exact sequences and takes values in pre- C^*-algebras

- compl is left-adjoint and preserves push-outs

remains to show: $I \rtimes_{\max} G \to A \rtimes_{\max} G$ is injective

- every rep of $I \rtimes^{\mathrm{alg}} G$ extends to rep of $A \rtimes^{\mathrm{alg}} G$

for se:

split induces split of $0 \to C_c(G, I) \to C_c(G, A) \to C_c(G, Q) \to 0$

- split extends to split under completion

– needs more analytic arguments

□

Corollary 3.98. *We have a left-exact descended functor* $- \rtimes G : \mathrm{KK}^G_{\mathrm{sep},!} \to \mathrm{KK}_{\mathrm{sep},!}$.

Corollary 3.99.

1. *Green's imprimitivity theorem: For* $H \subseteq G$ *closed:*
$$- \rtimes_? H \xrightarrow{\simeq} \mathtt{Ind}^G_H(-) \rtimes_? G : \mathrm{KK}^H_{\mathrm{sep},!} \to \mathrm{KK}^G_{\mathrm{sep},!} \ .$$

2. *For* $H \subseteq G$ *open and closed: We have adjunction*
$$\mathtt{Ind}^G_H : \mathrm{KK}^H_{\mathrm{sep},!} \leftrightarrows \mathrm{KK}^G_{\mathrm{sep},!} : \mathrm{Res}^G_H \ .$$

3. *Green-Julg Theorem: If* G *is compact, then we have an adjunction*
$$\mathrm{Res}_G : \mathrm{KK}_{\mathrm{sep},!} \leftrightarrows \mathrm{KK}^G_{\mathrm{sep},!} : - \rtimes G \ .$$

4. *Dual Green-Julg: If* G *is discrete, then we have an adjunction*
$$- \rtimes_{\max} G : \mathrm{KK}^G_{\mathrm{sep},!} \leftrightarrows \mathrm{KK}_{\mathrm{sep},!} : \mathrm{Res}_G \ .$$

3.3.4 Extension to from separable to all C^*-algebras

Definition 3.100. *We define:*

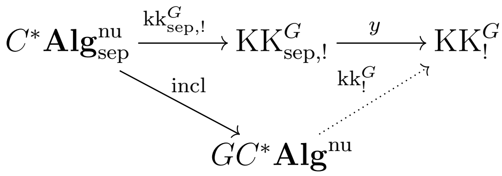
$$\mathrm{KK}^G_! := \mathtt{Ind}(\mathrm{KK}^G_{\mathrm{sep},!})$$

have canonical functor $y : \mathrm{KK}^G_{\mathrm{sep},!} \to \mathrm{KK}^G_!$

Definition 3.101. *We define:*

$$\mathrm{kk}_! : GC^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}^G_!$$

as the left Kan-extension



Proposition 3.102.

1. $\mathrm{KK}^G_!$ *and* $\mathrm{kk}_!$ *have symmetric monoidal refinements for* $\otimes_?$.

2. $$\mathbf{Fun}^{\mathtt{colim}}(\mathrm{KK}^G_!, \mathbf{D}) \overset{\mathrm{kk}^{G,*}_!}{\simeq} \mathbf{Fun}^{h,Gs,!,\mathrm{sfin}}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}) \tag{3.3}$$
   for any cocomplete stable ∞*-category*

3. $$\mathbf{Fun}^{\otimes,\mathtt{colim}}_{(\mathrm{lax})}(\mathrm{KK}^G_!, \mathbf{D}) \overset{\mathrm{kk}^{G,*}_!}{\simeq} \mathbf{Fun}^{\otimes,h,Gs,!,\mathrm{sfin}}_{(\mathrm{lax})}(GC^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$$
   for any cocomplete stable symmetric monoidal ∞*-category* $\mathbf{D}$.

standard notation

$$\mathrm{KK}^G := \mathrm{KK}^G_{\mathrm{se}}\ , \quad \mathrm{kk}^G := \mathrm{kk}^G_{\mathrm{se}}$$
$$\mathrm{E}^G_{\mathrm{sep}} := \mathrm{KK}^G_{\mathrm{ex}}\ , \quad \mathrm{e}^G := \mathrm{kk}^G_{\mathrm{ex}}$$

want to extend functors

C - a functor from $GC^*\mathbf{Alg}^{\mathrm{nu}}$ to $HC^*\mathbf{Alg}^{\mathrm{nu}}$

- for $A \to B$ define $C(A)^{C(B)}$ as image of $C(A) \to C(B)$

- assume: C preserves separable algebras

– then $C(A)^{C(B)}$ is separable provided A is separable

Definition 3.103. *We say that C is* `Ind`*-s-finitary if it has the following properties:*

1. *For every A in $GC^*\mathbf{Alg}^{\mathrm{nu}}$ the inductive system $(C(A')^{C(A)})_{A'\subseteq_{\mathrm{sep}}A}$ is cofinal in the inductive system of all invariant separable subalgebras of $C(A)$.*

2. *The canonical map $(C(A'))_{A'\subseteq_{\mathrm{sep}}A} \to (C(A')^{C(A)})_{A'\subseteq_{\mathrm{sep}}A}$ is an isomorphism in* `Ind`*$(HC^*\mathbf{Alg}^{\mathrm{nu}})$.*

Lemma 3.104. *Assume that C preserves separable algebras and satisfies Item 1. If C satisfies one of:*

1. *C preserves inclusions*

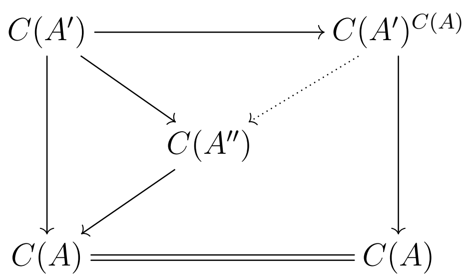
2. *C preserves countably filtered colimits*

then C is `Ind`*-s-finitary.*

Proof. Argument in case 2.

consider an invariant separable subalgebra A' of A

- gives the outer part of the following diagram



(3.4)

- poset of invariant separable subalgebras of A is countably filtered

C preserves countably filtered colimits

- $\mathtt{colim}_{A'\subseteq_{\mathrm{sep}}A}\, C(A') \cong C(A)$

– the left vertical arrow is the canonical inclusion into the colimit.

– let I be the kernel of $C(A') \to C(A')^{C(A)}$

– I is separable

– I is the kernel of $C(A') \to C(A)$.

– find an invariant separable subalgebra A'' of A such that I is annihilated by $C(A') \to C(A'')$

— use here countably filtered and annihilate a countable sets of generators of I

get dotted arrow.

- existence of A'' for given A' shows:

– the canonical map of inductive systems $(C(A'))_{A' \subseteq_{\mathrm{sep}} A} \to (C(A')^{C(A)})_{A' \subseteq_{\mathrm{sep}} A}$ has an inverse in $\mathtt{Ind}(\mathbf{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}))$.

[BELb, Lem. 4.3] □

Lemma 3.105. *If F is some s-finitary functor on $HC^*\mathbf{Alg}^{\mathrm{nu}}$ and C is $\mathtt{Ind}$-s-finitary, then the composition $F \circ C$ is an s-finitary functor on $GC^*\mathbf{Alg}^{\mathrm{nu}}$.*

Proof. A in $HC^*\mathbf{Alg}^{\mathrm{nu}}$

- must show: canonical morphism is an equivalence:

$$\operatorname*{\mathtt{colim}}_{A' \subseteq_{\mathrm{sep}} A} F(C(A')) \to F(C(A)) \tag{3.5}$$

Condition 3.103.2 implies equivalence:

$$\operatorname*{\mathtt{colim}}_{A' \subseteq_{\mathrm{sep}} A} F(C(A')) \xrightarrow{\simeq} \operatorname*{\mathtt{colim}}_{A' \subseteq_{\mathrm{sep}} A} F(C(A')^{C(A)})$$

Condition 3.103.1 implies equivalence:

$$\operatorname*{\mathtt{colim}}_{A' \subseteq_{\mathrm{sep}} A} F(C(A')^{C(A)}) \xrightarrow{\simeq} \operatorname*{\mathtt{colim}}_{B' \subseteq_{\mathrm{sep}} C(A)} F(B')$$

F is s-finitary: get equivalence

$$\operatorname*{\mathtt{colim}}_{B' \subseteq_{\mathrm{sep}} C(A)} F(B') \xrightarrow{\simeq} F(C(A))$$

composition of these equivalences is the desired equivalence (3.5). □

Proposition 3.106. *Assume*

1. *F preserve separable algebras*

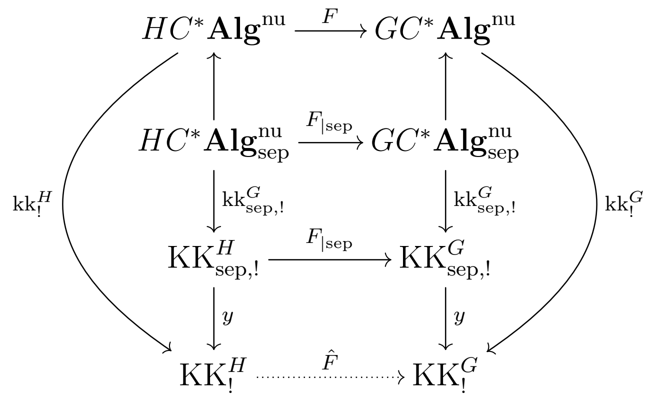
2. *$F_{|\mathrm{sep}}$ descends to $\mathrm{KK}_{\mathrm{sep},!}$*

3. *F is $\mathtt{Ind}$-s-finitary*

Then we have an essentially unique colimit- and compact object preserving factorization

$$\begin{array}{ccc} \mathrm{KK}^H_{\mathrm{sep},!} & \xrightarrow{F_{|\mathrm{sep}}} & \mathrm{KK}^G_{\mathrm{sep},!} \\ \downarrow y & & \downarrow y \\ \mathrm{KK}^H_{!} & \xrightarrow{\hat{F}} & \mathrm{KK}^G_{!} \end{array}$$

Proof.

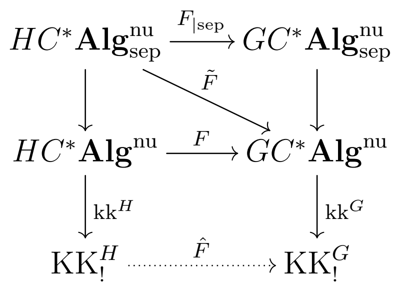


define $\hat{F}$ by universal property of $y : \mathrm{KK}^H_{\mathrm{sep},!} \to \mathrm{KK}^H_{!}$

- $\hat{F}$ preserves filtered colimits

- must show that "back face" of the cube commutes

-



- outer square commutes by construction

- the two triangles commute

- $\mathrm{kk}^G \circ \tilde{F}$ is s-finitary by Lemma 3.105

- $\hat{F} \circ \hat{\mathrm{kk}}^H$ is s-finitary by definition of kk^H and since $\hat{F}$ preserves filtered colimits

- $\hat{F} \circ \hat{\mathrm{kk}}^H$ is the left Kan extension of $\mathrm{kk}^G \circ \tilde{F}$

- $\mathrm{kk}^G \circ F$ is the left Kan extension of $\mathrm{kk}^G \circ \tilde{F}$

- hence both are equivalence.

$\square$

Proposition 3.107. Res^L_G, $\mathtt{Ind}^G_H$, $- \rtimes_{\max} G$ *and* $- \rtimes_r G$ *are* $\mathtt{Ind}$-*s-finitary and preserve separable algebras.*

Proof. preservation of separable algebras: clear (use that groups are second countable)

Res^L_G: $A' \subseteq \mathrm{Res}^L_G(A)$ G-invariant and separable

- cofinality

- A'' algebra generated by LA'

- is separable and L-invariant

- $A' \subseteq \mathrm{Res}^L_G(A'')$

Res^L_G - preserves inlcusions

- use Lemma 3.104

$\mathtt{Ind}^G_H$: preserves inclusions by same argument as Lemma 3.95

cofinality:

$B' \subseteq \mathtt{Ind}^G_H(A)$ separable

- $B' \subseteq C_0(\mathtt{supp}(\chi)) \otimes A$

- find separable $A' \subseteq A$ with $B' \subseteq C_0(\mathtt{supp}(\chi)) \otimes A'$

– use again that G is second countable

– Lemma 3.104

$\rtimes_{\max} G$:

- preserves filtered colimits

- cofinality (exercise)

– Lemma 3.104

$\rtimes_r G$:

- preserves inclusions

- cofinality (exercise)

– Lemma 3.104

$\square$

Corollary 3.108. *We have descended colimit- and compact object preserving functors*

1. *For any homomorphism $L \to G$:*
$$\mathrm{Res}^L_G : \mathrm{KK}^L_! \to \mathrm{KK}^G_! \ .$$

2. *For $H \subseteq G$ closed:*
$$\mathrm{Res}^L_G : \mathrm{KK}^L_! \to \mathrm{KK}^G_! \ .$$

3. $- \rtimes_r G : \mathrm{KK}^G \to \mathrm{KK}$ *for* $? \in \{r, \max\}$ *and* $- \rtimes_{\max} : \mathrm{E}^G \to \mathrm{E}$.

Corollary 3.109. *For* $!$ *in* $\{\mathrm{se}, \mathrm{ex}\}$*:*

1. *Green's imprimitivity theorem: For $H \subseteq G$ closed:*
$$- \rtimes_? H \xrightarrow{\simeq} \mathtt{Ind}^G_H(-) \rtimes_? G : \mathrm{KK}^H_! \to \mathrm{KK}^G_! \ .$$

2. *For $H \subseteq G$ open and closed: We have adjunction*
$$\mathtt{Ind}^G_H : \mathrm{KK}^H_! \leftrightarrows \mathrm{KK}^G_! : \mathrm{Res}^G_H \ .$$

3. *Green-Julg Theorem: If G is compact, then we have an adjunction*
$$\mathrm{Res}_G : \mathrm{KK}_! \leftrightarrows \mathrm{KK}^G_! : - \rtimes G \ .$$

4. *Dual Green-Julg: If G is discrete, then we have an adjunction*
$$- \rtimes_{\max} G : \mathrm{KK}^G_! \leftrightarrows \mathrm{KK}_! : \mathrm{Res}^G \ .$$

Proposition 3.110. Res^L_G *has symmetric monoidal refinement.*

Proof. have seen: $\mathrm{Res}^L_{G,|\mathrm{KK}^L_{\mathrm{sep}}}$ is symmetric monoidal

- $\mathtt{Ind} : \mathbf{Cat}^{ex}_\infty \to \mathbf{Pr}^L_{\mathrm{st}}$ is symmetric monoidal functor

- preserves algebras and algebra morphisms

- interpret symmetric monoidal categories and symmetric monoidal functors as commutative algebras an morphisms between them

□

4 Applications and calculations

4.1 K-homology

4.1.1 Basic Definitions

in general:

$\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is commutative ring:

– since $\mathbb{C}$ is commutative algebra and coalgebra

– composition product is second structure, a priori only associative

– in this case the same

Definition 4.1. *We define the equivariant K-theory spectrum $KU^G := \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ in* $\mathbf{CAlg}(\mathbf{Mod}(KU))$

KK^G is enriched in KU^G

G - compact group

- all irreducible unitary representations finite dimensional

- every unitary representation completely reducible (orthogonal sum of irreducible ones)

- $\hat{G}$ - set of equivalence classes of irreducible unitary rep's of G

- $L^2(G)$ has $G \times G$-action by left- and right translations

- $\pi \in \hat{G}$

– get homomorphism $V_\pi^* \otimes V_\pi \to L^2(G)$

– $v \otimes w \mapsto \langle v, \pi(g)w \rangle$

– check equivariance: $\pi(h)v \otimes \pi(l)w \mapsto \langle v, \pi(h^{-1}gl)w \rangle$

Proposition 4.2 (Peter-Weyl Theorem)**.**

$$\bigoplus_{\pi \in \hat{G}} V_\pi^* \otimes V_\pi \cong L^2(G)$$

as representation of $G \times G$*.*

Example 4.3. G - finite

- $|G| := \sum_{\pi \in \hat{G}} \dim(\pi)^2$

- can use this to show that one has found a complete set of representatives

consider representation ringoid:

- isoclasses if finite-dimensional (unitary) representations

- operations $\oplus$, $\otimes$

- form ring completion,

Definition 4.4. *The representation ring* $R(G)$ *is the ring completion of the ringoid of finite-dimensional representations.*

Lemma 4.5. *We have an isomorphism of groups* $R(G) \cong \mathbb{Z}[\hat{G}]$*.*

Example 4.6. C_2

- $\hat{C}_2 = \{1, \sigma\}$

- $\sigma^2 = 1$

- $R(C_2) \cong \mathbb{Z} \oplus \sigma\mathbb{Z}$

- $(n + \sigma m)(n' + \sigma m') = (nn' + mm') + \sigma(nm' + mn')$

- $R(C_2) \cong \mathbb{Z}[\zeta_2]$

Example 4.7. C_n

- choose nth root of unity, e.g. $\zeta_n := e^{\frac{2\pi i}{n}}$

- $\hat{C}_n \cong \mathbb{Z}/n\mathbb{Z}$

- for $[k] \in \mathbb{Z}/n\mathbb{Z}$ get

- $[l] \mapsto \zeta_n^l$

- $R(C_n) \cong \mathbb{Z}[\zeta_n]$ □

Example 4.8. $U(1)$

- $\widehat{U(1)} \cong \mathbb{Z}$

- $n \mapsto (u \mapsto u^n)$

- $R(U(1)) \cong \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[x, x^{-1}]$

Example 4.9. $G = SU(2)$

- $\hat{G}$ has basis $\pi_n := S^n(\mathbb{C}^2)/\mathtt{im}(\| - \|^2 S^{n+2}(\mathbb{C}^2))$

- $\dim(\pi_n) = n + 1$

- $\pi_n \otimes \pi_m \cong \pi_{n+m} + \pi_{n+m-2} + \dots$

- $R(G)$ has basis $(s_n)_{n\in\mathbb{N}}$ $s_n \cong S^n(\mathbb{C}^2)$ - not irreducible

- $s_n = \pi_n + \pi_{n-2} + \dots$

- $s_n s_m = s_{n+m}$

- $R(SU(2)) \cong \mathbb{Z}[x] \cong \mathbb{Z}[\mathbb{N}]$ □

Proposition 4.10. *If G is a compact group, then $KU_0^G \cong R(G)$ (as rings) and $KU_1^G \cong 0$.*

Proof. first calculate KU_*^G as a group

- Green-Julg: $KU^G = \mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \simeq \mathrm{KK}(\mathbb{C}, C^*(G)) \simeq K(C^*(G))$

- $C^*(G) \cong \bigoplus_{\pi\in\hat{G}} \mathtt{End}(V_\pi)$

- $K(C^*(G)) \simeq K(\bigoplus_{\pi\in\hat{G}} \mathtt{End}(V_\pi)) \simeq \bigoplus_{\pi\in\hat{G}} KU$

– use here: $K(\mathtt{End}(V_\pi)) \simeq K(\mathtt{Mat}_{\dim(\pi)}(\mathbb{C})) \simeq KU$

– $KU^G_* \cong \begin{cases} \bigoplus_{\pi\in\hat{G}} \mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases}$

– get $KU^G_* \cong R(G)$ as $\mathbb{Z}$-graded groups

(ρ, V_ρ) - finite-dimensional representation

- is $(\mathbb{C}, \mathbb{C})$-bimodule

- induces $[\rho] \in \mathrm{KK}^G_0(\mathbb{C}, \mathbb{C})$

– sum goes to sum

– tensor product goes to product

— get ring map $R(G) \to \mathrm{KK}^G_0(\mathbb{C}, \mathbb{C})$

must show that this is isomorphism

must show for π in $\hat{G}$

- $[\pi]$ goes to class of projection onto $1_\pi \in \mathtt{End}(V_\pi) \subseteq C^*(G)$

- under $- \rtimes G$ see that V_π goes to $(C^*(G), C^*(G))$-bimodule $V_\pi \rtimes G \cong L^2(G) \otimes V_\pi$

– under this identification:

– left G-action on both, $L^2(G)$ and V_π

– right G-action only on $L^2(G)$

- to complete the Green-Julg iso consider restriction along $\mathbb{C} \to C^*(G)$

– projection onto trivial subrepresentation

– insert Peter-Weyl for $L^2(G)$

– get $\mathbb{C}, C^*(G)$ -bimodule ${}^G(\bigoplus_{\pi'\in\hat{G}} V^*_{\pi'} \otimes V_{\pi'} \otimes V_\pi) \cong V_\pi$

- this is bimodule which represenents $\mathbb{C} \to 1_\pi$

□

Corollary 4.11. *If A is a G^*-C^*-algebra, then $K_*(A)$ is a module over $R(G)$.*

4.1.2 G-equivariant homology theories

we consider G**Top** - topological spaces with G-action and equivariant continuous maps

- it is topologically enriched

- distinguish a subclass of objects: G-CW-complexes

Definition 4.12. *An n-dimensional G-cell is a G-space of the form $G/H \times D^n$ for H closed in G.*

define G-CW-complexes inductively:

- let A be a G-space

Definition 4.13. *We consider A as -1-dimensional relative G-CW complex. An n-dimensional G-CW-complex X relative to A is a space obtained as a push-out (by attaching n-dimensional G-cells)*

$$\begin{array}{ccc} \coprod_{i\in I} G/H_i \times S^{n-1} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G/H_i \times S^{n-1} & \longrightarrow & X \end{array}$$

for some $n-1$-dimensional G-CW-complex Y. A G-CW-complex is a G-space which is has a filtration $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots$ by n-dimensional G-CW-complexes X_n such that X_{n+1} is obtained from X_n by attaching $n+1$-cells and $X \cong \mathtt{colim}_{n\in\mathbb{N}} X_n$.

G**CW** - full subcategory of G**Top** of G-CW complexes

- W_h - homotopy equivalences (use topological enrichment)

Definition 4.14. *We define the ∞-category of G-spaces $G\mathbf{Spc} := G\mathbf{CW}[W_h^{-1}]$ as the Dwyer-Kan localization of G-CW-complexes at homotopy equivalences.*

X in G**Top**

- H closed subgroup

- X^H - H-fixed points in X

$f : X \to Y$ - a morphism in G**Top**

Definition 4.15. *f is a G-weak equivalence, if $f^H : X^H \to Y^H$ is a weak equivalence in* **Top**.

W_{we} - weak equivalence in $G\mathbf{Top}$

Theorem 4.16. *The canonical map $G\mathbf{CW}[W_h^{-1}] \to G\mathbf{Top}[W_{we}^{-1}]$ is an equivalence.*

Corollary 4.17. $G\mathbf{Spc} \simeq G\mathbf{Top}[W_{we}^{-1}]$.

consider $G\mathbf{Orb}$ - full subcategory of $G\mathbf{Top}$ on orbits of G

- is topologically enriched

- presents an ∞-category (also denoted by $G\mathbf{Orb}$)

X in $G\mathbf{Top}$

- $S \in G\mathbf{Orb}$

- $X(S) := \ell\mathtt{Hom}_{G\mathbf{Top}}(S, X)$ in **Spc**

- get functor

$$G\mathbf{Top} \to \mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}) \simeq \mathbf{PSh}(G\mathbf{Orb}) \ , \quad X \mapsto X(-)$$

Theorem 4.18 (Elemendorf's theorem)**.** *The functor $G\mathbf{Top} \to \mathbf{PSh}(G\mathbf{Orb})$ presents the Dwyer-Kan localization of $G\mathbf{Top}$ at the weak equivalences.*

Corollary 4.19. $G\mathbf{Spc} \simeq \mathbf{PSh}(G\mathbf{Orb})$

Remark 4.20. $BG \simeq \mathtt{Aut}_{G\mathbf{Orb}}(G)$

$G\mathbf{Top} \to \mathbf{PSh}(G\mathbf{Orb}) \xrightarrow{\mathrm{ev}_G} \mathbf{Fun}(BG, \mathbf{Spc})$

- this is a further localization

- inverts maps whose underlying map is a homotopy equivalence

- $\mathbf{Fun}(BG, \mathbf{Spc})$ is the home of Borel equivariant homotopy theory

□

Definition 4.21. *An equivariant homology theory is a functor $E : G\mathbf{Orb} \to \mathbf{M}$ for a stable cocomplete target* **M**

get colimit preserving functor $E : \mathbf{PSh}(G\mathbf{Orb}) \to \mathbf{M}$

- get functor $E : G\mathbf{Top} \to \mathbf{M}$ which preserves weak equivalences and whose factorization over $\mathbf{PSh}(G\mathbf{Orb})$ preserves colimits

- will all be denoted by E

- for X in $G\mathbf{Top}$

$$E(X) \simeq \int_{G\mathbf{Orb}} X(S) \otimes E(S)$$

Definition 4.22. *An equivariant cohomology theory is a functor* $E : G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{M}$ *for a stable complete target* $\mathbf{M}$.

get limit preserving functor $E : \mathbf{PSh}(G\mathbf{Orb})^{\mathrm{op}} \to \mathbf{M}$

- get functor $E : G\mathbf{Top}^{\mathrm{op}} \to \mathbf{M}$ which preserves weak equivalences and whose factorization over $\mathbf{PSh}(G\mathbf{Orb})^{\mathrm{op}}$ preserves limits

- will all be denoted by E

- for X in $G\mathbf{Top}$

$$E(X) \simeq \int^{G\mathbf{Orb}^{\mathrm{op}}} E(S)^{X(S)}$$

4.1.3 Equivariant K-theory for compact groups

G - a compact group

- have functor $G\mathbf{Orb}^{\mathrm{op},\delta} \to GC^*\mathbf{Alg}^{\mathrm{nu}}$: $S \mapsto C_0(S)$ (consider $G\mathbf{Orb}$ as discrete category)

- use here compactness of G in order to ensure that morphisms in $G\mathbf{Orb}$ are proper and therefore preserve C_0-functions

now $G\mathbf{Orb}$ and $GC^*\mathbf{Alg}^{\mathrm{nu}}$ as enriched

- the functor is enriched

- factorizes over $G\mathbf{Orb}^{\mathrm{op}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}_h$

- apply kk^G_h

- get functor $K^G : G\mathbf{Orb}^{\mathrm{op}} \to \mathrm{KK}^G$

- define $K_G := \underline{\mathrm{KK}}^G(K^G, \mathbb{C}) : G\mathbf{Orb} \to \mathrm{KK}^G$

Definition 4.23. *The functors K^G and K_G represent G-equivariant KK^G-valued K-theory and K-homology.*

B in KK^G

- can introduce coefficients in B:

- $K^G_B := K^G \otimes B$

- $K_{G,B} := \underline{\mathrm{KK}}^G(K^G, B)$

– if B is a commutative algebra, then K^G_B takes values in commutative rings

– since $C_0(S)$ is a commutative algebra in $GC^*\mathbf{Alg}^{\mathrm{nu}}$

calculate values on orbits

- use: $C_0(G/H) \simeq \mathtt{Ind}^G_H(\mathbb{C})$

- $\mathtt{Ind}^G_H(A) \otimes B \cong \mathtt{Ind}^G_H(A \otimes \mathrm{Res}^G_H(B))$

– get - $K^G_B(G/H) \simeq C_0(G/H) \otimes B \simeq \mathtt{Ind}^G_H(\mathrm{Res}^G_H(B))$

- $K_{G,B}(G/H) \simeq \mathrm{Coind}^G_H(\mathrm{Res}^G_H(B))$

consider $G\mathrm{LCH}_{\mathrm{prop}}$ - locally compact G-spaces and proper maps

$X \mapsto \mathrm{kk}^G(C_0(X))$

- B in KK^G

Proposition 4.24. *If X is homotopy equivalent to a retract of a finite G-CW complex, then $\mathrm{kk}^G(C_0(X)) \otimes B \simeq K^G_B(X)$ and $\underline{\mathrm{KK}}^G(C_0(X), B) \simeq K_{G,B}(X)$.*

Proof. the class of X for which this is an equivalence has the following closure properties:

- contains $G\mathbf{Orb}$

- is invariant under homotopy equivalence

- is invariant under retracts

- is invariant under attaching G-cells

hence contains all locally compact spaces X which are homotopy equivalent to a retract of a finite G-CW complex

use:

- $G\mathrm{LCH}^{\mathrm{fd}}_{\mathrm{prop}}$ - homotopy retracts of finite G-CW complexes

- $G\mathrm{LCH}^{\mathrm{fd}}_{\mathrm{prop}} \to \mathbf{PSh}(G\mathbf{Orb})^{\omega}$ is localization at homotopy equivalence

- $\mathbf{Fun}^{\mathrm{Rex}}\mathbf{PSh}(G\mathbf{Orb})^{\omega}, \mathbf{M}) \simeq \mathbf{Fun}(G\mathbf{Orb}, \mathbf{M})$ for finitely cocomplete and idempotent complete target

- $F, F' : G\mathrm{LCH}^{\mathrm{fd}}_{\mathrm{prop}} \to \mathbf{M}$

- both homotopy invariant and excisive for cofibrant closed decompositions

- an equivalence $F_{|G\mathbf{Orb}} \simeq F'_{|G\mathbf{Orb}}$ extends essentially uniquely to an equivalence

□

absolute K-homology (in analogy to the usage of the "absolute" in arithmetic)

- $\mathbf{Mod}(KU^G)$ - valued K-theory and K-homology

- set $\mathrm{K}^G_B := \mathrm{KK}^G(\mathbb{C}, K^G_B) : G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Mod}(KU^G)$

- $\mathrm{K}_{G,B} := \mathrm{KK}^G(\mathbb{C}, K_{G,B}) : G\mathbf{Orb} \to \mathbf{Mod}(KU^G)$

Corollary 4.25. *If X is homotopy equivalent to a retract of a finite G-CW complex, then*

$$\mathrm{K}^G_B(X) \simeq K(C_0(X) \otimes B) \ , \quad \mathrm{K}_{G,B}(X) \simeq \mathrm{KK}^G(C_0(X), B) \ .$$

- $\pi_*\mathrm{K}^G_B(X)$ and $\pi_*\mathrm{K}_{G,B}(X)$ are modules over $R(G)$

$\mathcal{F}$ - a set of subgroups of G

Definition 4.26. *$\mathcal{F}$ is called a family of subgroups if it is invariant under conjugation and forming subgroups.*

Example 4.27.
1. $\mathcal{C}yc$
2. $\mathcal{A}ll$
3. $\mathcal{C}omp$ - compact subgroups
4. $\mathcal{F}in$ - finite subgroups
5. $\{e\}$ - trivial subgroup
6. $\mathcal{P}rop$ - proper
7. $\mathcal{VC}yc$ - virtually cyclic

fix family $\mathcal{F}$ of subgroups

- define ideal $I_{\mathcal{F}} := \bigcap_{H\in\mathcal{F}}(\mathtt{ker}(R(G) \to R(H))$

Example:

$I := I_{\{e\}}$ - dimension ideal

assuem G finite

- γ - conugacy class in G

- $\mathcal{F}(\gamma)$ - family of all $H \subseteq G$ with $H \cap \gamma = \emptyset$

- $(\gamma) \subseteq R(G)$ - ideal of ρ with $\mathtt{tr}\rho(\gamma) = 0$

- $L_{(\gamma)} : \mathbf{Mod}(KU^G) \leftrightarrow \mathbf{Mod}(KU^G)_{(\gamma)}$: incl

– symmetric monoidal Bousfield localization at $(KU^G \xrightarrow{\alpha} KU^G)_{\alpha\in R(G)\setminus\gamma}$

Lemma 4.28. $\mathrm{K}_{G,B}(-)_{(\gamma)}$ *vanishes on* $F(\gamma)$*.*

Proof. H in $\mathcal{F}(\gamma)$

- can find η in $R(G)$ with

– $\eta_{|H} = 0$

– $\mathtt{Tr}(\eta)(g) \neq 0$ for all g in γ

— hence $\eta \notin (\gamma)$

- η acts on $\mathrm{K}_{G,B}(G/H)_{(\gamma)}$ by $\eta_{|H} = 0$

- η acts invertibly on $\mathrm{K}_{G,B,(\gamma)}(G/H)$

– hence $\mathrm{K}_{G,B}(G/H)_{(\gamma)} = 0$ $\square$

X - G space

- X^γ - fixed points

- inclusion $X^\gamma \to X$

Theorem 4.29 (Segal localization). *If X^γ admits an invariant open neighbourhood such hat $X^\gamma \to N$, then*

$$\mathrm{K}_{G,B}(X^\gamma)_{(\gamma)} \to \mathrm{K}_{G,B}(X)_{(\gamma)}$$

is an equivalence

Proof. $X^{(\gamma)} \subseteq N$ - open invariant neighbourhood

- have push-out

$$\begin{array}{ccc} & & X^\gamma \\ & & \downarrow \\ N \setminus X^\gamma & \longrightarrow & N \\ \downarrow & & \downarrow \\ X \setminus X^\gamma & \longrightarrow & X \end{array}$$

- have push-out square

$$\begin{array}{ccc} & & K_{G,B}(X^\gamma)_{(\gamma)} \\ & & \simeq\downarrow \\ \mathrm{K}_{G,B}(N \setminus X^\gamma)_{(\gamma)} & \longrightarrow & K_{G,B}(N)_{(\gamma)} \\ \downarrow & & \downarrow \\ \mathrm{K}_{G,B}(X \setminus X^\gamma)_{(\gamma)} & \longrightarrow & \mathrm{K}_{G,B}(X)_{(\gamma)} \end{array}$$

left vertical arrow is $0 \to 0$

- right vertical arrow is equivalence

□

consider equivariant K-cohomology

- $\mathrm{K}^G_{B,*}(X)$ is $R(G)$-module

- $\mathcal{F}$ - a family of subgroups of G

Proposition 4.30. *If X is an n-dimensional G-CW complex with stabilizers in $\mathcal{F}$, then*

$$I^n_{\mathcal{F}} \pi_* \mathrm{K}^G_B(X) \cong 0$$

Proof. preparation:

assume: $H \in \mathcal{F}$

claim: $I_{\mathcal{F}} \pi_* \mathrm{K}^G_B(G/H) \cong 0$

- x in $I_{\mathcal{F}}$

- $x \otimes \mathrm{kk}^G(C_0(G/H)) \simeq \mathtt{Ind}^G_H(\mathrm{Res}^G_H(x)) = 0$

argue by induction by n

X_n - n-skeleton

long exact sequence

$\pi_* \mathrm{K}^G_B(X_n, X_{n-1}) \to \pi_* \mathrm{K}^G_B(X_n) \to \pi_* \mathrm{K}^G_B(X_{n-1}) \to \pi_{*-1} \mathrm{K}^G_B(X_n, X_{n-1})$

outer terms are annihilated by $I_{\mathcal{F}}$

- $\pi_* \mathrm{K}^G_B(X_{n-1})$ annihilated by $I^{n-1}_{\mathcal{F}}$

- z a class in $\pi_* \mathrm{K}^G_B(X_n)$

- i in $I^{n-1}_{\mathcal{F}}$

- iz comes from $\pi_* \mathrm{K}^G_B(X_n, X_{n-1})$

- one more application of element of $I_{\mathcal{F}}$ annihilates class

□

an $R(G)$-module M is $I_{\mathcal{F}}$-complete if

$M \to \mathtt{lim}_n M/I^n M := M\hat{}_I$

is an isomorphism

Corollary 4.31. *If X is a G-CW complex with stabilizers in $\mathcal{F}$ and $\mathtt{lim}^1 \pi_1 \mathrm{K}_B^G(X_n) \cong 0$, then $\pi_0 \mathrm{K}_B^G(X)$ is $I_{\mathcal{F}}$-complete*

Proof. always have Milnor sequence

$$0 \to \overset{1}{\mathtt{lim}}\, \pi_{*-1} \mathrm{K}_B^G(X_n) \to \pi_* \mathrm{K}_B^G(X) \to \mathtt{lim}\, \pi_* \mathrm{K}_B^G(X_n) \to 0$$

- by assumption $\pi_0 \mathrm{K}_B^G(X) \cong \mathtt{lim}\, \pi_0 \mathrm{K}_B^G(X_n)$

- $\mathtt{lim}_m\, \pi_0 K_B^G(X)/I_{\mathcal{F}}^m \cong \mathtt{lim}_{m,n}\, \pi_0 \mathrm{K}_B^G(X_n)/I_{\mathcal{F}}^m \pi_0 \mathrm{K}_B^G(X_n) \cong \mathtt{lim}_n\, \pi_0 \mathrm{K}_B^G(X_n) \simeq \pi_0 K_B^G(X)$

□

always have map $R(G) \to \pi_0 \mathrm{K}^G(X)$, $i \mapsto x \cdot 1$

- induced from $X \to *$

- get map $R(G)\hat{}_{I_{\mathcal{F}}} \to \pi_0 \mathrm{K}_B^G(X)$

Theorem 4.32 (Atiyah-Segal completion)**.** *$R(G)\hat{}_{I_{\{e\}}} \to \pi_* \mathrm{K}_B^G(BG)$ as isomorphism.*

Proof. later □

better approach:

- completeness as a property of M in $\mathbf{Mod}(KU^G)$

$x \in R(G)$

- $M \xrightarrow{x} M \to M/x$

- define completion at x by $\hat{M_x} := \texttt{lim}_n M/x^n$

$I \subseteq R(G)$ - an ideal

- need I to be finitely generated

- $I = (x_1, \dots, x_n)$

- define I-completion

- $\hat{M_I} := (\dots(\hat{M_{x_1}})\hat{}_{x_2} \dots)\hat{}_{x_n}$

- is independent of choice of generators

want $M \mapsto \hat{M_I}$ as left-adjoint of Bousfield localization

- M in $\mathbf{Mod}(KU^G)$ is I-torsion if M is in $\mathbf{Mod}(KU^G)^{\mathrm{perf}}$ and every element in $\pi_* M$ is annihilated by I^n for some n

- A in $\mathbf{Mod}(KU^G)$ is I-acyclic if $A \otimes_{KU^G} M \simeq 0$ for all I-torsion modules

- it is enough to check $(\dots(KU^G/x_1)/x_2)\dots)/x_n$ for the generators x_i of I

- i.e. $A[x_1^{-1}, \dots, x_n^{-1}] \simeq 0$

- $f : N \to N'$ in $\mathbf{Mod}(KU^G)$ is called a I-local equivalence if its cofibre is I-acyclic

- M is I-complete if $\texttt{map}(f, M)$ is an equivalence for all I-local equivalences

- have Bousfield localization $L_I : \mathbf{Mod}(KU^G) \to L_I\mathbf{Mod}(KU^G)$

- $L_I(M) \simeq \hat{M_I}$

for Bousfield localization $\mathbf{Mod}(KU^G) \to L_I\mathbf{Mod}(KU^G)$ of $\mathbf{Mod}(KU^G)$ at $(K(x) \to KU^G)_{x \in R(G) \setminus I}$

- I-adic completion

[GM97, Sec. 4]

Theorem 4.33. *If X is a CW-complex with stabilizers in $\mathcal{F}$, then $\mathrm{K}_B^G(X)$ is I-complete.*

Proof. $L_I\mathbf{Mod}(KU^G)$ is closed under limits

- $K_B^G(X)$ is a limit over K_B^G on finite subcomplexes

- if Y is finite G-CW complex with stabilizers in $\mathcal{F}$ then $K_B^G(Y)$ is I-complete □

4.1.4 Locally finite K-homology

G locally compact group

- $G\mathrm{LCH}_{\mathrm{prop}}$ - category of locally compact Hausdorff spaces with G-action and proper maps

- have functor $C_0(-) : G\mathrm{LCH}_{\mathrm{prop}}^{\mathrm{op}} \to GC^*\mathbf{Alg}^{\mathrm{nu}}$

- B in KK^G

- can consider $K_{c,B}^G : \mathrm{kk}(C_0(-)) \otimes B : G\mathrm{LCH}_{\mathrm{prop}}^{\mathrm{op}} \to \mathrm{KK}^G$

Definition 4.34. *The functor $K_{c,B}^G : G\mathrm{LCH}_{\mathrm{prop}}^{\mathrm{op}} \to \mathrm{KK}^G$ is called the compactly supported equivariant K-theory with coefficients in B*

Definition 4.35. *The functor $K_{G,B}^{lf} := \underline{\mathrm{KK}}^G(C_0(-), B) : G\mathrm{LCH}_{\mathrm{prop}} \to \mathrm{KK}^G$ is called the locally finite equivariant K-homology with coefficients in B*

Proposition 4.36. *$K_{c,B}^G$ and $K_B^{G,lf}$ are homotopy invariant and excisive for G-invariant cofibrant decompositions into closed subspaces.*

Remark 4.37. absolute versions

$$\mathrm{K}_{G,B}^{lf}(-) := \mathrm{KK}^G(C_0(-), B) : G\mathrm{LCH}_{\mathrm{prop}} \to \mathbf{Mod}(KU)$$

$$\mathrm{K}_{c,B}^G(-) := \mathrm{KK}^G(\mathbb{C}, C_0(-) \otimes B) : G\mathrm{LCH}_{\mathrm{prop}}^{\mathrm{op}} \to \mathbf{Mod}(KU)$$

assume: B is separable

- $\mathrm{K}_{c,B}^G(-)$ sends countable disjoint unions of second countable spaces into coproducts

- $\mathrm{K}_{G,B}^{lf}(-)$ sends countable disjoint unions of second countable spaces into products provided B is in $\mathrm{KK}_{\mathrm{sep}}$

– values: for G discrete (or more generally H clopen):

- use $(\mathtt{Ind}_H^G, \mathrm{Res}_H^G)$-adjunction

$$\mathrm{K}^{lf}_{G,B}(G/H) \simeq \mathrm{KK}^G(C_0(G/H), B) \simeq \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}_H^G(B))$$

– if H is in addition compact

$$\mathrm{K}^{lf}_{G,B}(G/H) \simeq \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}_H^G(B)) \simeq K(\mathrm{Res}_H^G(B) \rtimes H)$$

□

these are not equivariant homology or cohomology theories

- "wedge axiom" not satisfied

- can force an equivariant homology theory

$G\mathrm{LCH}^{G\mathrm{fin}}_{\mathrm{prop}}$ - spaces which are homotopy equivalent to finite G-CW complexes

Definition 4.38. *We define the representable KK^G-theory as the left Kan extension*

$$\begin{array}{ccc} G\mathrm{LCH}^{G\mathrm{fin}}_{\mathrm{prop}} & \xrightarrow{\mathrm{KK}^G(C_0(-)\otimes A, B)_{|G\mathrm{LCH}^{G\mathrm{fin}}_{\mathrm{prop}}}} & \mathbf{Mod}(KU) \\ & \searrow \qquad \nearrow_{R\mathrm{KK}^G(-,A,B)} & \\ & G\mathbf{Top} & \end{array}$$

special case: $R\mathrm{K}_{G,B}(-) := R\mathrm{KK}^G(-, \mathbb{C}, B)$

Proposition 4.39. *$R\mathrm{KK}^G(-, A, B)$ is an equivariant homology theory*

values on orbits:

$$R\mathrm{K}_{G,B}(G/H) \simeq \begin{cases} K(\mathrm{Res}_H^G(B) \rtimes H) & H \in \mathcal{C}omp \\ \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}_H^G(B)) & H \notin \mathcal{C}omp \end{cases}$$

Remark 4.40. warning this is not Kasparov's definition of $R\mathrm{KK}^G(X, A, B)$

- the latter uses $C_0(X)$-equivariant KK^G-theory of $A \otimes C_0(X)$ and $B \otimes C_0(X)$

- our definition is made to be a homology theory

– this is not clear (probably not true) for Kasparov's theory

4.2 Assembly maps

4.2.1 The Kasparov assembly map

G - locally compact group

Problem 4.41. *Does* $- \rtimes_r G : \mathrm{KK}^G \to \mathrm{KK}$ *has a left adjoint?*

Example 4.42. G compact:

- Green-Julg:

$$\mathrm{Res}_G : \mathrm{KK} \leftrightarrows \mathrm{KK}^G : - \rtimes_r G$$

- left adjoint in this case is Res_G

- $- \rtimes_r G$ preserves all limits □

in general:

Remark 4.43.

$\mathcal{C}, \mathcal{D}$ - left exact ∞-categories

- $R : \mathcal{C} \to \mathcal{D}$ - finite limit preserving functor

- apply $\mathrm{Pro} : \mathbf{Cat}^{\mathrm{lex}} \to \mathbf{Pr}^R$ (actually an equivalence)

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{R} & \mathcal{D} \\ \downarrow y_{\mathcal{C}} & & \downarrow y_{\mathcal{D}} \\ \mathrm{Pro}(\mathcal{C}) & \xrightarrow{\hat{R}} & \mathrm{Pro}(\mathcal{D}) \end{array}$$

- $\hat{R}$ preserves all limits

- $\hat{R}$ has left-adjoint $\hat{L}$

$\mathtt{Map}_{\mathcal{D}}(D, R(C)) \simeq \mathtt{Map}_{\mathrm{Pro}(\mathcal{D})}(D, y_{\mathcal{D}}(R(C))) \simeq \mathtt{Map}_{\mathrm{Pro}(\mathcal{D})}(D, \hat{R}(y_{\mathcal{C}}(C))) \simeq \mathtt{Map}_{\mathrm{Pro}(\mathcal{C})}(\hat{L}(D), y_{\mathcal{C}}(C)) \simeq \mathtt{colim}\,\mathtt{Map}_{\mathcal{C}}(\hat{L}(D), C)$

- here in last term interpret $\hat{L}(D)$ is a pro-system $(C_i)_{i \in I}$ in $\mathcal{C}$

- $\mathtt{Map}_{\mathcal{C}}(\hat{L}(D), C)$ is an inductive system $(\mathtt{Map}_{\mathcal{C}}(C_i, C))_{i \in I}$ in $\mathbf{Spc}$

- colimit is over I

$\square$

$- \rtimes_r G : \mathrm{KK}^G \to \mathrm{KK}$ preserves finite limits

- admits pro-left adjoint: $\hat{\mathrm{Res}}_G : \mathrm{Pro}(\mathrm{KK}) \leftrightarrows \mathrm{Pro}(\mathrm{KK}^G) : \widehat{- \rtimes_r G}$

- $\mathtt{colim}\, \mathrm{KK}^G(\hat{\mathrm{Res}}_G(A), B) \simeq \mathrm{KK}(A, B \rtimes_r G)$

Baum-Connes conjecture predicts candidate for $\hat{\mathrm{Res}}_G$:

Definition 4.44. *A classifying space $E_{\mathcal{F}}G$ for a family of subgroups $\mathcal{F}$ is a G-CW complex with*

$$E_{\mathcal{F}}G(G/H) \simeq \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}$$

in this definition: $E_{\mathcal{F}}G$ is a topological space

- use the notation also for homotopical object in $G\mathbf{Top}[W^{-1}]$, $G\mathbf{Spc}$ or $\mathbf{PSh}(G\mathbf{Orb})$

Lemma 4.45. *A classifying space $E_{\mathcal{F}}G$ (as CW-complex) exists.*

Proof. use Elmendorf:

- $i : G_{\mathcal{F}}\mathbf{Orb} \to G\mathbf{Orb}$

– $E_{\mathcal{F}}G \simeq i_! *_{\mathcal{F}}$

– $*_{\mathcal{F}}$ - final in $\mathbf{PSh}(G_{\mathcal{F}}\mathbf{Orb})$

$GCW[W_h^{-1}] \simeq G\mathbf{Spc} \simeq \mathbf{PSh}(G\mathbf{Orb})$

there exists G-CW-complex representing this homotopy type $i_! *_{\mathcal{F}}$ $\square$

Lemma 4.46. *If X is a G-CW complex with stabilizers in $\mathcal{F}$, then $\mathtt{Hom}_{G\mathbf{Top}}(X, E_{\mathcal{F}}G)$ is contractible.*

Proof. assumption on X:

- $X(-) \simeq i_! i^* X(-)$ for $i : G_{\mathcal{F}}\mathbf{Orb} \to G\mathbf{Orb}$

- i is fully faithful

- $i^* i_! \simeq \mathtt{id}_{\mathbf{PSh}(G_{\mathcal{F}}\mathbf{Orb})}$

- $i^* E_{\mathcal{F}} G \simeq *_{\mathcal{F}}$

use again $GCW[W_h^{-1}] \simeq G\mathbf{Spc} \simeq \mathbf{PSh}(G\mathbf{Orb})$

$$\begin{aligned} \ell\mathtt{Hom}_{G\mathbf{Top}}(X, E_{\mathcal{F}}G) &\simeq \mathtt{Map}_{\mathbf{PSh}(G\mathbf{Orb})}(X(-), E_{\mathcal{F}}G) \\ &\simeq \mathtt{Map}_{\mathbf{PSh}(G\mathbf{Orb})}(i_! i^* X(-), E_{\mathcal{F}}G) \\ &\simeq \mathtt{Map}_{\mathbf{PSh}(G_{\mathcal{F}}\mathbf{Orb})}(i^* X(-), i^* E_{\mathcal{F}}G) \\ &\simeq \mathtt{Map}_{\mathbf{PSh}(G_{\mathcal{F}}\mathbf{Orb})}(i^* X(-), *_{\mathcal{F}}) \\ &\simeq * \end{aligned}$$

□

Corollary 4.47. *The classifying space $E_{\mathcal{F}}G$ is unique up to contractible choice.*

choose G-CW complex $E_{\mathcal{F}}G$

Definition 4.48. *Let $\mathcal{E}_{\mathcal{F}}G$ denote the inductive system of G-finite subcomplexes of $E_{\mathcal{F}}G$ and inclusions.*

$\mathcal{E}_{\mathcal{F}}G$ is filtered

define

$$\hat{\mathrm{Res}}_G(A) \simeq (\mathrm{kk}^G(C_0(X)) \otimes \mathrm{Res}_G A)_{X \in \mathcal{E}_{\mathcal{C}omp}G}$$

$$\mathtt{colim}\, \mathrm{KK}^G(\hat{\mathrm{Res}}_G(A), B) \simeq \mathtt{colim}_{X \in \mathcal{E}_{\mathcal{F}}G} \mathrm{KK}^G(C_0(X) \otimes \mathrm{Res}_G A, B) \simeq R\mathrm{KK}^G(E_{\mathcal{C}omp}G, \mathrm{Res}_G A, B)$$

in order to identify $\hat{\mathrm{Res}}_G(-)$ as pro-adjoint must construct natural transformation

$$R\mathrm{KK}^G(E_{\mathcal{C}omp}G, \mathrm{Res}_G A, B) \to \mathrm{KK}(A, B \rtimes_r G)$$

- natural in B

assume now: X in $G\mathrm{LCH}_{\mathrm{prop}}$ with proper G-action such that X/G is compact will construct Kasparov's projection $p : \mathbb{C} \to C_0(X) \rtimes G$

Lemma 4.49. *There exists function χ in $C_c(X)$ with $\int_G \chi^2(g^{-1}x)\mu(g) = 1$ for all x.*

Proof.

for any $[x]$ in X/G choose preimage x in X and positive function χ_x in $C_c(X)$

- by compactness of X/G: can choose finite family $x_1, \dots, x_n$ such that image of $\bigcup_{i=1}^n \mathtt{supp}(\chi_x)$ in X/G is all of X/G

- set $\tilde{\chi} := \sum_{i=1}^n \chi_{x_i}$

- set $\rho(x) := \int_G \chi^2(g^{-1}x)\mu(g)$

- this function is positive and G-invariant

- set $\chi := \frac{\tilde{\chi}}{\sqrt{\rho}}$

- χ has the required properties □

from now on G unimodular (for simplicity):

- $g \mapsto (x \mapsto \chi(x)\chi(g^{-1}x))$ is element in $C_c(G, C_0(X))$

- by properness of action

- consider as element p_χ of $C_0(X) \rtimes_r G$

$$\begin{aligned} p_\chi^2(h, x) &= \int_G \chi(x)\chi(g^{-1}x)\chi(g^{-1}x)\chi((g^{-1}h^{-1})g^{-1}x)\mu(g) \\ &= \int_G \chi(x)\chi(g^{-1}x)\chi(g^{-1}x)\chi(h^{-1}x)\mu(g) \\ &= \chi(x)\chi(h^{-1}x) \\ &= p_\chi(x, h) \end{aligned}$$

check also: $p_\chi^* = p_\chi$: $p_\chi(g^{-1}x, g^{-1}) = \chi(g^{-1}x)\chi(gg^{-1}x) = p_\chi(g, x)$

Definition 4.50. *p_χ is called the Kasparov projection*

element of $\mathrm{KK}_0(\mathbb{C}, C_0(X) \rtimes_r G)$

Lemma 4.51. *The space $R(X)$ of χ in $C_c(X)$ with $\int_G \chi(g^{-1}x)\mu(g) = 1$ is contractible.*

Proof. Exercise

- see later

- will show: $\mathtt{sing}R(X)$ is trivial Kan complex ☐

Corollary 4.52. *The class p_χ is independent of the choice of χ.*

notation p_X

Definition 4.53. *The composition*

$$\mu^{Kasp}_{X,A,B} : \mathrm{KK}^G(C_0(X)\otimes \mathrm{Res}_G A, B) \xrightarrow{-\rtimes G} \mathrm{KK}((C_0(X)\otimes \mathrm{Res}_G A)\rtimes_r G, B\rtimes_r G) \xrightarrow{p_X\otimes A\circ} \mathrm{KK}(A, B\rtimes_r G)$$

is called the Kasparov assembly map for X with coefficients on B.

want a map of pro systems (natural in B)

$(\mathrm{KK}^G(C_0(X) \otimes \mathrm{Res}_G A, B))_{X\in\mathcal{E}_{CompG}} \to \mathrm{KK}(A, B \rtimes_r G)$

- must refine $\mu^{Kasp}_{X,A,B}$ this to natural transformation in X and B

$f : X \to Y$ proper G-equivariant

- $f^* : R(Y) \to R(X)$

- $\chi \in R(Y)$

the following commutes

$$\begin{array}{ccc}
A & \xrightarrow{(p_{f^*\chi}\otimes A)\rtimes_r G} & (C_0(X) \otimes A) \rtimes_r G \\
\| & & \downarrow{\scriptstyle (f^*\otimes A)\rtimes_r G} \\
A & \xrightarrow{(p_\chi\otimes A)\rtimes_r G} & (C_0(Y) \otimes A) \rtimes_r G
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{KK}^G(C_0(X) \otimes \mathrm{Res}_G A, B) & \xrightarrow{-\rtimes_r G} & \mathrm{KK}((C_0(X) \otimes \mathrm{Res}_G A) \rtimes_r G, B \rtimes_r G) & \xrightarrow{p_{f^*\chi}\otimes A} & \mathrm{KK}(A, B \rtimes_r G) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \| \\
\mathrm{KK}^G(C_0(Y) \otimes \mathrm{Res}_G A, B) & \xrightarrow{-\rtimes_r G} & \mathrm{KK}((C_0(Y) \otimes \mathrm{Res}_G A) \rtimes_r G, B \rtimes_r G) & \xrightarrow{p_\chi\otimes A} & \mathrm{KK}(A, B \rtimes_r G)
\end{array}$$

must improve this idea

- must get rid of choice of χ

superscript pc inducates proper cocompact G-action

Proposition 4.54. *We have a natural transformation of functors from* $G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}} \times \mathrm{KK}^{G,\mathrm{op}} \times \mathrm{KK} \to \mathbf{Mod}(KU)$

$$\mathrm{KK}^G(C_0(-) \otimes A, B) \to \mathtt{const}_{\mathrm{KK}(A, B\rtimes_r G)} \ .$$

Proof. $R : (G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}})^{\mathrm{op}} \to \mathbf{Set}$

- $X \mapsto R(X)$

- have natural transformation of functors $(G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}})^{\mathrm{op}} \to \mathbf{Set}$

$$p : R \to \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathbb{C}, C_0(-) \rtimes G)$$

- $X \mapsto (\chi \mapsto p_\chi)$

- naturality expresses: $f^* p_\chi = p_{f^*\chi}$

compose with $\Omega^\infty \mathrm{KK}$, interpret $R(-)$ with values in $\mathbf{Spc}$

- get natural transformation of functors $(G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}})^{\mathrm{op}} \to \mathbf{Spc}$

– $p : R \to \Omega^\infty \mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G)$

apply $(\Sigma^\infty_+, \Omega^\infty)$-adjunction

- get natural transformation of functors $(G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}})^{\mathrm{op}} \to \mathbf{Sp}$

– $p : \Sigma^\infty_+ R \to \mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G)$

consider functors $p, q : G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}} \times \Delta \to G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}}$

- $q : (X, [n]) \mapsto X \times \Delta^n$

- $p : (X, [n]) \mapsto X$

– $\Delta^n \to *$ induces natural transformation $q \to p$

$E : (G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}})^{\mathrm{op}} \to \mathbf{Sp}$ any functor

- define $\mathcal{H}(E) := q_! q^* E$ (homotopification)

- $\mathcal{H}(E)(X) \simeq \mathtt{colim}_{\Delta^{\mathrm{op}}} E(X \otimes \Delta^n)$

- $q_! p^* E(X) \simeq \mathtt{colim}_{\Delta^{\mathrm{op}}} E(X) \simeq E(X)$

- have natural transformation $p^* E \to q^* E$

- get $q_! p^* E \to q_! q^* E$

- hence $E \to \mathcal{H}(E)$

- call E homotopy invariant if $\mathtt{pr}_X^* : E(X) \to E(X \times \Delta^1)$ is an equivalence

Proposition 4.55. *E is homotopy invariant if and only of $E \to \mathcal{H}(E)$ is an equivalence.*

Proof. Exercise! □

Lemma 4.56. *$R \to *$ induces an equivalence $\mathcal{H}(\Sigma^\infty_+ R) \to \mathtt{const}_S$*

Proof. must show:

- $\mathtt{colim}_{\Delta^{\mathrm{op}}} \Sigma^\infty_+ R(X \otimes \Delta^n) \simeq S$

- $\mathtt{colim}_{\Delta^{\mathrm{op}}} R(X \otimes \Delta^n) \simeq *$ (in **Spc**, since Σ^∞_+ preserves colimits)

- $R(X \otimes \Delta^-)$ is simplicial space

- is levelwise discrete since R takes values in sets

- hence $R(X \otimes \Delta^-)$ is simplicial set

- $\mathtt{colim}_{\Delta^{\mathrm{op}}} R(X \otimes \Delta^n) \simeq |R(X \otimes \Delta^-)|$ - realization

suffices to show

- $R(X \otimes \Delta^-) \to *$ is trivial Kan fibration

- any $\chi \in R(X \otimes \partial\Delta^n)$ extends to $\tilde{\chi} \in R(X \otimes \Delta^n)$

- set e.g. $\tilde{\chi}(\sigma t) = \sqrt{\sigma\chi^2(x,t) + (1-\sigma)\chi^2(x,t_0)}$

– $t \in \partial\Delta$

- σt in Δ^n - barizentric coordinates

– t_0 - zeroth vertex of Δ^n

□

use that $\mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G)$ is homotopy invariant

- $\mathtt{const}_S \simeq \mathcal{H}(\Sigma^\infty_+ R) \to \mathcal{H}(\mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G)) \xleftarrow{\simeq} \mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G)$

$\mathtt{const}_S \to \mathrm{KK}(\mathbb{C}, C_0(-) \rtimes_r G) \to \mathtt{map}(\mathrm{KK}((C_0(-) \otimes A) \rtimes G, B), \mathrm{KK}(A, B \rtimes_r G))$

- second map is composition

- this yields desired natural transformation

$$\mathrm{KK}((C_0(-) \otimes A) \rtimes G, B) \to \mathtt{const}_{\mathrm{KK}(A, B\rtimes_r G)} : G\mathrm{LCH}^{pc}_{\mathrm{prop}} \to \mathbf{Mod}(KU)$$

□

restrict $R\mathrm{KK}^G(-, \mathrm{Res}_G A, B)$ to $G\mathbf{Top}_{/E_{\mathcal{C}omp}G}$

- the objects in $G\mathrm{LCH}^{G\mathrm{fin}}_{\mathrm{prop}}$ in this slice are in $G\mathrm{LCH}^{\mathrm{pc}}_{\mathrm{prop}}$

- get natural transformation

$$\mu^{Kasp}_{A,B} : R\mathrm{KK}^G(-, \mathrm{Res}_G A, B) \to \mathtt{const}_{\mathrm{KK}(A, B\rtimes_r G)}$$

Conjecture 4.57 (A generalized version of the Baum-Connes Conjecture)**.**

$$\mu^{Kasp}_{E_{\mathcal{C}omp}G,A,B} : R\mathrm{KK}^G(E_{\mathcal{C}omp}G, \mathrm{Res}_G A, B) \to \mathrm{KK}(A, B \rtimes_r G)$$

is an equivalence.

it presents $\hat{\mathrm{Res}}_G(A) \simeq (\mathrm{kk}^G(C_0(X)) \otimes \mathrm{Res}_G A)_{X \in \mathcal{E}_{\mathcal{C}omp}G}$ as pro-left adjoint of $- \rtimes_r G$

Conjecture 4.58 (Baum-Connes conjecture for G and B)**.** *The assembly map*

$$\mu^{Kasp}_{E_{\mathcal{C}omp}G,\mathbb{C},B} : R\mathrm{KK}^G(E_{\mathcal{C}omp}G, \mathrm{Res}_G \mathbb{C}, B) \to \mathrm{KK}(\mathbb{C}, B \rtimes_r G)$$

is an equivalence.

it is known to be false in general

- but still no counter example for $B = \mathbb{C}$

- if G is compact, then can take constant function

- in this case the Baum Connes conjecture is true: This is the Green-Julg theorem

4.2.2 The Meyer-Nest approach

in this section: G is discrete

- there is a version for locally compact groups

- it depends on generalization of the $(\mathtt{Ind}, \mathrm{Res})$-adjunction

– this has not been discussed in the course

Definition 4.59. *Define $\mathcal{CC}$ as the full subcategory of A in KK^G with $\mathrm{Res}^G_H(A) \simeq 0$ for all H in $\mathcal{C}omp$*

- the objects of $\mathcal{CC}$ are called weakly acyclic objects

- a morphism in KK^G is called a weak equivalence if its fibre is weakly acyclic

Lemma 4.60. *$\mathcal{CC}$ is a thick localizing tensor ideal*

Proof. Res^G_H is symmetric monoidal and preserves colimits □

Definition 4.61. *Define $\mathcal{CI}$ as the localizing subcategory generated by $\mathtt{Ind}^G_H(A)$ for all H in $\mathcal{C}omp$ and A in KK^H .*

Lemma 4.62. *$\mathcal{CI}$ is a tensor ideal.*

Proof. $\mathtt{Ind}^G_H(A) \otimes B \simeq \mathtt{Ind}^G_H(A \otimes \mathrm{Res}^G_H(B))$ □

- the objects of $\mathcal{CI}$ are called compactly induced objects

Example 4.63. $\mathrm{kk}^G(C_0(G/H))$ in $\mathcal{CI}$

X - a finite G-CW-complex with compact stabilizers

- then $C_0(X) \in \mathcal{CI}$ □

Lemma 4.64. *The category $\mathcal{CC}$ is the right complement of $\mathcal{CI}$, in particular*

$$\mathtt{map}_{\mathrm{KK}^G}(\mathcal{CI}, \mathcal{CC}) \simeq 0 \ .$$

Proof. $(\mathtt{Ind}, \mathrm{Res})$ - adjunction □

- it is at this point where we use discreteness of G

Lemma 4.65. *We have a smashing right Bousfield localization*

$$\mathrm{incl} : \mathcal{CI} \leftrightarrows \mathrm{KK}^G : P \ .$$

Proof. $\mathcal{CI}$ is localizing

- shows existence of adjunction

– is Dwyer-Kan equivalence at the weak equivalences

must show: smashing

– $P(A) \to A$ - counit

– $N(A) \to P(A) \to A$ cofibre sequence

– $N(A) \in \mathcal{CC}$

– since $\mathrm{KK}^G(Q, P(A) \to A)$ is equivalence for all Q in $\mathcal{CI}$

- $P(A) \simeq P(\mathbf{1}) \otimes A$

– $P(\mathbf{1}) \otimes A \in \mathcal{CI}$ (since $\mathcal{CI}$ is tensor ideal)

– $P(\mathbf{1}) \otimes A \to A$ is weak equivalence (since $\mathcal{CC}$ is a tensor ideal)

□

Definition 4.66. *The morphism $\alpha : P(\mathbf{1}) \to \mathbf{1}$ is called the Dirac morphism.*

Definition 4.67. *The map*

$$\mu^{MN}_{G,A,B} : \mathrm{KK}(A, P(B) \rtimes_r G) \to \mathrm{KK}(A, B \rtimes_r G)$$

is called the Meyer-Nest assembly map.

Proposition 4.68. *The Mayer-Nest and the Kasparov assembly maps are equivalent.*

Proof.

$$\begin{array}{ccc} R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, P(B)) & \xrightarrow{\simeq} & R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, B) \\ \simeq \downarrow \mu^{Kasp}_{G,A,P(B)} & & \downarrow \mu^{Kasp}_{G,A,B} \\ \mathrm{KK}(A, P(B) \rtimes G) & \xrightarrow{\mu^{MN}_{G,A,B}} & \mathrm{KK}(A, B \rtimes G) \end{array}$$

upper horizontal equivalence:

- $R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, N(B)) \simeq 0$

– $R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, N(B))$ is colimit of $\mathrm{KK}^G(C_0(X) \otimes A, N(B))$ for X finite G-CW complex with compact stabilizers

– $\mathrm{kk}^G(C_0(X) \otimes A) \in \mathcal{CI}$

right vertical equivalence: Oyono-Oyono (for discrete G), Chabert-Echterhoff for general G

- sketch:

- suffices to show equivalence for $\mathtt{Ind}^G_H(C)$ in place of B

$$\mathrm{KK}^G(C(X) \otimes \mathrm{Res}_G(A), \mathtt{Ind}^G_H(C)) \simeq \mathrm{KK}^H(C(\mathrm{Res}^G_H(X)) \otimes \mathrm{Res}_H(A), C)$$

- colimit over $X \subseteq \mathcal{E}_{\mathcal{C}omp}G$ calculates homology of $E_{\mathcal{C}omp}H \simeq *$

- $\mathrm{KK}^H(\mathrm{Res}_H(A), C) \simeq \mathrm{KK}(A, B \rtimes H) \simeq \mathrm{KK}(A, \mathtt{Ind}^G_H(C) \rtimes_r G)$

– Green imprimitivity □

dual Dirac

G - a discrete group

Lemma 4.69. *The following assertions are equivalent:*

1. *There exists* $\beta : \mathbf{1} \to P(\mathbf{1})$ *such that* $\beta \circ \alpha \simeq \mathtt{id}$.
2. $\mathrm{KK}^G(\mathcal{CC}, \mathcal{CI}) \simeq 0$
3. $\mathrm{KK}^G \simeq \mathcal{CI} \times \mathcal{CC}$

Definition 4.70. *A morphism* $\beta : \mathbf{1} \to P(\mathbf{1})$ *as in Lemma 4.69.1 is called a dual Dirac morphism and the composition* $\gamma := \alpha \circ \beta : \mathbf{1} \to \mathbf{1}$ *is called the* γ*-element.*

one says that G admits a γ-element

Proof. γ is idempotent

- $\gamma\mathcal{CC} = 0$

– use $\mathcal{CI} \otimes \mathcal{CC} \simeq 0$

– $(A \to P(A) \to A) \otimes \mathcal{CC} \simeq 0$

- $(1 - \gamma)_{|\mathcal{CI}} = 0$

– use: $P(A) \to A$ is equivalence for $A \in \mathcal{CI}$

– then $A \to P(A)$ is also equivalence

– $\gamma A = \mathtt{id}_A$

$1 \Rightarrow 2$:

$A \in \mathcal{CC}$

- $A = \gamma A + (1 - \gamma)A$

- $\gamma A = 0$

- $\mathrm{KK}^G((1 - \gamma)A, \mathcal{CI}) = \mathrm{KK}^G(A, (1 - \gamma)\mathcal{CI}) = 0$

$2 \Rightarrow 3$

- clear since also $\mathrm{KK}^G(\mathcal{CI}, \mathcal{CC}) \simeq 0$

$3 \Rightarrow 1$

$\mathbf{1}$ decomposes $P(\mathbf{1}) \oplus \mathbf{1}_{\mathcal{CC}}$

- take $\beta : \mathbf{1} \to P(\mathbf{1})$ the projection

□

Corollary 4.71. *If $\gamma = 1$, then the Baum-Connes conjecture with coefficients for G holds.*

Proof. $\mathrm{KK}^G \simeq \mathcal{CI}$

- $P(A) \to A$ is identity

□

Corollary 4.72. *If G admits a γ-element, then*

$$\mu^{Kasp}_{G,\mathbb{C},B} : R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, B) \to R\mathrm{KK}^G(E_{\mathcal{C}omp}G, A, B)$$

is split injective.

Proof. $\mu^{MN}_{G,A,B}$ admits a left inverse

□

injectivity is relevant: implies e.g. Novikov conjecture

Remark 4.73. existence of γ-element is usually shown by providing explicit candidate for β

Theorem 4.74 ([KS03])**.** *If G is discrete, acts isometrically and properly on a weakly bolic, weakly geodesic metric space of bounded coarse geometry, then G admits a γ-element.*

- a simply-connected complete non-positvely curved Riemannian manifold of bounded sectional curvature is an example of such a space

- Euclidean buildings with uniformly bounded ramification

□

4.2.3 The Davis Lück functor

consider

$G_{\mathcal{C}omp}\mathbf{Orb} \to \mathbf{Mod}(KU)$

- $S \mapsto \mathrm{KK}^G(C_0(S), B)$

– value is defined on all of $G\mathbf{Orb}$

– but not functorial for non-proper maps $G/H \to G/L$, i.e. if L/H is not compact

– value for compact H:

$$\mathrm{KK}^G(C_0(G/H), B) \simeq \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}^G_H(B)) \simeq K(B \rtimes_r H)$$

Problem 4.75. *Extend this to a functor* $G\mathbf{Orb} \to \mathbf{Mod}(KU)$.

- value at $*$ is $K(B \rtimes_r G)$

- defines equivariant homology theory

in the following describe solution if G is discrete

- first construction due to Davis-Lück [DL98] (with corrections by M. Joachim [Joa03])

$GC^*\mathbf{Cat}^{\mathrm{nu}}$ - category of C^*-categories with G-action

- construct $\mathbf{V} : \mathbf{Set} \to C^*\mathbf{Cat}^{\mathrm{nu}}$:

- describe C^*-category $\mathbb{C}[S]$:

– objects: elements of s

– morphisms: $\mathtt{Hom}_{\mathbb{C}[S]}(s, s') = \begin{cases} \mathbb{C} & s = s' \\ 0 & else \end{cases}$

- $f : S \to S'$

– induces obvious functor $s \mapsto f(s)$

go from C^*-categories to algebras

have adjunction

$$A^f : C^*\mathbf{Cat}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl}$$

- or with G-action

$$A^f : GC^*\mathbf{Cat}^{\mathrm{nu}} \leftrightarrows GC^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl}$$

- $\mathbb{C}[-] : G\mathbf{Set} \xrightarrow{\mathbf{V}} GC^*\mathbf{Cat}^{\mathrm{nu}} \xrightarrow{A^f} GC^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{\mathrm{kk}^G} \mathrm{KK}^G$

Proposition 4.76. $\mathrm{kk}^G(\mathbb{C}[S]) \simeq \mathrm{kk}^G(C_0(S))$

Proof. uses another functor

$Re : surveyA : C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{inj}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$

- subscript means: functors must be injective on objects

– $A^0(\mathbf{C}) := \bigoplus_{C,C' \in \mathbf{C}} \mathtt{Hom}_{\mathbf{C}}(C, C')$

– matrix multiplication

– is a pre-C^*-algebra

– $A(\mathbf{C})$ - closure of $A^0(\mathbf{C})$

- $A^f \to A$ - natural transformation (by universal property of A^f)

Proposition 4.77 (M. Joachim [Joa03])**.** $\mathrm{kk}^G(A^f(\mathbf{C})) \to \mathrm{kk}^G(A(\mathbf{C}))$ *is an equivalence.*

$A(\mathbb{C}[S]) \cong C_0(S)$

□

- not natural in S

- left-hand side is covariant

- right hand side is contravariant

Definition 4.78. *We define the Davis-Lück functor*

$$K^{DL}_{G,B} : G\mathbf{Orb} \to \mathrm{KK}^G$$

by

$$K^{DL}_{G,B} : G\mathbf{Orb} \xrightarrow{\mathbb{C}[-]} GC^*\mathbf{Cat}^{\mathrm{nu}} \xrightarrow{\mathrm{kk}^G} \mathrm{KK}^G \xrightarrow{-\otimes B} \mathrm{KK}^G \xrightarrow{-\rtimes_r G} \mathrm{KK}$$

$$\mathrm{K}^{DL}_{G,B} := \mathrm{KK}(-, K^{DL}_{G,B})$$

absolute version

Theorem 4.79. *There is an equivalence*

$$(\mathrm{K}^{DL}_{G,B})_{|G_{\mathbf{Fin}}\mathbf{Orb}} \simeq \mathrm{KK}^G(C_0(-), B)_{|G_{\mathbf{Fin}}\mathbf{Orb}}$$

Proof. this is a version of Paschke duality [BELa]

assume: H compact, discrete

$$\begin{aligned}\mathrm{K}^{DL}_{G,B}(G/H) &\simeq \mathrm{KK}(\mathbb{C}, (\mathbb{C}[G/H] \otimes B) \rtimes_r G) \simeq \mathrm{KK}(\mathbb{C}, (\mathtt{Ind}^G_H(\mathbb{C}) \otimes B) \rtimes_r G) \\ &\simeq \mathrm{KK}(\mathbb{C}, (\mathtt{Ind}^G_H(\mathrm{Res}^G_H(B)) \rtimes_r G) \simeq \mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H B \rtimes H) \\ &\simeq \mathrm{KK}^H(\mathrm{Res}_H\mathbb{C}, \mathrm{Res}^G_H B) \simeq \mathrm{KK}^G(C_0(G/H), B)\end{aligned}$$

- suffices to construct this equivalence natural in G/H

- is not easy

□

Corollary 4.80. $\mathrm{K}^{DL}_{G,B} \simeq R\mathrm{K}_{G,B}$ *on G-CW-complexes with compact stabilizers*

$\mathrm{K}^{DL}_{G,B}$ represents an equivariant homology theory

- $\mathrm{K}^{DL}_{G,B} \simeq R\mathrm{K}_{G,B}$ on G-CW-complexes with compact stabilizers

discuss now Davis-Lück assembly map

- $E : G\mathbf{Orb} \to \mathbf{M}$ any functor

- $\mathbf{M}$ cocomplete

- $\mathcal{F}$ - any family of subgroups

- $i : G_{\mathcal{F}}\mathbf{Orb} \to G\mathbf{Orb}$ - inclusion

– have adjunction $i_! : \mathbf{Fun}(G_{\mathcal{F}}\mathbf{Orb}, \mathbf{M}) \leftrightarrows \mathbf{Fun}(G\mathbf{Orb}, \mathbf{M}) : i^*$

– have counit $i_! i^* E \to E$

Definition 4.81. *The map* $\mathrm{Asmb}_{\mathcal{F},E} : i_! E(*) \to E(*)$ *is called the Davis-Lück assembly map associated to* E *and* $\mathcal{F}$

$\mathrm{Asmb}_{\mathcal{F},E} : \mathtt{colim}_{S \in G_{\mathcal{F}}\mathbf{Orb}} E(S) \to E(*)$

- in terms of homology theory

$E(E_{\mathcal{F}}G) \to E(*)$ induced by $E_{\mathcal{F}}G \to *$

Theorem 4.82 ([Kra20], [BELa])**.** *The Kasparov and Davis-Lück assembly maps are equivalent.*

$$\begin{array}{ccc} R\mathrm{K}_{G,B}(E_{\mathcal{C}omp}G) & \xrightarrow{\mu^{Kasp}_{E_{\mathcal{C}omp}G,\mathbb{C},B}} & R\mathrm{K}_{G,B}(*) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{K}^{DL}_{G,B}(E_{\mathcal{C}omp}G) & \xrightarrow{\mathrm{Asmb}_{\mathcal{C}omp,\mathrm{K}^{DL}_{G,B}}} & \mathrm{K}^{DL}_{G,B}(*) \end{array}$$

study dependence on B

- $K_G : \mathrm{KK}^G \to \mathbf{Fun}(G\mathbf{Orb}, \mathrm{KK})$

- $B \mapsto K^{DL}_{G,B}$

$i^G_H : H\mathbf{Orb} \to G\mathbf{Orb}$ - induction functor

- $i^G_H(S) := G \rtimes_H S$

Theorem 4.83 ([Kra20], [BELa])**.** *For any subgroup* H *of* G *we have a commutative square*

$$\begin{array}{ccc} \mathrm{KK}^H & \xrightarrow{K^{DL}_H} & \mathbf{Fun}(H\mathbf{Orb}, \mathrm{KK}) \\ \downarrow \mathtt{Ind}^H_G & & \downarrow i^G_{H,!} \\ \mathrm{KK}^G & \xrightarrow{K^{DL}_G} & \mathbf{Fun}(G\mathbf{Orb}, \mathrm{KK}) \end{array}$$

Corollary 4.84.

$$\begin{array}{ccc} \mathrm{K}^{DL}_{H,B}(E_{\mathbf{Fin}}H) & \xrightarrow{\mathrm{Asmb}_{\mathbf{Fin},\mathrm{K}^{DL}_{H,B}}} & K(B \rtimes_r H) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{K}^{DL}_{G,\mathtt{Ind}^G_H(B)}(E_{\mathbf{Fin}}G) & \xrightarrow{\mathrm{Asmb}_{\mathbf{Fin},\mathrm{K}^{DL}_{G,\mathtt{Ind}^G_H(B)}}} & K(\mathtt{Ind}^G_H(B) \rtimes_r G) \end{array}$$

Corollary 4.85. *If* $\mathrm{Asmb}_{\mathbf{Fin},\mathrm{K}^{DL}_{G,\mathbb{C},B}}$ *is an equivalence for all* B *in* KK^G*, then* $\mathrm{Asmb}_{\mathbf{Fin},\mathrm{K}^{DL}_{H,\mathbb{C},A}}$ *is an equivalence for all* A *in* KK^H*.*

The Baum-Connes conjecture with coefficients is inherited by subgroups.

4.3 The index class

4.3.1 KK-theory for graded algebras

in order construct index classes of Dirac operators naturally need graded C^*-algebras and corresponding KK-theory

we first introduce the corresponding structures

- we consider complex G-C^*-algebras

- we will interpret C_2-graded G-C^*-algebras as $G_2 := G \times C_2$-equivariant C^*-algebras

- the tensor product is modified to $\hat{\otimes}$

- Koszul sign rules

consider $G_2C^*\mathbf{Alg}^{\mathrm{nu}}$

- $A \in G_2C^*\mathbf{Alg}^{\mathrm{nu}}$

- have the following structure

– $\sigma \in C_2$ - non-trivial element

– $A \cong A_0 \oplus A_1$ as $\mathbb{C}$-vector space, eigenspace decomposition for σ

– A_0 - eigenvalue 1

– A_1 - eigenvalue -1

— write elements as $a_0 + a_1$

— A_0 is subalgebra

— $A_1 A_0 \subseteq A_1$, $A_0 A_1 \subseteq A_1$

— $A_1 A_1 \subseteq A_0$

graded tensor product on $G_2 C^* \mathbf{Alg}^{\mathrm{nu}}$:

change symmetry: $\hat{\otimes}$

- $\hat{\otimes}^{\mathrm{alg}} : G_2 C^* \mathbf{Alg}^{\mathrm{nu}} \to G_2{}^* \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$

- underlying bifunctor on $\otimes$

- symmetry: $s_{A,B} : A \hat{\otimes}^{\mathrm{alg}} B \to B \hat{\otimes}^{\mathrm{alg}} A$:

$$s_{A,B}((a_0 + a_1) \otimes b_0 + b_1)) = (b_0 \otimes a_0 - b_1 \otimes a_1) + (b_1 \otimes a_0 + b_0 \otimes a_1)$$

- this is the tensor product imported from C_2-graded vector spaces

– unit, associator and relations imported, so do not have to check

now check: $A \hat{\otimes}^{\mathrm{alg}} B$ is G_2-pre C^*-algebra

- form minimal or maximal completion

– yields $\hat{\otimes}_{\min}$ and $\hat{\otimes}_{\max}$

Lemma 4.86. *The functor* $\mathrm{kk}^{G_2} : G_2 C^* \mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}^{G_2}$ *has a symmetric monoidal refinement for* $\hat{\otimes}$*.*

Proof. need first to descend $\hat{\otimes}$ to $\mathrm{KK}^{G_2}_{\mathrm{sep}}$

- then extend to KK^{G_2}

- consider to version: minimal and maximal

- it is bicontinuous

– hence descends to homotopy localization

- it is associative

– hence descends to $\mathbb{K}_{G_2}$-stabilization

Lemma 4.87.

1. *$\hat{\otimes}_?$ is semi-exact for semiexact sequences of graded algebras for $? \in \{\min, \max\}$.*

2. *$\hat{\otimes}_{\max}$ is exact.*

Proof. exercise □

- $\hat{\otimes}$ descends to semiexact localization

$\hat{\otimes}$ preserves group objects

- by associativity

- $\hat{\otimes}$ descends to $\mathrm{KK}^{C_2}_{\mathrm{sep}}$

tensor unit of $\hat{\otimes}$ is $\mathbb{C}$

- trivially graded

now extend along `Ind`-completion

- arguments as in the ungraded case □

have functor

$\mathrm{Res}^G_{G_2} : \mathrm{KK}^G \to \mathrm{KK}^{G_2}$

- is symmetric monoidal

Example 4.88 (Examples of graded C^*-algebras)**.**

$\mathbb{C}$ with the trivial grading

- is the tensor unit of $\hat{\otimes}$

$\hat{\mathtt{Mat}}_2(\mathbb{C})$

- $2x2$-matrices with even odd grading

- is $\mathtt{End}(\mathbb{C} \oplus \mathbb{C}^{\mathrm{op}})$

Clifford algebra

- $\mathtt{Cl}^1 \cong \mathbb{C}[\sigma]/(\sigma^2 = 1)$

- $\deg(\sigma) = 1$

- $\sigma^* = \sigma$

- is isomorphic to $C^*(\hat{C}_2)$ as C_2-algebra

Lemma 4.89. *We have an isomorphism* $\mathtt{Cl}^1 \hat{\otimes} \mathtt{Cl}^1 \cong \hat{\mathtt{Mat}}_2(\mathbb{C})$ *in* $G_2C^*\mathbf{Alg}^{\mathrm{nu}}$.

Proof. - generators are τ and σ

– let σ act on $\mathtt{Cl}^1$ by left multiplication

– let τ act by $iz\sigma$ (z the grading operator)

– $iz\sigma^* = -i\sigma z = iz\sigma$

– $\tau\sigma + \sigma\tau = iz\sigma\sigma + \sigma iz\sigma = iz - iz = 0$

– $\tau\sigma = iz\sigma\sigma = iz$

— $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

— $\tau\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

— $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

— $\tau = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ □

$\hat{S}$

- C_2 acts on $\mathbb{R}$ by multiplication by -1

- $\hat{S} := C_0(\mathbb{R})$ with induced action in $C_2C^*\mathbf{Alg}^{\mathrm{nu}}$

- have semisplit exact sequence

$$0 \to C_0((0,\infty)) \otimes \mathtt{Cl}^1 \to \hat{S} \xrightarrow{\epsilon} \mathbb{C} \to 0$$

- $\epsilon : \hat{S} \to \mathbb{C}$ is $f \mapsto f(0)$

- $C_0(0,\infty) \otimes \mathtt{Cl}^1 \to \hat{S}$ sends $f_0 + \sigma f_1$ to $t \mapsto f_0(|t|) + \mathtt{sign}(t) f_1(|t|)$ $\hat{S}$ is represented on $L^2(\mathbb{R})$

- as multiplication operator

- Hilbert space again with flip action □

$\hat{S}$ is a coalgebra

counit:

$\epsilon : \hat{S} \to \mathbb{C}$ - evaluation at 0

$\hat{S}\hat{\otimes}\hat{S}$ acts on $L^2(\mathbb{R})\hat{\otimes}L^2(\mathbb{R})$

- this is $L^2(\mathbb{R})\hat{\otimes}L^2(\mathbb{R}) \cong L^2(\mathbb{R}^2)$ with the grading given by the flip action again

- define $\Delta : \hat{S} \to \hat{S}\hat{\otimes}\hat{S}$

- formally $f(x) \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}x)$

$\mathbb{R}^2$ has coordinates x_0, x_1

- on $L^2(\mathbb{R}^2)$ have operators

- x_0, x_1 - multiplication by coordinates

- have operators z_0, z_1 - grading operators

- $z_i\phi = \pm\phi$ depending on whether ϕ is even or odd in x_i

- $z_0\phi(x_0, x_1) := \frac{1}{2}((\phi(x_0, x_1) + \phi(-x_0, x_1)) - (\phi(x_0, x_1) - \phi(-x_0, x_1))$

- z_1 analogous

- define $\hat{x}_0 := x_0$

- $\hat{x}_1 := z_0 x_1$

- then

- $\hat{x}_0\hat{x}_1 + \hat{x}_1\hat{x}_0 = 0$

- consider unbounded odd operator $\hat{x}_0 + \hat{x}_1$ on $L^2(\mathbb{R}^2)$

- is selfadjoint

- define $\hat{S} \to B(L^2(\mathbb{R}^2))$

- $f \mapsto f(\hat{x}_0 + \hat{x}_1)$ by functional calculus

- this takes values in $\hat{S}\hat{\otimes}\hat{S}$

$\Delta : \hat{S} \to \hat{S}\hat{\otimes}\hat{S}$ is coproduct

obvious: $\epsilon \otimes \mathtt{id} : \hat{S} \to \hat{S}\hat{\otimes}\hat{S} \to \hat{S}$ is identity

- $x \mapsto \hat{x}_0 + \hat{x}_1 \to x$

Lemma 4.90. $(\hat{S}, \epsilon, \Delta)$ *is a commutative coalgebra in* $C_2C^*\mathbf{Alg}^{\mathrm{nu}}$.

Definition 4.91. *We define* $\hat{\mathrm{KK}}^G := \mathrm{Comod}_{\mathrm{KK}^{G_2}}(\mathrm{kk}^G(\hat{S}))$

have functor

$\mathrm{KK}^{G_2} \to \hat{\mathrm{KK}}^G, \quad A \mapsto \hat{S}\hat{\otimes}A$ - free comodule

define $\hat{\mathrm{kk}}^G : G_2C^*\mathbf{Alg}^{\mathrm{nu}} \to \hat{\mathrm{KK}}^G$ as composition

$$\hat{\mathrm{kk}}^G : G_2C^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{\mathrm{kk}^{G_2}} \mathrm{KK}^{G_2} \xrightarrow{\hat{S}\hat{\otimes}-} \hat{\mathrm{KK}}^G$$

Corollary 4.92. $\hat{\mathrm{KK}}^G(A, B) \simeq \mathrm{KK}^{G_2}(\hat{S} \otimes A, B)$.

this is here consequence of definition

- in the classical literature $\hat{\mathrm{KK}}^G_*(A,B)$ was define by Kasparov in terms of cycles and relations

- this formula is then a theorem by U. Haag [Haa99, Thm. 3.8]

$\hat{\mathrm{kk}}^G$ is symmetric monoidal functor

- comparison with ungraded case

$$\begin{array}{ccc} GC^*\mathbf{Alg}^{\mathrm{nu}} & \xrightarrow{\mathrm{Res}^G_{G_2}} & G_2C^*\mathbf{Alg}^{\mathrm{nu}} \\ \downarrow{\scriptstyle \mathrm{kk}^G} & & \downarrow{\scriptstyle \hat{\mathrm{kk}}^G} \\ \mathrm{KK}^G & \xrightarrow{i} & \hat{\mathrm{KK}}^G \end{array}$$

i from universal property of kk^G

- is symmetric monoidal

Proposition 4.93. *i is fully faithful.*

Proof.

$$\begin{aligned} \hat{\mathrm{KK}}^G(i(A), i(B)) &\simeq \mathtt{map}_{\mathrm{Comod}(\hat{S})}(\hat{S}\hat{\otimes}A, \hat{S}\hat{\otimes}B) \\ &\simeq \mathrm{KK}^{G_2}(\hat{S}\hat{\otimes}A, \mathrm{Res}^G_{G_2}B) \\ &\simeq \mathrm{KK}^G((\hat{S}\rtimes C_2)\otimes A, B) \\ &\simeq \mathrm{KK}^G(A,B) \end{aligned}$$

to this end show that $\hat{S}\rtimes C_2 \simeq \mathbf{1}$

- use exact sequence in $C_2C^*\mathbf{Alg}^{\mathrm{nu}}$

$$0 \to C_0((0,\infty))\hat{\otimes}\mathtt{Cl}^1 \to \hat{S} \to \mathbb{C} \to 0$$

- induces exact sequence in $C^*\mathbf{Alg}^{\mathrm{nu}}$

$$0 \to (C_0((0,\infty))\hat{\otimes}\mathtt{Cl}^1)\rtimes C_2 \to \hat{S}\rtimes C_2 \to \mathbb{C}\rtimes C_2 \to 0$$

- all algebras in bootstrap class

- apply K-theory

- discuss long exact sequence and show that

$$K_*(\hat{S} \rtimes C_2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases}$$

- conclude $\mathrm{kk}(\hat{S} \rtimes C_2) \simeq \mathbf{1}$

□

Lemma 4.94. *In* $\hat{\mathrm{KK}}^G$ *we have equivalence* $S(\mathbb{C}) \simeq \mathtt{Cl}^1$.

4.3.2 The index class

locally finite K-homology captures index classes

X - metric space with G-action by isometries

- H separable Hilbert space with unitary G-action

- $\phi : C_0(X) \to B(H)$ equivariant homomorphism

Definition 4.95. *The pair* (H, ϕ) *is called an equivariant* X*-controlled Hilbert space.*

Example 4.96.

choose G-invariant measure μ on X

- $H := L^2(X, \mu)$

- G-action by translations

– is isometric since μ is invariant

- $\phi : C_0(X) \to B(H)$ - action by multiplication operators

(H, ϕ) is equivariant X-controlled Hilbert space □

fix (H, ϕ) - equivariant X-controlled Hilbert space

- consider A in $B(H)^G$ - G-invariant operator

Definition 4.97. *The operator A is called controlled if there exists $R > 0$ such that if for all f, f' in $C_0(X)$ with $d(\texttt{supp}(f), \texttt{supp}(f')) > R$, we have $\phi(f)A\phi(f') = 0$. The infimum of these R is called the propagation of A.*

Definition 4.98. *A is locally compact if $\phi(f)A, A\phi(f) \in K(H)$ for all f in $C_0(X)$.*

Example 4.99 (integral operators)**.**

consider continuous function $k : X \times X \to \mathbb{C}$

- G-invariant: $k(gx, gy) = k(x, y)$ for all x, y in X and g in G

- assume k defines bounded integral operator on $L^2(X, \mu)$:

– $(A\psi)(x) := \int_X k(x, y)\psi(y)\mu(y)$

– $A \in B(H)^G$

- the boundedness condition is complicated in general

– but here is a simple case: if X/G is compact, then A is defined

- A is locally compact

– e.g.: $\phi(f)A$ factorizes as $L^2(X, \mu) \to C_{\texttt{supp}(f)}(U) \to L^2(X, \mu)$

— second map is compact

— first map is bounded (uses continuity of of k and finite propagation)

– hence A is locally compact

- assume: $k(x, y) = 0$ for $d(x, y) \geq R$

– then A is controlled with propagation R

Definition 4.100. *We define the Roe algebra $C^*(X, H, \phi)^G$ to be the C^*-algebra generated by the controlled and locally compact operators on H.*

Remark 4.101. in our example: the Roe algebra is generated by integral operators as above □

Definition 4.102. *The equivariant X-controlled Hilbert space (H, ϕ) is called ample if it absorbs any other X-controlled Hilbert space by a controlled equivariant unitary inclusion.*

this means:

- if (H', ϕ') is any X-controlled Hilbert space, then there exists isometry $U : H' \to H$ such that U is controlled

Remark 4.103 (existence of ample X-controlled Hilbert spaces)**.**

G trivial

- assume: $X = \mathtt{supp}(\mu)$

- then $(L^2(X, \mu) \otimes \ell^2, \phi \otimes \mathtt{id}_{\ell^2})$ is ample

- if there exists $R > 0$ such that $\dim(L^2(B(R, x), \mu)) = \infty$ for all x in X, then $(L^2(X, \mu), \phi)$ itself is ample

- for non-trivial G:

– it is more complicated [BE17, Prop. 4.2]

– requires assumptions on X

Proposition 4.104 ([BE17, Prop. 8.1 + 4.2])**.** *If X is the underlying metric space of a complete Riemannian G-manifold with a proper G-action, then X admits an equivariant ample X-controlled Hilbert space.*

□

assume: (H, ϕ) is ample

$C^*(X, H, \phi)^G$ contains any other $C^*(X, H', \phi')^G$ as corner

- full corner if (H', ϕ') is also ample

– $K(C^*(X, H, \phi)^G)$ is then independent of (H, ϕ)

Definition 4.105. *$K\mathcal{X}(X) := K(C^*(X, H, \phi)^G$ is called the coarse K-homology of X.*

Remark 4.106 (relation with equivariant coarse K-homology)**.**

for details: [BE17, Sec. 5], [BE23]

- there exists an equivariant coarse homology theory

$$K\mathcal{X}^G : G\mathbf{BC} \to \mathbf{Mod}(KU)$$

- $G\mathbf{BC}$ - category of G-bornological coarse spaces

– a metric space X with isometric G-acation represents an object of $G\mathbf{BC}$

assume X is very proper (e.g. underlying metric space of a complete Riemannian G-manifold with a proper G-action)

- then X admits an ample equivariant X-controlled Hilbert space (H, ϕ)

- $K(C^*(X, H, \phi)) \simeq K\mathcal{X}^G(X)$

- $f : X \to X'$ a proper controlled map

– controlled means: for all $S > 0$ exists $R > 0$ such that $d(x, y) < S$ implies $d'(f(x), f(y)) < R$.

– induces morphism in $G\mathbf{BC}$

– by functoriality get

– $f_* : K\mathcal{X}(X) \to K\mathcal{X}(X')$

functoriality cam be described in terms Roe algebras

- (H, ϕ) is X-controlled

- $f_*(H, \phi) := (H, \phi \circ f^*)$ is X' -controlled

- f_* induced by $C^*(X, H, \phi)^G \to C^*(X', H, \phi \circ f_*) \xrightarrow{U_*} C^*(X', H', \phi')$

– for choice of ample (H', ϕ')

– for $U : (H, \phi \circ f^*) \to (H', \phi')$ controlled □

Example 4.107 (Clifford algebras)**.**

V - an Euclidean vector space

- $\mathtt{Cl}(V)$ - C^*-algebra generated by V under $vw + wv = -2\langle v, w\rangle$ and $v^* = -v$

- is C_2-graded such that v in V is odd

- $\mathtt{Cl}^n := \mathtt{Cl}(\mathbb{R}^n)$

G - compact Lie group

- V - finite-dimensional unitary G-representation

Proposition 4.108 (Kasparov)**.** *In* $\hat{\mathrm{KK}}^G$ *we have* $\hat{\mathrm{kk}}^G(C_0(V)) \simeq \hat{\mathrm{kk}}^G(\mathtt{Cl}(V))$

$$\hat{\mathrm{KK}}_0^G(A \otimes \mathtt{Cl}^n, B) \simeq \hat{\mathrm{KK}}_0^G(A \otimes C_0(\mathbb{R}^n), B) \simeq \mathrm{KK}_{-n}^G(A, B) \qquad \square$$

M complete Riemannian manifold with isometric G-action

Definition 4.109. *An equivariant degree n Dirac bundle on M is a C_2-graded bundle of $\mathtt{Cl}^n$-right modules $E \to M$ with a metric and a connection ∇^E and a bilinear map $c : T^*M \otimes E \to E$ (the Clifford multiplication) such that*

1. *For Y in $T^*_m M$ the map $c(Y) : E_m \to E_m$ is odd and $\mathtt{Cl}^n$-linear.*
2. *$c(Y)^* = -c(Y)$ and $c(Y)^2 = -\|Y\|$*
3. *∇^E is hermitean, grading-preserving, and $[\nabla^E_X, c(Y)] = c(\nabla^{LC}_X Y)$ (compatibility with Levi-Civita connection)*
4. *For v in $\mathbb{R}^n$ the right-multiplication $\cdot v$ is odd, parallel, and satisfies $v^* = -v$.*
5. *All structures a G-invariant*

Example 4.110 (*Spin^c* Dirac operator)**.**

define Lie group $Spin^c(n)$

- $\mathtt{Cl}^n \cong \mathtt{Cl}(\mathbb{R}^n)$

- $SO(n)$ acts on $\mathbb{R}^n$

- $Spin^c \subseteq \mathtt{Cl}^{n,*}$

- subgroup of unitaries generated by $U(1)1_{\mathtt{Cl}^n}$ and xy for unit vectors x, y in $\mathbb{R}^n$

construct $Spin^c \to SO(n)$

- $u \mapsto u - u^*$

- preserves subspace $\mathbb{R}^n \subseteq \mathtt{Cl}^n$

- have exact sequence

$$0 \to U(1) \to Spin^c(n) \to SO(n) \to 0$$

M - oriented manifold

- $P \to M$ - $SO(n)$-principal bundle of oriented frames

Definition 4.111. *A $Spin^c$-structure is a reduction of structure groups of P to $Spin^c(n)$*

in detail: it is given by:

- $Q^c \to M$ - a $Spin^c$-principal bundle

- an isomorphism $Q^c \times_{Spin^c(n)} SO(n) \cong P$

- $S^c := Q^c \times_{Spin^c} \mathtt{Cl}^n$ is bundle of right $\mathtt{Cl}^n$-modules

- have $(\mathbb{R}^n)^* \otimes \mathtt{Cl}^n \to \mathtt{Cl}^n$ - left multiplication (and dualization using metric)

– induces Clifford multiplication $c : TM^* \otimes S^c \to S^c$ induced by left multiplication

- choose connection ∇^{S^c} on S^c which refines Levi-Civita connection

Proposition 4.112. *(S^c, ∇^{S^c}, c) is a Dirac bundle of degree* $\dim(M)$.

$Spin(n) \subseteq Spin^c(n)$ - a two-fold covering of $SO(n)$

Definition 4.113. *A Spin structure is a reduction of the structure group of Q^c to $Spin(n)$.*

- get Dirac bundle $S := Q \times_{Spin(n)} \mathtt{Cl}^n$

- has an additional real structure

- in this case ∇^S is unique: called the Spin connection □

concider Dirac bundle (E, c, ∇^E) of degree n

Definition 4.114. *The Dirac operator associated to the Dirac bundle is defined as the composition*

$$D := c \circ \nabla : \Gamma(S) \to \Gamma(M, T^*M \otimes S) \to \Gamma(S)$$

- it is $\mathtt{Cl}^n$-linear

first order G-invariant Differential operator

- $\sigma(D)^2(\xi) = \|\xi\|^2$

Lemma 4.115. *D is formally selfadjoint on $L^2(M, E)$*

an unbounded operator is essentially selfadjoint if its closure is selfadjoint

Lemma 4.116. *D is essentially selfadjoint with domain $\Gamma_0(X, S)$ on $H := L^2(X, S)$*

consider $H := L^2(M, E)$ as equivariant M-controlled Hilbert space

- can form e^{itD} - wave operator, unitary in $B(H)^G$

Theorem 4.117 (finite propagation speed)**.** *e^{itD} is controlled with propagation $|t|$*

$f \in C_0(\mathbb{R})$

- assume $\hat{f} \in C_c(\mathbb{R})$

– fix R with $\mathtt{supp}(\hat{f}) \subseteq [-R, R]$

- $\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\xi} dt$

- $f(D) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{itD} dt$ has propagation R

- $f(D)$ is G-invariant

- $f(D)$ is locally compact by Rellichs theorem

- conclude: $f(D) \in C^*(M, H, \phi)^G$

by density: $f(D) \in C^*(M, H, \phi)^G$ for any f in $C_0(\mathbb{R})$

- get homomorphism $\hat{S} \to C^*(M, H, \phi)^G$

- extends to $i(D) : \hat{S} \hat{\otimes} \mathtt{Cl}^n \to C^*(M, H, \phi)^G$

Definition 4.118. *The class of* $i(D)$ *in* $\mathrm{KK}(\hat{S}\hat{\otimes}\mathtt{Cl}^n, C^*(M,H,\phi)^G) \cong \hat{K}_{-n}(C^*(M,H,\phi)^G)$ *is called the equivariant coarse index class* $\mathtt{index}\mathcal{X}(D)$ *of* D*.*

if G acts properly, then $\mathtt{index}\mathcal{X} \in K\mathcal{X}^G_{-n}(M)$ naturally

Example 4.119. special case:

- M compact

- G trivial

- $C^*(M,H,\phi)^G \cong K$

- get class $\mathtt{index}\mathcal{X}(D)$ in $K_{-n}(K) \cong \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

this is usual index of Dirac operator

Definition 4.120 (Atiyah-Singer)**.** *The index of the Spin-Dirac operator is given by* $\langle \hat{A}(TM), [M]\rangle$*.*

here $\hat{A}(TM)$ - a characteristic class of TM

- can be expressed in terms of Pontrjagin classes (Chern class of $TM \otimes \mathbb{C}$)

there is a similar formula for the general case:

- $E \cong S \otimes V$

– for V - an auxiliary bundle (with metric and connection)

- $\mathtt{index}\mathcal{X}(D^E) = \langle \hat{A}(TM) \cup \mathbf{Ch}(V), [M]\rangle$

see [BGV04] for details □

Remark 4.121 (the K-homology class of a Dirac operator)**.**

there is a more basic class $[D] \in \mathrm{KK}^G(C_0(M) \otimes \mathtt{Cl}_n, \mathbb{C})$

- it is called the K-homology class of D

- is a class in $\mathrm{K}^G_{\mathbb{C},-n}(M)$

represented by a graded Kasparov module $(L^2(M,E), F, \phi)$

- $\phi : C_0(M) \otimes \mathtt{Cl}^n \to B(H)$ action by multiplication operators

- $F := \frac{D}{\sqrt{1+D^2}}$

- use [Mey00, Sec. 5 and 7] in order translate Kasparov modules to maps from $\hat{S}\hat{\otimes}C_0(M) \otimes \mathtt{Cl}^n$ to $B(H)$

the coarse way:

$K\mathcal{X}^G_{-n-1}(\mathcal{O}^\infty(M)) \simeq \mathrm{K}^G_{\mathbb{C},-n}(M)$

- $\mathcal{O}^\infty(M) = \mathbb{R} \times M$

- warped product metric

- $\tilde{g} = dt^2 + f(t)g$, $f(t) = 1$ for $t < 0$ and $f(t) = t^2$ for $t >> 0$

- canonical $\tilde{D}$ extension of D

– a selfadjoint deformation of $e_{n+1}\partial_t + D$

- is $\mathtt{Cl}_{n+1}$-equivariant

$[D]$ corresponds to $\mathtt{index}\mathcal{X}(\tilde{D})$ under isomorphism above

- for details on this approach: [Bun18]

□

back to the general case:

- D for a Dirac bundle

Lemma 4.122. *If the spectrum of D has a gap at 0, the* $\mathtt{index}\mathcal{X}(D) = 0$.

Proof. assume gap at 0

- $f(D)$ does not depend on values of f near 0

- $f \mapsto f(D)$ extends from $f \in C_0(\mathbb{R})$ to $C_0(-\infty, 0] \oplus C_0[0, \infty)$

- $\hat{\mathrm{K}}\mathrm{K}(C_0(-\infty, 0] \oplus C_0[0, \infty) \otimes \mathtt{Cl}_n, C^*(M, H, \phi)^G) = 0$

– since $C_0(-\infty, 0] \oplus C_0[0, \infty)$ is contractible □

Example 4.123 (application to spin Dirac operator).

M - oriented Riemannian complete spin

- G acts by automorphisms

- D - spin Dirac operator

- $D^2 = \Delta + \frac{s}{4}$ (Lichnerowicz formula)

- s - scalar curvature function

- if $s \geq c > 0$, then $\sigma(D) \cap (-c, c) = \emptyset$

- $\mathtt{index}\mathcal{X}(D) = 0$

Remark 4.124. $\mathtt{index}\mathcal{X}(D)$ only depends on the smooth spin manifold and coarse class of the metric

- if $\mathtt{index}\mathcal{X}(D) \neq 0$, then there is no metric with uniformly positive scalar curvature on the coarse equivalence class

Example 4.125. $\mathbb{R}^n$ with flat metric

- known: $\mathtt{index}\mathcal{X}(D) \neq 0$

– construct non-trivial pairings with K-theory classes on Higson corona

– see [Bun23, Ex. 7.6]

- there is no metric in the coarse class of the flat metric of uniformly positive scalar curvature

every $\mathbb{Z}^n$-periodic metric is in this class

Corollary 4.126. *T^n does not admit a metric of positive scalar curvature*

Remark 4.127. M compact spin

- $\mathtt{index}\mathcal{X}(D) = \langle \hat{A}(TM), [M] \rangle$ is a smooth invariant of M

– does not depend on metric

- $\alpha(M) \neq 0$ obstructs the existence of metric with positive scalar curvature □

Example 4.128 (coarse K-theory of free cocompact G-spaces).

assume:

- G acts cocompactly and freely on X

- (H, ϕ) - ample

Lemma 4.129. $C^*(X, H, \phi)^G \cong C_r^*(G) \otimes K$

$K\mathcal{X}^G(X) \cong K(C_r^*(G)$

a formal way to see this:

- $G_{can,min} \to X$, $g \mapsto gx_0$ is a coarse equivalence

- $K\mathcal{X}^G(G_{can,min}) \simeq K(C_r^*(G))$ by explicit calculation

□

4.3.3 Consequences of the Baum-Connes conjecture

for more information see: [MV03], [GAJV19],

Example 4.130 (The Gromov-Lawson-Rosenberg conjecture)**.**

G - a group

- M closed connected $Spin$-manifold with $\pi_1(M) = G$

– $n := \dim(M)$

- $\bar{M} \to M$ universal covering

- choose metric on M

– get G-invariant metric on $\bar{M}$

- $\bar{D}^{spin}$ - Spin-Dirac operator

- $\texttt{index}\mathcal{X}(\bar{D}^{spin}) \in K\mathcal{X}_{-n}(\bar{M}) \cong K_{-n}(C_r^*(G))$

since work with spin: all this has real version

- define $\alpha_G(M) := \texttt{index}\mathcal{X}(\bar{D}^{spin}) \in KO_{-n}(C_{r,\mathbb{R}}^*(G))$

Corollary 4.131. *If M admits psc-metric, then $\alpha_G(M) = 0$.*

Conjecture 4.132 (Gromov-Lawson-Rosenberg)**.** *If $\alpha_G(M) = 0$, then M admits a psc metric.*

has counter examples by Th. Schick

need modification:

- consider Bott manifold B:

- compact, spin, $\dim(B) = 8$, $\pi_1(B) = 1$

- $\mathtt{index}\mathcal{X}(D_B^{spin}) = \beta_{\mathbb{R}} \in KO_{-8}(\mathbb{R})$ Bott element - invertible element

- $\alpha_G(M)\beta_{\mathbb{R}} = \alpha_G(M \times B)$

Conjecture 4.133 (modified Gromov-Lawson-Rosenberg conjecture)**.** *If $\alpha_G(M) = 0$, then $M \times B^d$ admits a psc metric for sufficiently large d.*

have map equivariant map $f : \bar{M} \to EG$

- unique up to homotopy

- $[\bar{D}^{spin}] \in \mathrm{KKO}^G_{-n}(C_0(\bar{M}, \mathbb{R}), \mathbb{R}) \cong KO_{-n}(M)$ - equivariant K-homology class of $\bar{D}^{spin}$

- $f_*[\bar{D}^{spin}] \in R\mathrm{KKO}^G_n(EG, \mathbb{R}, \mathbb{R}) \cong KO_{-n}(BG)$

- under $KO_*(BG)_{\mathbb{Q}} \cong H_*(BG, \mathbb{Q}[p])$ with $|p| = 4$ this class is

Atiyah-Singer index theorem: $f_*[\bar{D}^{spin}]_{\mathbb{Q}} = f_*([M] \cap \hat{A}(TM))$

Conjecture 4.134 (Gromov-Lawson-Rosenberg)**.** *If $\bar{M}$ admits a metric of positive scalar curvature, then $f_*[\bar{D}^{spin}] = 0$. In particular $(f_*([M] \cap \hat{A}(TM)) = 0$.*

- higher $\hat{A}$-genera of M vanish

– in general: even if D is invertible the class $[D]$ can be non-zero

-$\mu^{Kasp}_{G,\mathbb{R},\mathbb{R}}(D^{spin}) = \alpha_G(M) \in KO_{-n}(C^*_{\mathbb{R},r}(G))$ - real version of Kasparov assembly map

Corollary 4.135. *Assume that $\mu^{Kasp}_{G,\mathbb{R},\mathbb{R}}$ (the real version) is injective (e.g. G admits a γ-element). Then if M admits a psc metric, then $f_*[\bar{D}^{spin}] = 0$ in $KO_{-n}(BG)$.*

this says that $f_*[\bar{D}^{spin}] = 0$ is necessary condition

- $f_*[\bar{D}^{spin}] = 0$ in $KO_{-n}(BG)$ is very close to existence of psc metric

- e.g. for trivial group: Stolz

□

Example 4.136 (signature operator)**.**

M oriented

$\dim(M) = 2l$ even

$E = \bigoplus_{i=0}^{n} \Lambda^i T^*M$

- has Dirac bundle structure of degree 0

- grading on p-forms by $i^{p(p-1)+l}*$ on $\Lambda^p T^*M$

– there exists a Dirac bundle structure

- Dirac operator $d + d^* = D^{\mathtt{sign}}$

- get class $\mathtt{index}\mathcal{X}(D^{\mathtt{sign}}) \in K\mathcal{X}_0^G(M)$

Proposition 4.137. *If M is compact and l is even, then* $\mathtt{index}\mathcal{X}(D^{\mathtt{sign}}) = \mathtt{sign}(M)$*.*

fix G

- consider M compact connected manifold with $G = \pi_1(M)$

- $\bar{M} \to M$ universal covering

- G-action

- $f : M \to BG$ classifying map

- $D^{\mathtt{sign}}$ gives rise to class $[D^{\mathtt{sign}}] \in \mathrm{KK}_0(C(M), \mathbb{C}) \cong K_0(M)$ - K-homology

Conjecture 4.138 (Novikov-Conjecture)**.** *The class $f_*[D^{\mathtt{sign}}]_{\mathbb{Q}}$ in $K_0(BG)_{\mathbb{Q}}$ only depends on the homotopy type of M.*

under $K_*(BG)_{\mathbb{Q}} \cong H_{\mathtt{ev}}(M, \mathbb{Q})$

- $f_*[D^{\mathtt{sign}}]_{\mathbb{Q}} = f_*([M] \cap L(TM))$

– $L(TM)$ - characteristisc class of tangent bundle

– apriori depends on smooth structure

– actually only on topological manifold

Conjecture 4.139 (Novikov-Conjecture). *The class $f_*([M] \cap L(TM))$ in $H_{\mathtt{ev}}(BG, \mathbb{Q})$ only depends on the homotopy type of M.*

- $\bar{D}^{\mathtt{sign}}$ - signature operator on $\bar{M}$

- $K^G(C_0(\bar{M}), \mathbb{C}) \cong K(C(M), \mathbb{C})$

- $[\bar{D}^{\mathtt{sign}}] = [D^{\mathtt{sign}}]$ under this iso

Theorem 4.140 (Mischenko-Fomenko). *The class $\mathtt{index}\mathcal{X}(\bar{D}^{\mathtt{sign}}) \in K_0(C_r^*(G))$ is a homotopy invariant of $\bar{M}$.*

Corollary 4.141. *If $\mu^{Kasp}_{G,\mathbb{C},\mathbb{C}}$ is rationally injective, then the Novikov conjecture holds for G.*

□

Example 4.142 (L^2-index theorem).

M closed compact, connected

- $\pi_1(M) = G$

- D - Dirac operator of degree 0

- $\mathtt{index}\mathcal{X}(D) \in K\mathcal{X}_0(M) \cong \mathbb{Z}$

- $\bar{M}$ - universal covering

- $\bar{D}$ - G-invariant

- $\mathtt{index}\mathcal{X}(\bar{D}) \in K\mathcal{X}_0(\bar{M}) \cong K_0(C_r^*(G))$

$\mathtt{tr} : C_r^*(G) \to \mathbb{C}$

- $f \mapsto f(e)$

- is faitful: $a \in C^*$, $a \geq 0$ and $\mathtt{tr}(a) = 0$ implies $a = 0$

- $\mathtt{tr}(1) = 1$

get induced map $\mathtt{tr} : K_0(C^*_r(G)) \to \mathbb{R}$

- $[p] \mapsto \mathtt{tr}(p)$

- extend $\mathtt{tr}$ to matrix algebras

Theorem 4.143 (Atiyah L^2-index theorem)**.**

$$\mathtt{tr}(\mathtt{index}\mathcal{X}(\bar{D})) = \mathtt{index}\mathcal{X}(D) \ .$$

□

Example 4.144 (Kadison-Kaplansky conjecture)**.**

Conjecture 4.145. *If G is torsion-free, then $C^*_r(G)$ does only have the trivial projections 0 and 1.*

Proposition 4.146. *If $\mu^{Kasp}_{G,\mathbb{C},\mathbb{C}}$ is surjective, then the Kadison-Kaplansky conjecture holds.*

Proof. claim: if p is projection in $C^*_r(G)$, then $\mathtt{tr}(p) \in \mathbb{Z}$

assume claim:

- note: $0 \leq p \leq 1$

- hence $\mathtt{tr}(p) \in \{0, 1\}$

- trace faithful

- hence $p \in \{0, 1\}$

show claim:

$p = \mu^{Kasp}_{G,\mathbb{C},\mathbb{C}}(x)$

- $x \in R\mathrm{KK}_0(EG, \mathbb{C}, \mathbb{C})$

- there exists $Spin^c$-manifold M of even dimension

- exists map $f : M \to BG$ (classifying $\bar{M}$)

- $\bar{M} \to EG$

- $x = f_*([D^{spin^c}])$

- $\mu^{Kasp}_{G,\mathbb{C},\mathbb{C}}(x) = \mathtt{index}\mathcal{X}(\bar{D}^{Spin^c})$ in $K_0(C^*_r(G))$

- Atyiah L^2-index theorem $\mathtt{tr}(p) = \mathtt{tr}\ \mathtt{index}\mathcal{X}(\bar{D}^{Spin^c}) = \mathtt{index}\mathcal{X}(D^{Spin^c}) \in \mathbb{Z}$

□

why do we need G to be torsion-free:

assume G has torsion element g

- order n

- $q := \frac{1}{n}\sum_{i=0}^{n-1} h^n$ is non-trivial projection

- $\mathtt{tr}(q) = \frac{1}{n}$

- so assumption on torsion of G is necessary

Question: Does $\mathtt{tr} : K(C^*_r(G)) \to \mathbb{R}$ take values in $1/n\mathbb{Z}$ where n is the is the common multiple of torsion

Corollary 4.147 (A consequence of Kadison-Kaplansky)**.** $\mathbb{Q}[G]$ *has no non-trivial idempotent*

Example 4.148 (Zero-in-the -spectrum conjecture)**.**

M - compact aspherical

Conjecture 4.149. 0 *is in the spectrum of of one of the Hodge Laplacians on* $\bar{M}$

$G = \pi_1(M)$

Proposition 4.150. *injectivity of the Assembly map implies the zero-in Zero-in-the -spectrum conjecture*

Proof. assume: $\dim(M)$ is even

note: $(\bar{D}^{\mathtt{sign}})^2 = \bigoplus_{n=0}^{\dim(M)} \Delta_n$

argue by contradiction

- then $\bar{D}^{\mathtt{sign}}$ is invertible

use: $[D^{\mathtt{sign}}] \neq 0$ in $K_0(M)$

- even rationally by Atiyah-Singer

- since $[M] \cap L(TM) \neq 0$

- look at degree-dim(M)-component which is $[M]$

$\mu^{Kasp}_{G,\mathbb{C},\mathbb{C}}([D^{\mathtt{sign}}]) = \mathtt{index}\mathcal{X}(\bar{D}^{\mathtt{sign}}) = 0$

contradiction

for even case cross with circle

□

Farber-Weinberger: there exists non-aspherical examples with no zero in the spectrum

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