# Orbifold index and equivariant $K$-homology

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**Abstract**

Let $G$ be countable group and $M$ be a proper cocompact even-dimensional $G$-manifold with orbifold quotient $\bar{M}$. Let $D$ be a $G$-invariant Dirac operator on $M$. It induces an equivariant $K$-homology class $[D] \in K_0^G(M)$ and an orbifold Dirac operator $\bar{D}$ on $\bar{M}$. Composing the assembly map $K_0^G(M) \to K_0(C^*(G))$ with the homomorphism $K_0(C^*(G)) \to \mathbb{Z}$ given by the representation $C^*(G) \to \mathbb{C}$ of the maximal group $C^*$-algebra induced from the trivial representation of $G$ we define $\mathrm{index}([D]) \in \mathbb{Z}$. In the second section of the paper we show that $\mathrm{index}(\bar{D}) = \mathrm{index}([D])$ and obtain explicit formulas for this integer. In the third section we review the decomposition of $K_0^G(M)$ in terms of the contributions of fixed point sets of finite cyclic subgroups of $G$ obtained by W. Lück. In particular, the class $[D]$ decomposes in this way. In the last section we derive an explicit formula for the contribution to $[D]$ associated to a finite cyclic subgroup of $G$.

# Contents

# 1 Introduction

Let $G$ be countable group and $M$ be a proper cocompact even-dimensional $G$-manifold with orbifold quotient $\bar{M}$. In the literature, orbifolds which can be represented as a global quotient of a smooth manifold by a proper action of a discrete group are often called good orbifolds.

Let $D$ be a $G$-invariant Dirac operator on $M$ acting on sections of a $G$-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded Dirac bundle $F \to M$. It induces an equivariant $K$-homology class $[D] \in K_0^G(M)$ and an orbifold Dirac operator $\bar{D}$ on $\bar{M}$ with index $\mathrm{index}(\bar{D}) \in \mathbb{Z}$. In the following we briefly describe these objects.

We can identify $\bar{D}$ with the restriction of $D$ to the subspace of $G$-invariant sections $C^\infty(M,F)^G$. The operator $\bar{D}$ is an example of an elliptic operator on an orbifold. Index theory for elliptic operators on orbifolds has been started with [Kaw81] (see also [Kaw79], [Kaw78] for special cases, and [Far92b], [Far92c], [Far92a] for alternative approaches). In particular, we have $\dim\ \ker(\bar{D}) < \infty$, and we can define

$$\mathrm{index}(\bar{D}) := \dim\ \ker(\bar{D}^+) - \dim\ \ker(\bar{D}^-)\ .$$

In the present paper we use the analytic definition of equivariant $K$-homology using equivariant $KK$-theory

$$K^G(M) := KK^G(C_0(M), \mathbb{C})\ .$$

The class $[D] \in KK^G(C_0(M), \mathbb{C})$ is represented by the Kasparov module $(\mathcal{E}, \mathcal{F})$ with $\mathcal{E} := L^2(M,F)$ and $\mathcal{F} := D(D^2+1)^{-1/2}$ (see Subsection 2.1 for more details).

Let $C^*(G)$ denote the unreduced group $C^*$-algebra of $G$. In general, the theory of the present paper would not work with the reduced group $C^*$-algebra $C_r^*(G)$. The key point is that finite-dimensional unitary representations of $G$ extend to representations of $C^*(G)$, but not to $C_r^*(G)$ in general.

We now consider the assembly map

$$ass : K_0^G(M) \to K_0(C^*(G))\ .$$

We use an analytic description of the assembly map which is part of Definition 2.1, and we refer to [MN06], [DL98] and [BM04] for modern treatements of assembly maps in general.

Composing the assembly map with the homomorphism $I_1 : K_0(C^*(G)) \to K_0(\mathbb{C}) \cong \mathbb{Z}$ given by the representation $1 : C^*(G) \to \mathbb{C}$ induced from the trivial representation of $G$ we define

$$\mathrm{index}([D]) := I_1 \circ ass([D]) \in \mathbb{Z} \ .$$

As a special case of the first main result Theorem 2.2 we get the equality

$$\mathrm{index}(\bar{D}) = \mathrm{index}([D]) \ . \tag{1}$$

Theorem 2.2 deals with the slightly more general case where the trivial representation *triv* of $G$ is replaced by an arbitrary finite-dimensional unitary representation of $G$. We think, that equation (1) was known to specialists, at least as a folklore fact.

The next result of the present paper is a nice local formula for index($[D]$). The main feature of local index theory is that one can calculate the index of a Dirac operator on a closed smooth manifold in terms of an integral of a local index form. A standard reference for local index theory is the book [BGV92]. Local index theory generalizes to Dirac operators on orbifolds. The index formulas in [Kaw81] and [Far92b] express the index of the Dirac operator on the orbifold as a sum of integrals of local index forms over the various strata. In the case of a good orbifold $G\backslash M$ the strata correspond to the fixed point manifolds $M^g$ of the elements $g \in G$. There are various ways to organize these contributions. For the purpose of the present paper we need a formula which expresses the index as a sum of contributions associated to the conjugacy classes of finite cyclic subgroups of $G$. We will state this formula in Corollary 2.4 (we refrain from giving a detailed statement here since this would require the introduction of too much of notation). In principle one could deduce the formula given in Corollary 2.4 by reorganising the previous results [Kaw81] and [Far92b]. But we found it simpler to prove the formula directly using the heat equation approach to local index theory and the local calculations from equivariant index theory [BGV92].

The proper cocompact $G$-manifold $M$ can be given the structure of a finite $G$-CW-complex. The equivariant $K$-homology of proper $G$-CW-complexes has been studied intensively in connection with the Baum-Connes conjecture. Rationally, $K^G(M)$ decomposes as a sum of contributions of conjugacy classes $(C)$ of finite cyclic subgroups $C \subset G$ (see (7) for a detailed statement). This decomposition is a consequence of a result of [Lüc02b] which is finer since it only requires to invert the primes dividing the orders of the finite subgroups of $G$. We thus can write $[D]$ as a sum of contributions $[D](C)$ where $(C)$ runs over the set of conjugacy classes of finite cyclic subgroups of $G$. Our last result Theorem 4.3 is the calculation of $[D](C)$. In the proof we use the index formula Corollary 2.4 as follows. By a result of [LO01b] the equivariant $K$-theory $K^0_G(M)$ has a description in terms of finite-dimensional $G$-equivariant vector bundles $E \to M$. We first derive a cohomological index formula Theorem 4.1 for the pairing of a $K$-homology class coming from a finite cyclic subgroup $C \subset G$ with the class $[E] \in K^0_G(M)$. In the proof we use the relation (1).

We then observe that the pairing of $[D]$ with $[E]$ is the index of the twisted operator $[D_E]$ which can be written as a sum of contributions of conjugacy classes of finite subgroups by 2.4. We obtain $[D](C)$ be a comparison of the formulas in Theorem 4.1 and Corollary 2.4 and variation of $E$.

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# 2 Assembly and orbifold index

## 2.1 The equivariant $K$-homology class of an invariant Dirac operator

Let $G$ be a countable discrete group. Let $M$ be a smooth proper cocompact $G$-manifold, i.e. a $G$-manifold such that the stabilizer $G_x$ is finite for all $x \in M$, and $G\backslash M$ is compact. We further assume that $M$ is equipped with a complete $G$-invariant Riemannian metric $g^M$ and a $G$-homogeneous Dirac bundle $(F, \nabla^F, \circ, (.,.)_F)$. Here $\circ : TM \otimes F \to F$ is the Clifford multiplication, $\nabla^F$ is a Clifford connection, $(.,.)_F$ is the hermitian scalar product, and these structures satisfy the usual compatibiliy conditions (see [BGV92], Ch.3) and are, in addition, $G$-invariant.

For simplicity we assume that $\dim(M)$ is even and that the Dirac bundle is $\mathbb{Z}/2\mathbb{Z}$-graded. In fact, the odd-dimensional case can easily be reduced to the even dimensional case by taking the product with $S^1$.

We use equivariant $KK$-theory in order to define equivariant $K$-homology. Thus let $KK^G$ be the equivariant $KK$-theory introduced in [Kas88] (see also [Bla98]). Let $C_0(M)$ be the $G$-$C^*$algebra of continuous functions on $M$ vanishing at infinity. Then by definition $K^G_0(M) = KK^G(C_0(M), \mathbb{C})$. The Dirac operator $D$ associated to the invariant Dirac bundle $F$ induces a class $[D] \in K^G_0(M)$ as follows. We form the $\mathbb{Z}/2\mathbb{Z}$-graded $G$-Hilbert space $\mathcal{E} := L^2(M, F)$. Then $C_0(M)$ acts on $\mathcal{E}$ by multiplication. Furthermore, we consider the bounded $G$-invariant operator $\mathcal{F} := D(D^2 + 1)^{-1/2}$ which is defined by applying the function calculus to the unique (see [Che73]) selfadjoint extension of $D$. Then $[D]$ is represented by the Kasparov module $(\mathcal{E}, \mathcal{F})$.

[1]The factor $\frac{1}{\mathrm{ord}(g)}$ in (2.3) was missing.

## 2.2 Descent and index

Let $C^*(G)$ denote the (non-reduced) group $C^*$-algebra of $G$. It has the universal property, that any unitary representation of $G$ extends to representation of $C^*(G)$. In particular, if $\rho : G \to U(V_\rho)$ is an unitary representation of $G$ on a finite-dimensional Hilbert space $V_\rho$, then there is an extension $\rho : C^*(G) \to \mathrm{End}(V_\rho)$. On the level of $K$-theory it induces a homomorphism (using Morita invariance and $K_0(\mathbb{C}) \cong \mathbb{Z}$) $I_\rho : K_0(C^*(G)) \to K_0(\mathrm{End}(V_\rho)) \cong \mathbb{Z}$. In particular, if $\rho = 1$ is the trivial representation, then we also write $I := I_1$. Note that $I_\rho$ can be written as a Kapsarov product $\otimes_{C^*(G)}[\rho]$, where $[\rho] \in KK(C^*(G), \mathrm{End}(V(\rho)))$ is represented by the Kasparov module $(V_\rho, 0)$.

Let $C^*(G, C_0(M))$ be the (non-reduced) cross product of $G$ with $C_0(M)$. Then there is the descent homomorphism $j^G : K_0^G(M) \cong KK^G(C_0(M), \mathbb{C}) \to KK(C^*(G, C_0(M)), C^*(G))$ introduced in [Kas88], 3.11. Following [GHT00] we choose any cut-off function $\chi \in C_c^\infty(M)$ with values in $[0,1]$ such that $\sum_{g\in G} g^*\chi^2 \equiv 1$. Then we define the projection $P \in C^*(G, C_0(M))$ by $P(g) = (g^{-1})^*\chi\chi$. Let $[P] \in K_0(C^*(G, C_0(M)) \cong KK(\mathbb{C}, C^*(G, C_0(M)))$ be the class induced by $P$, which is independent of the choice of $\chi$.

**Definition 2.1** *We define* $\mathrm{index}_\rho : K_0^G(M) \to \mathbb{Z}$ *to be the composition*

$$K_0^G(M) \xrightarrow{j^G} KK(C^*(G, C_0(M)), C^*(G)) \xrightarrow{[P]\otimes_{C^*(G,C_0(M))}} KK(\mathbb{C}, C^*(G, C_0(M))) \xrightarrow{I_\rho} \mathbb{Z}\ .$$

In particular, we set $\mathrm{index} := \mathrm{index}_1$.

## 2.3 Index and Orbifold index

The quotient $\bar{M} := G\backslash M$ is a smooth compact orbifold carrying an orbifold Dirac bundle $\bar{F} := G\backslash F$ with associated orbifold Dirac operator $\bar{D}$. In our case the space of smooth sections $C^\infty(\bar{M}, \bar{F})$ can be identified with the $G$-invariant sections $C^\infty(M, F)^G$. Then $\bar{D}$ coincides with the restriction of $D$ to this subspace. It is well-known that $\dim(\ker \bar{D}) < \infty$ so that we can define the index $\mathrm{index}(\bar{D}) := \dim_s \ker(\bar{D}) \in \mathbb{Z}$, where the subscript "$_s$" indicates hat we take the super dimension.

If $\rho : G \to U(V_\rho)$ is a finite-dimensional unitary representation of $G$, then we define the orbifold bundle $\bar{V}(\rho) := G\backslash M \times V_\rho$ and let $\bar{D}_\rho$ be the twisted operator associated to $\bar{F} \otimes \bar{V}(\rho)$. The space $C^\infty(\bar{M}, \bar{F} \otimes \bar{V}(\rho))$ can be identified with $(C^\infty(M, F) \otimes V_\rho)^G$ such that $\bar{D}_\rho$ is the restriction of $D \otimes 1$ to this subspace. Still we can define $\mathrm{index}(\bar{D}_\rho)$.

**Theorem 2.2** $\mathrm{index}(\bar{D}_\rho) = \mathrm{index}_\rho([D])$

*Proof.* We first apply $j^G$ to the Kasparov module $(L^2(M, F), \mathcal{F})$ representing $[D]$. According to [Kas88], 3.11., $j^G([D])$ is represented by $(C^*(G, L^2(M, F)), \tilde{\mathcal{F}})$, where $C^*(G, L^2(M, F))$

is a $C^*(G)$-right-module admitting a left action by $C^*(G, C_0(M))$. It is a closure of the space of finitely supported functions $f : G :\longrightarrow L^2(M, F)$. The operator $\tilde{\mathcal{F}}$ is given by $(\tilde{\mathcal{F}} f)(g) = (\mathcal{F} f)(g)$. The $C^*(G)$-valued scalar product is given by $\langle f_1, f_2\rangle(g) = \sum_{h\in G}\langle f_1(h), f_2(hg)\rangle$. Furthermore, the left action of $C^*(G, C_0(M))$ is given by $(\phi f)(g) = \sum_{h\in G} \phi(h)(hf)(g)$.

Using associativity of the Kasparov product we can compute $\mathrm{index}_\rho$ by first applying $\otimes_{C^*(G)}[\rho]$ and then $[P]\otimes_{C^*(G,C_0(M))}$. Using that $C^*(G, L^2(M, F)) \otimes_{C^*(G)} V_\rho \cong L^2(M, F) \otimes V_\rho$ by $f \otimes v \mapsto \sum_{g\in G} f(g)\rho(g)v$ we conclude that $j^G([D]) \otimes_{C^*(G)} [\rho]$ is represented by the Kasparov module $(L^2(M, F) \otimes V_\rho, \hat{\mathcal{F}})$, where $\hat{\mathcal{F}} = \mathcal{F} \otimes \mathrm{id}_{V_\rho}$. The left-action of $C^*(G, C_0(M))$ is given by $(\phi f) = \sum_{h\in G} \phi(h)(h \otimes \rho(h))f$.

Finally we compute $[P] \otimes_{C^*(G,C_0(M))} \left(j^G([D]) \otimes_{C^*(G)} [\rho]\right)$. We represent $[P]$ by the Kasparov module $(PC^*(G, C_0(M)), 0)$. We must understand $PC^*(G, C_0(M)) \otimes_{C_0(M)} (L^2(M, F) \otimes V_\rho)$.

There is a natural unitary inclusion $L : L^2(\bar{M}, \bar{F} \otimes \bar{V}(\rho)) \hookrightarrow L^2(M, F) \otimes V_\rho$. If $f \in L^2(\bar{M}, \bar{F} \otimes \bar{V}(\rho))$ is considered as an element $\hat{f}$ of $(L^2_{loc}(M, F) \times V_\rho)^G$ in the natural way, then $L(f) := \chi\hat{f}$. The projection $LL^*$ onto the range of $L$ is given by

$$LL^*(f) = \sum_{g\in G} (g^{-1})^*\chi g f \ .$$

It now follows from the definition of $P$ that

$$\begin{aligned} PC^*(G, C_0(M)) \otimes_{C^*(G,C_0(M))} (L^2(M, F) \otimes V_\rho) &= P(L^2(M, F) \otimes V_\rho) \\ &\overset{L^*}{\cong} L^2(\bar{M}, \bar{F} \otimes \bar{V}(\rho)) \end{aligned}$$

The operator $\bar{D}$ has a natural selfadjoint extension (also denoted by $\bar{D}$ such that we can form $\bar{\mathcal{F}} := \bar{D}(1 + \bar{D}^2)^{-1/2}$. We claim that $[P] \otimes_{C^*(G,C_0(M))} \left(j^G([D]) \otimes_{C^*(G)} [\rho]\right)$ is represented by the Kapsarov module $(L^2(\bar{M}, \bar{F} \otimes \bar{V}(\rho)), \bar{\mathcal{F}})$. The assertion of the Theorem immediately follows from the claim. In order to show the claim we employ the characterization of the Kasparov product in terms of connections (see [Kas88], 2.10). In our situation we have only to show that $\bar{\mathcal{F}}$ is a $\hat{\mathcal{F}}$-connection.

For Hilbert-$C^*$-modules $X, Y$ over some $C^*$-algebra $A$ let $L(X, Y)$ and $K(X, Y)$ denote the spaces of bounded and compact adjoinable $A$-linear operators (see [Bla98] for definitions). For $\xi \in PC^*(G, C_0(M))$ we define $\theta_\xi \in L(L^2(M, F) \otimes V_\rho, PL^2(M, F) \otimes V_\rho)$ by $\theta_\xi(f) = \xi f$. Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ are selfadjoint we only must show that $\theta_\xi \circ \hat{\mathcal{F}} - (L\bar{\mathcal{F}}L^*) \circ \theta_\xi \in K(L^2(M, F) \otimes V_\rho, PL^2(M, F) \otimes V_\rho)$. We have $\xi\hat{\mathcal{F}} - (L\bar{\mathcal{F}}L^*)\xi = [\xi, \hat{\mathcal{F}}] + (\hat{\mathcal{F}} - L\bar{\mathcal{F}}L^*)P\xi$. Since $[\xi, \hat{\mathcal{F}}]$ is compact it suffices to show that $(\hat{\mathcal{F}} - (L\bar{\mathcal{F}}L^*)P$ is compact. We consider $\tilde{D} := (1 - P)D(1 - P) + L\bar{D}L^*$. Then we have $\tilde{D} = D + Q$, where $Q$ is a zero order non-local operator. Let $\tilde{\mathcal{F}} := \tilde{\mathcal{D}}(1 + \tilde{\mathcal{D}}^2)^{-1/2}$. Then $(\hat{\mathcal{F}} - L\bar{\mathcal{F}}L^*)P = (\hat{\mathcal{F}} - \tilde{\mathcal{F}})P$. Let $\tilde{\chi} \in C_c^\infty(M)$ be such that $\chi\tilde{\chi} = \chi$. Then we have $(\hat{\mathcal{F}} - \tilde{\mathcal{F}})P = (\hat{\mathcal{F}} - \tilde{\mathcal{F}})\tilde{\chi}P$. Therefore it suffices to show that $(\hat{\mathcal{F}}-\tilde{\mathcal{F}})\tilde{\chi}$ is compact. This can be done using the integral representations for $\hat{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ as in [Bun95]. $\square$

## 2.4 The local index theorem

In this the present subsection we derive a local index theorem which is a formula for $\mathrm{index}_\rho([D])$ in terms of integrals of characteristic forms over the various singular strata of $\bar{M}$.

Let $W \in C^\infty(M\times M, F\boxtimes F^*)^G$ be a an invariant section which satisfies an estimate

$$|W(x,y)| \le C\exp(-c\mathrm{dist}(x,y)^2) \tag{2}$$

for some $c>0$, $C<\infty$. Since $\bar{M}$ is compact the manifold $M$ has bounded geometry, and in particular, it has at most exponential volume growth. Therefore, $W$ defines an integral operator $\bar{W}$ on $L^2(\bar{M},\bar{F}\otimes\bar{V}_\rho)$ by

$$\bar{W}f(x) := \int_M (W(x,y)\otimes \mathrm{id}_{V_\rho})f(y)dy\ .$$

This operator is in fact of trace class. We claim that

$$\mathrm{Tr}\,\bar{W} = \int_{\bar{M}}\sum_{g\in G}\mathrm{tr}(W(x,gx)g_x)dx\ \mathrm{tr}\rho(g)\ , \tag{3}$$

where $g_x$ denotes the linear map $g_x : F_x \to F_{gx}$. In order to see the claim note that $\mathrm{Tr}\,\bar{W} = \mathrm{Tr}\,LW\bar{L}^*$, and $R := LWL^*$ is the integral operator on $L^2(M,F)\otimes V_\rho$ given by the integral kernel $R(x,y) = \sum_{g\in G}\chi(x)W(x,gy)g_y\chi(y)\otimes\rho(g)$.

Again, since $M$ and $F$ have bounded geometry the heat kernel $W_t$, $t>0$, i.e. the integral kernel of $\exp(-tD^2)$, satisfies the Gaussian estimate (2). Moreover, $\bar{W}_t$ is precisely $\exp(-t\bar{D}^2)$. By the McKean-Singer formula we have

$$\mathrm{index}(\bar{D}_\rho) = \mathrm{Tr}_s\bar{W}_t$$

for any $t>0$, where $\mathrm{Tr}_s$ is the super trace. We obtain the local index formula by evaluating $\lim_{t\to 0}\mathrm{Tr}_s\bar{W}_t$.

If $g\in G$, then let $M^g$ denote the fixed point submanifold of $g$. If $M^g\neq\emptyset$, then $g$ is of finite order. Furthermore, let $Z_G(g)$ denote the centralizer of $g$ in $G$. Then $Z_G(g)\backslash M^g$ is compact. For $g\in G$ let $(g)\in C(G)$ denote the conjugacy class of $g$, where $C(G)$ denotes the set of conjugacy classes. By $\mathcal{F}(G)$ we denote the set of elements of finite order, and by $\mathcal{F}C(G)$ we denote the set of conjugacy classes of $G$ of finite order.

The formula (3) can we rewritten as follows.

$$\begin{aligned}
\mathrm{Tr}_s\bar{W} &= \int_{\bar{M}} \sum_{g\in G} \mathrm{tr}_s(W(x,gx)g_x)dx\ \mathrm{tr}\rho(g) \\
&= \sum_{(g)\in C(G)} \int_{G\backslash M} \sum_{h\in Z_G(g)\backslash G} \mathrm{tr}_s(W(x,hgh^{-1}x)(hgh^{-1})_x)dx\ \mathrm{tr}\rho(hgh^{-1}) \\
&= \sum_{(g)\in C(G)} \int_{Z_G(g)\backslash M} \mathrm{tr}_s(W(x,gx)g_x)dx\ \mathrm{tr}\rho(g)\ .
\end{aligned}$$

If $W = W_t$ is the heat kernel, then due to the usual gaussian estimates the integral $\int_{Z_G(g)\backslash M} \mathrm{tr}_s(W(x,gx)g_x)dx$ localizes at $Z_G(g)\backslash M^g$ as $t\to 0$. There is a $Z_G(g)$-invariant density $U(g)\in C^\infty(M^g,|\Lambda^{max}|T^*M^g)^{Z_G(g)}$ which is locally determined by the Riemannian structure $g^M$ and the Dirac bundle $F$ such that

$$\lim_{t\to\infty} \int_{Z_G(g)\backslash M} \mathrm{tr}_s(W_t(x,gx)g_x)dx = \frac{1}{\mathrm{ord}(g)} \int_{Z_G(g)\backslash M^g} U(g)\ .$$

An explicit formula for $U(g)$ is given in [BGV92], Ch. 6.4, and it will be recalled below. We conclude that

$$\mathrm{index}_\rho([D]) = \sum_{(g)\in C\mathcal{F}(G)} \frac{1}{\mathrm{ord}(g)} \int_{Z_G(g)\backslash M^g} U(g)\mathrm{tr}\rho(g)\ .$$

The fixed point manifold $M^g$ is a totally geodesic Riemannian submanifold of $M$ with induced metric $g^{M^g}$. Let $R^{M^g}$ denote its curvature tensor. We define the form $\hat{\mathbf{A}}(M^g)\in\Omega(M^g,\mathrm{Or}(M^g))$ by

$$\hat{\mathbf{A}}(M^g) = \det{}^{1/2}\left(\frac{R^{M^g}/4\pi\mathrm{i}}{\sinh(R^{M^g}/4\pi\mathrm{i})}\right)\ ,$$

where $\mathrm{Or}(M^g)$ denote the orientation bundle (the orientation bundle occurs since we must choose an orientation in order to define $\det^{1/2}$).

Furthermore, we define the $G$-equivariant bundle $F/S := \mathrm{End}_{\mathrm{Cliff}(TM)}(F)$. It comes with a natural connection $\nabla^{F/S}$. By $R^{F/S}$ we denote its curvature. Following [BGV92], 6.13, we define the form $\mathbf{ch}(g,F/S)\in\Omega(M^g,\Lambda^{max}N\otimes\mathrm{Or}(M))$ by

$$\mathbf{ch}(g,F/S) = \frac{2^{\mathrm{codim}_M(M^g)}}{\sqrt{\det(1-g^N)}}\mathrm{str}(\sigma_{\mathrm{codim}_M(M^g)}(g^F)\exp(-R_0^{F/S}/2\pi\mathrm{i}))\ .$$

Here $g^N$ is the restriction of $g$ to the normal bundle $N$ of $M^g$. Note that $\det(1-g^N)>0$ so that $\sqrt{\det(1-g^N)}$ is well-defined. Furthermore $g^F$ is the action of $g$ on the fibre of $F_{|M^g}$. Since $g^F$ commutes with $\mathrm{Cliff}(TM^g)$ it corresponds to an element of $\mathrm{Cliff}(N)\otimes\mathrm{End}_{\mathrm{Cliff}(M)}(F)$. $\sigma_{\mathrm{codim}_M(M^g)} : \mathrm{Cliff}(N)\to\Lambda^{max}N$ is the symbol map so that $\sigma_{\mathrm{codim}_M(M^g)} g^F \in \mathrm{End}_{\mathrm{Cliff}(M)}(F) \otimes \Lambda^{max} N$. Furthermore, the restriction $R_0^{F/S}$ of the curvature $R^{F/S}$ to $M^g$ is a section of $\Omega(M^g, \mathrm{End}_{\mathrm{Cliff}(M)}(F)_{|M^g})$. The super trace $\mathrm{str} : \mathrm{End}_{\mathrm{Cliff}(M)}(F) \to \mathbb{C} \otimes \mathrm{Or}(M)$ is defined by $\mathrm{str}(W) = \mathrm{tr}_s(\Gamma W)$, where $\Gamma = \mathrm{i}^{n/2}\mathrm{vol}_M$ is the chirality operator defined using the orientation of $M$.

Let $T_N : \Lambda^{max} N \to \mathbb{C} \otimes \mathrm{Or}(N)$ be the normal Beresin integral, where $\mathrm{Or}(N)$ is the bundle of normal orientations. Then we have

$$U(g) := [T_N(\frac{\hat{\mathbf{A}}(M^g)\mathbf{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi\mathrm{i}))})]_{max} .$$

Here $R^N$ is the curvature tensor of $N$, $\frac{1}{\det^{1/2}(1-g^N \exp(-R^N))} \in \Omega(M^g, \mathrm{Or}(M^g))$, and $[.]_{max}$ takes the part of maximal degree. In order to interpret the right-hand side as a density on $M^g$ we identify $\Lambda^{max} T^* M^g \otimes \mathrm{Or}(M^g)^2 \otimes \mathrm{Or}(N) \otimes \mathrm{Or}(M)$ with $|\Lambda^{max}| T^* M^g$ in the canonical way.

**Theorem 2.3**

$$\mathrm{index}_\rho([D]) = \sum_{(g) \in C\mathcal{F}(G)} \frac{\mathrm{tr}\rho(g)}{\mathrm{ord}(g)} \int_{Z_G(g)\backslash M^g} [T_N(\frac{\hat{\mathbf{A}}(M^g)\mathbf{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi\mathrm{i}))})]_{max}$$

## 2.5 Cyclic subgroups

We now reformulate the local index theorem in terms of contributions of conjugacy classes of cyclic subgroups. Let $\mathcal{F}Cyc(G)$ denote the set of finite cyclic subgroups. If $C \in \mathcal{F}Cyc(G)$, then let $\mathrm{gen}(C)$ denote the set of its generators. The normalizer $N_G(C)$ and the Weyl group $W_G(C) := N_G(C)/Z_G(C)$ acts on $\mathrm{gen}(C)$. There is a natural map $p : \mathcal{F}(G) \to \mathcal{F}Cyc(G)$, $g \mapsto < g >$ which factors over conjugacy classes $\bar{p} : C\mathcal{F}(G) \to C\mathcal{F}Cyc(G)$. If $(C) \in C\mathcal{F}Cyc(G)$, then $\bar{p}^{-1}(C)$ can be identified with $W_G(C)\backslash \mathrm{gen}(C)$.

Note that $M^g = M^{<g>}$, i.e. it only depends on the cyclic subgroup generated by $g$. Similarly, $Z_G(g) = Z_G(< g >)$. So we obtain

**Corollary 2.4**

$$\mathrm{index}_\rho([D]) = \sum_{(C) \in C\mathcal{F}Cyc(G)} \frac{1}{|C|} \sum_{g \in W_G(C)\backslash \mathrm{gen}(C)} \int_{Z_G(C)\backslash M^C} U(g)\mathrm{tr}\rho(g)$$

## 2.6 Cap product and twisting

We define $K_G^0(M) := KK^G(\mathbb{C}, C_0(M))$. If $E$ is a $G$-equivariant complex vector bundle, then let $[E] \in K_G^0(M)$ denote the class represented by the Kasparov module $(C_0(M, E), 0)$, where we define the $C_0(M)$-valued scalar product on $C_0(M, E)$ after choosing a $G$-invariant hermitean metric $(.,.)_E$.

Since $C_0(M)$ is commutative any right $C_0(M)$-module is a left- $C_0(M)$-module in a natural way. If we apply this to Kasparaov modules we obtain a map

$$a : KK^G(\mathbb{C}, C_0(M)) \to KK^G(C_0(M), C_0(M)) \ .$$

**Definition 2.5** *The cap-product $K^0_G(M) \otimes K^G_0(M) \to K^G_0(M)$ is defined by*

$$v \cap x := a(v) \otimes_{C_0(M)} x \ .$$

If we choose on $(E, (.,.))$ a hermitian connection $\nabla^E$, then we can form the twisted Dirac bundle $E \otimes F$ with associated Dirac operator $D_E$. The following fact is well-known. An elementary proof (for trivial $G$) can be found e.g. in [Bun95].

**Proposition 2.6** $[D_E] = [E] \cap [D]$

## 2.7 A cohomological index formula for twisted operators

Let $R^E$ denote the curvature of the connection $\nabla^E$. For a finite cyclic subgroup $C \subset G$ let $R^E_0$ denote the restriction of $R^E$ to $M^C$. If $g \in \mathrm{gen}(C)$ , then we have

$$\mathbf{ch}(g, E \otimes F/S) = \mathbf{ch}(g, F/S) \cup \mathbf{ch}(g, E) \ ,$$

where $\mathbf{ch}(g, E) = \mathrm{tr} g^E \exp(-R^E_0/2\pi \mathrm{i})$. Here $g^E$ denotes the action of $g$ on the fibre of $E$. Thus we can write

$$U_E(g) := [T_N(\frac{\hat{\mathbf{A}}(M^g)\mathbf{ch}(g, F/S) \cup \mathbf{ch}(g, E)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi \mathrm{i}))})]_{max} \ .$$

We can write $U_E(g) = [\hat{U}(g) \cap \mathbf{ch}(g, E)]_{max}$, where

$$\hat{U}(g) = T_N(\frac{\hat{\mathbf{A}}(M^g)\mathbf{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi \mathrm{i}))}) \ . \qquad (4)$$

The cohomology $H^*(Z_G(C)\backslash M^C, \mathbb{C})$ of the orbifold $Z_G(C)\backslash M^C$ can be computed using the complex of invariant differential forms $(\Omega^*(M^C)^{Z_G(C)}, d)$. Furthermore, the homology $H_*(Z_G(C)\backslash M^C, \mathbb{C})$ can be identified with the dual of the cohomology, i.e. $H_*(Z_G(C)\backslash M^C, \mathbb{C}) \cong H^*(Z_G(C)\backslash M^C, \mathbb{C})^*$. The closed form $\hat{U}(g) \in \Omega^*(M^g, \mathrm{Or})$ now defines a homology class $[\hat{U}(g)] \in H_*(Z_G(C)\backslash M^C, \mathbb{C})$ such that $[\hat{U}(g)]([\omega]) = \int_{Z_G(C)\backslash M^C} [\hat{U}(g) \cap \omega]_{max}$ for any closed form $\omega \in \Omega^*(M^C)^{Z_G(C)}$.

Let $[\mathbf{ch}(g, E)] \in H^*(Z_G(C)\backslash M^C, \mathbb{C})$ denote the cohomology class represented by the closed form $\mathbf{ch}(g, E)$.

**Theorem 2.7**

$$\mathrm{index}_\rho([E] \cap [D]) = \sum_{(C) \in C\mathcal{F}Cyc(G)} \frac{1}{|C|} \sum_{g \in W_G(C)\backslash \mathrm{gen}(C)} \langle [\mathbf{ch}(g, E)], [\hat{U}(g)] \rangle \mathrm{tr} \rho(g)$$

# 3 Chern characters

## 3.1 The cohomological Chern character

In this Subsection we review the construction of the Chern character given in [LO01a]. There the equivariant $K$-theory is introduced using a classifying space $\mathbf{K}_G\mathbb{C}$. If $X$ is a proper $G$-CW complex, then $\mathbf{K}_G^0(X) := [X, \mathbf{K}_G\mathbb{C}]_G$, where $[.,.]_G$ denotes the set of homotopy classes of equivariant maps.

Let $\mathbb{K}_G(X)$ be the Grothendieck group of $G$-equivariant complex vector bundles. Then there is a natural homomorphism $b : \mathbb{K}_G(X) \to \mathbf{K}_G^0(X)$, which is an isomorphism if $X$ is finite ([LO01a], Prop. 1.5).

If $H$ is a finite group, then let $R_\mathbb{C}(H)$ denote the complex representation ring of $H$ with complex coefficients. The character gives a natural identification of $R_\mathbb{C}(H)$ with the space of complex-valued class functions on $H$, i.e. $\mathbb{C}(C(H))$.

Since we want to work with differential forms later on we simplify matters by working with complex coefficients (the constructions in [LO01a] are finer since they work over $\mathbb{Q}$). For any finite subgroup $H \subset G$ the construction [LO01a], (5.4), provides a homomorphism

$$\mathbf{ch}_X^H : K_G^0(X) \to H^*(Z_G(H)\backslash X^H) \otimes \mathbb{C}(C(H)) \ .$$

For our purpose it suffices to understand $\mathbf{ch}_X^H(b(\{E\}))$, where $E$ is a $G$-equivariant complex vector bundle over $X$, and $\{E\}$ denotes its class in $\mathbb{K}_G(X)$. First of all note that $E_{|X^H}$ is a $N_G(H)$-equivariant bundle over $X^H$. We can further write $E_{|X^H} = \sum_{\phi\in\hat{H}} \mathrm{Hom}_H(V_\phi, E_{|X^H}) \otimes V_\phi$, where $\mathrm{Hom}_H(V_\phi, E_{|X^H})$ is a $Z_G(H)$-equivariant bundle over $X^H$. We therefore obtain an element of $\mathbb{K}^0_{Z_G(H)}(X^H) \otimes R(H)$. We now apply the composition

$$\mathbb{K}^0_{Z_G(H)}(X^H) \stackrel{\mathrm{pr}^*}{\to} \mathbb{K}^0_{Z_G(H)}(EG \times X^H) \stackrel{\cong}{\to} \mathbb{K}^0_1(EG \times_{Z_G(H)} X^H)$$
$$\stackrel{\mathbf{ch}}{\to} H^*(EG \times_{Z_G(H)} X^H, \mathbb{C}) \stackrel{(\mathrm{pr}^*)^{-1}}{\to} H^*(Z_G(H)\backslash X^H, \mathbb{C})$$

to the first component, and the character $R(H) \to \mathbb{C}(C(H))$ to the second. The result belongs to $H^*(Z_G(H)\backslash X^H, \mathbb{C}) \otimes \mathbb{C}(C(H))$ and is $\mathbf{ch}_X^H(b(\{E\}))$.

If $C$ is a finite cyclic subgroup, then let $r : \mathbb{C}(C(C)) \to \mathbb{C}(\mathrm{gen}(C))$ be the restriction map. Note that $W_G(C)$ acts on $\mathbb{C}(\mathrm{gen}(C))$ as well as on $H^*(Z_G(C)\backslash X^C, \mathbb{C})$. The result [LO01a], Lemma 5.6, now asserts that if $X$ is finite, then

$$\prod_{(C)\in C\mathcal{F}Cyc(G)} (1\otimes r)\mathbf{ch}_X^C : \mathbf{K}_G^0(X)_\mathbb{C} \to \prod_{(C)\in C\mathcal{F}Cyc(G)} \left(H^{ev}(Z_G(C)\backslash X^C, \mathbb{C}) \otimes \mathbb{C}(\mathrm{gen}(C))\right)^{W_G(C)} \quad (5)$$

is an isomorphism.

## 3.2 Differential forms

In the present subsection we give a description of the equivariant Chern character using differential forms. Let $M$ be a smooth proper $G$-manifold and $E$ be a $G$-equivariant complex vector bundle over $M$. Then we can find a $G$-invariant hermitian metric $(.,.)_E$ and a $G$-invariant metric connection $\nabla^E$. Let $R^E$ denote the curvature of $\nabla^E$. We define the closed $G$-invariant form $\mathbf{ch}(E) \in \Omega(M)^G$ by $\mathbf{ch}(E) := \mathrm{tr}\exp(-R^E/2\pi\mathrm{i})$. It represents a cohomology class $[\mathbf{ch}(E)] \in H^*(G\backslash M, \mathbb{C})$. Furthermore, we have the class $\mathbf{ch}_M^{\{1\}}(b(\{E\}))$, which is given by the following composition

$$\begin{aligned} &\mathbb{K}^0_G(M) \stackrel{\mathrm{pr}_1^*}{\to} \mathbb{K}^0_G(EG \times M) \stackrel{\cong}{\to} \mathbb{K}^0_1(EG \times_G M) \\ &\stackrel{\mathbf{ch}}{\to} H^*(EG \times_G M, \mathbb{C}) \stackrel{(\mathrm{pr}_2^*)^{-1}}{\to} H^*(G\backslash M, \mathbb{C}) \ . \end{aligned} \tag{6}$$

**Lemma 3.1** $[\mathbf{ch}(E)] = \mathbf{ch}_M^{\{1\}}(b(\{E\}))$

*Proof.* We show that $\mathbf{ch}_M^{\{1\}}(b(\{E\}))$ can be represented by the form $\mathbf{ch}(E)$. To do so we employ an approximation $j : \tilde{E}G \to EG$, where $\tilde{E}G$ is a free $G$-manifold and the $G$-map $j$ is $\dim(M)+1$-connected. This existence of such approximations will be shown in Subsection 3.3. Then we can define $\mathbf{ch}_M^{\{1\}}(b(\{E\}))$ by (6) but with $EG$ replaced by $\tilde{E}G$. It is now clear that $\mathrm{pr}_2^*\mathbf{ch}(E) = \mathbf{ch}(G\backslash\mathrm{pr}_1^* E)$. □

Let $C \subset G$ be a finite cyclic subgroup. Furthermore, let $[\mathbf{ch}(g, E)] \in H^*(Z_G(C)\backslash M^C, \mathbb{C})$ denote the cohomology class represented by $\mathbf{ch}(g, E)$. The function $\mathrm{gen}(C) \ni g \mapsto [\mathbf{ch}(g, E)]$ can naturally be considered as an element $[\mathbf{ch}(., E)] \in H^*(Z_G(C)\backslash M^C, \mathbb{C}) \otimes \mathbb{C}(\mathrm{gen}(C))$ which is in fact $W_G(C)$-equivariant.

**Proposition 3.2** $[\mathbf{ch}(., E)] = (1 \otimes r)\mathbf{ch}_M^C(b(\{E\}))$.

*Proof.* First of all note that $R_0^E$ is the curvature of $E_{|M^C}$. Furthermore, the decomposition $E_{|M^C} = \sum_{\phi\in\hat{C}} E(\phi) \otimes V_\phi$ is preserved by $R_0^E$, where $E(\phi) = \mathrm{Hom}_C(V_\phi, E_{|M^C})$. Let $R^{E(\phi)}$ be the restriction of the curvature to the subbundle $E(\phi) \otimes V_\phi$. We get for $g \in \mathrm{gen}(C)$

$$\begin{aligned} (1\otimes r)\mathbf{ch}_M^C(b\{E\})(g) &\stackrel{def.}{=} \sum_{\phi\in\hat{C}} \mathbf{ch}_{M^C}^{\{1\}}(b\{E(\phi)\})\mathrm{tr}\phi(g) \\ &\stackrel{Lemma 3.1}{=} \sum_{\phi\in\hat{C}} [\mathbf{ch}(E(\phi))]\mathrm{tr}\ \phi(g) \\ &= \sum_{\phi\in\hat{C}} [\mathrm{tr}\exp(-R^{E(\phi)}/2\pi\mathrm{i})]\mathrm{tr}\phi(g) \\ &= [\mathrm{tr} g^E \exp(-R_0^E/2\pi\mathrm{i})] \\ &= [\mathbf{ch}(g, E)] \ . \end{aligned}$$

□

## 3.3 Smooth approximations of $CW$-complexes

The goal of this subsection is to show that the approximation $j : \tilde{E}G \to EG$ used in the proof of Lemma 3.1 exists. We start with the following general result.

**Proposition 3.3** *If $X$ is a countable finite-dimensional $CW$-complex, then there exists a smooth manifold $M$ and a homotopy equivalence $M \xrightarrow{\sim} X$.*

*Proof.* Let $X$ be a finite-dimensional $CW$-complex. Following [Bro62] we call a manifold with boundary $(\bar{M}, \partial\bar{M})$ a tubular neighbourhood of $X$ if there exists a continuous map $F : \partial\bar{M} \to X$ such that the underlying topological space of $\bar{M}$ is the mapping cylinder $C(F) = \partial\bar{M} \times [0,1] \cup_F X$ of $F$, the inclusion $\partial\bar{M} \times [0,1) \hookrightarrow M$ is smooth, and the inclusion $X \hookrightarrow \bar{M}$ is smooth on each open cell of $X$.

Let

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n = X$$

be the filtration of $X$ by skeletons. We obtain $X^{i+1}$ from $X^i$ by attaching a countable number of $i+1$-cells.

It is clear that the collection of points $X^0$ admits a tubular neighbourhood of any given dimension. Let us now assume that there exists an $i$-dimensional $CW$-complex $Y$ together with a homotopy equivalence $h : Y \xrightarrow{\sim} X^i$ and a $m$-dimensional tubular neighbourhood $(W, \partial W)$, $F : \partial W \to Y$ of $Y$, such that $m \geq 2n + 3$.

Via a homotopy inverse of $h$ the attaching data for $X^i \subseteq X^{i+1}$ yields the attaching data $\tilde{\chi}_\alpha : S^i \to Y$, $\alpha \in J$, for a countable collection $J$ of $i+1$-cells. Let $\tilde{Z}$ be the result of attaching these cells to $Y$. Then we have a homotopy equivalence $\tilde{Z} \xrightarrow{\sim} X^{i+1}$.

We consider $\mathbb{R}$ with the cell-structure given by its decomposition into unit intervals. The product $(W \times \mathbb{R}, \partial W \times \mathbb{R})$ is a tubular neighbourhood of $Y \times \mathbb{R}$ in a natural way with retraction $F \times \mathrm{id}_\mathbb{R} : \partial W \times \mathbb{R} \to Y \times \mathbb{R}$. The projection $Y \times \mathbb{R} \to Y$ is a homotopy equivalence.

We now fix an inclusion $J \subseteq \mathbb{Z}$ and define attaching maps $\chi_\alpha := \tilde{\chi}_\alpha \times \{\alpha\} : S^i \to Y \times \mathbb{R}$. Let $\hat{Z}$ denote the complex obtained by attaching the cells to $Y \times \mathbb{R}$. Our choice of attaching maps is made such that these $i+1$-cells are attached to the $i$-skeleton of $Y$. We have a homotopy equivalence $\hat{Z} \xrightarrow{\sim} \tilde{Z}$.

In order to improve the attaching maps we argue as in the proof of [Bro62, Theorem II]. Since $2\dim(Y) + 1 = 2i + 1 \leq 2n + 3 \leq m = \dim(W)$ we can deform the attaching map $\tilde{\chi}_\alpha$ slightly so that its image is disjoint from $Y$. To do so we adapt the method of the proof of [Whi55, Theorem 11a], and we use the assumption that the open cells of

$Y$ are smoothly embedded. Using the mapping cylinder structure $W \setminus Y \cong \partial W \times [0,1)$ we can further deform the attaching map such that it maps to $\partial W$. Finally, using that $2i+1 \leq \dim(\partial W) = m-1$ we can deform it to an embedding (see [Whi36, Theorem II]) into $\partial W$. Again for dimension reasons, the normal bundle of this embedding is trivial. We still denote this deformed attaching map by $\tilde{\chi}_\alpha$, and we obtain a new deformed map $\chi_\alpha := \tilde{\chi}_\alpha \times \{\alpha\} : S^i \to \partial W \times \{\alpha\} \subseteq \partial W \times \mathbb{R}$.

For each $\alpha \in J$ we now perform the procedure of attaching a handle to $W \times \mathbb{R}$ described in [Bro62, Sec .2]. We can arrange the construction such that for $\alpha \in J$ it takes place on $W \times (\alpha - 1/4, \alpha + 1/4)$.

The result of this construction is a manifold with boundary $(N, \partial N)$ of dimension $m+1$ containing an $i+1$-dimensional $CW$-complex $Z$, and a map $N \to Z$ which represents $(N, \partial N)$ as a tubular neighbourhood of $Z$, and we have a homotopy equivalence $Z \xrightarrow{\sim} \hat{Z}$.

After we finite iteration of this construction we obtain a manifold with boundary $(\bar{M}, \partial\bar{M})$ which is a tubular neighbourhood of a $CW$-complex $\tilde{X}$ which admits a homotopy equivalence $\tilde{h} : \tilde{X} \xrightarrow{\sim} X$. The mapping cylinder structure on $\bar{M}$ gives rise to a projection $p : \bar{M} \to \tilde{X}$. We now consider the smooth manifold $M := \bar{M} \setminus \partial\bar{M}$. The composition $\tilde{h} \circ p_{|M} : M \to X$ is a homotopy equivalence from a smooth manifold to $X$. □

We can now construct the smooth $n$-connected approximation $j : \tilde{E}G \to EG$, where $n \geq 2$. We start with a countable $CW$-complex $BG$ of the homotopy type of the classifying space of $G$. For example, we can take the standard simplicial model. It is countable since $G$ is countable.

We consider the $n$-skeleton $BG^n \subseteq BG$. It is a finite-dimensional countable $CW$-complex. By Proposition 3.3 we can find a smooth manifold $\tilde{B}G$ together with a homotopy equivalence $\hat{j} : \tilde{B}G \to BG^n$. Let $\bar{j} : \tilde{B}G \to BG$ denote the composition of $\hat{j}$ with the inclusion $BG^n \hookrightarrow BG$. By construction $\bar{j}$ is $n$-connected.

Since $\bar{j}$ induces an isomorphism of fundamental groups it lifts to a $n$-connected map of universal coverings $j : \tilde{E}G \to EG$.

.

## 3.4 The homological Chern character

In this subsection we review the construction of the homological Chern character given in [Lüc02a], [Lüc02b]. Let $X$ be a proper $G$-CW-complex. The main constituent of the Chern character is a homomorphism

$$\mathbf{ch}_H^X : H_{ev}(Z_G(H)\backslash X^H, \mathbb{C}) \otimes R_{\mathbb{C}}(H) \to K_0^G(X)$$

for any finite subgroup $H \subset G$.

$$\begin{array}{rcl}
H_{ev}(Z_G(H)\backslash X^H,\mathbb{C}) \otimes R_{\mathbb{C}}(H) & \stackrel{(\mathrm{pr}_2)_*^{-1}\otimes \mathrm{id}}{\to} & H_{ev}(EG \times_{Z_G(H)} X^H,\mathbb{C}) \otimes R_{\mathbb{C}}(H) \\
& \stackrel{\mathbf{ch}^{-1}\otimes \mathrm{id}}{\to} & K_0(EG \times_{Z_G(H)} X^H)_{\mathbb{C}} \otimes R_{\mathbb{C}}(H) \\
& \stackrel{\cong}{\to} & K_0^{Z_G(H)}(EG \times X^H)_{\mathbb{C}} \otimes K_0^H(*)_{\mathbb{C}} \\
& \stackrel{mult}{\to} & K_0^{Z_G(H)\times H}(EG \times X^H)_{\mathbb{C}} \\
& \stackrel{\mathrm{Ind}_{Z_G(H)\times H}^G}{\to} & K_0^G(\mathrm{Ind}_{Z_G(H)\times H}^G(EG \times X^H))_{\mathbb{C}} \\
& \stackrel{\mathrm{Ind}_{Z_G(H)\times H}^G(\mathrm{pr}_2)_*}{\to} & K_0^G(\mathrm{Ind}_{Z_G(H)\times H}^G X^H)_{\mathbb{C}} \\
& \stackrel{m_*}{\to} & K_0^G(X)
\end{array}$$

Here $\mathbf{ch}$ is the homological Chern character, $\mathrm{Ind}_{Z_G(H)\times H}^G$ denotes the induction functor, and $m : \mathrm{Ind}_{Z_G(H)\times H}^G X^H = G \times_{Z_G(H)\times H} X^H \to X$ is the $G$-map $(g,x) \mapsto gx$.

Let $C \subset G$ be a finite cyclic subgroup. Then we have a natural inclusion $r^* : \mathbb{C}(\mathrm{gen}(C)) \to R_{\mathbb{C}}(C) \cong \mathbb{C}C$ such that the image consists of functions which vanish on $C \setminus \mathrm{gen}(C)$. Note that $\mathbb{C}(\mathrm{gen}(C))$ and $H_*(Z_G(H)\backslash X^H,\mathbb{C})$ are left and right $W_G(C)$-modules in the natural way. It follows from [Lüc02b], Thm. 0.7, that

$$\oplus_{(C)\in C\mathcal{F}Cyc(G)} \mathbf{ch}_C^X(1\otimes r^*) : \bigoplus_{(C)\in C\mathcal{F}Cyc(G)} H_{ev}(Z_G(C)\backslash X^C,\mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\mathrm{gen}(C)) \to K_0^G(X)_{\mathbb{C}} \tag{7}$$

is an isomorphism.

# 4 Explicit decomposition of $K$-homology classes

## 4.1 An index formula

Let $E$ be a $G$-equivariant vector bundle over $X$. If $A \in H_*(Z_G(H)\backslash X^H,\mathbb{C}) \otimes R_{\mathbb{C}}(H)$, then we can ask for a formula for $\mathrm{index}_\rho([E] \cap \mathbf{ch}_H^X(A))$ in terms of $\mathbf{ch}_X^H(b(\{E\}))$. Let $\epsilon : R(H) \to \mathbb{Z}$ be the homomorphism which takes the multiplicity of the trivial representation. It extends to a group homomorphism $\epsilon_{\mathbb{C}} : R_{\mathbb{C}}(H) \to \mathbb{C}$. Using the ring structure of $R_{\mathbb{C}}(H)$ and the pairing between homology and cohomology we obtain a natural pairing

$$\begin{array}{rcl}
\langle .,. \rangle_\rho & : & \big(H_*(Z_G(H)\backslash X^H,\mathbb{C}) \otimes R_{\mathbb{C}}(H)\big) \otimes \big(H^*(Z_G(H)\backslash X^H,\mathbb{C}) \otimes R_{\mathbb{C}}(H)\big) \\
& \to & R_{\mathbb{C}}(H) \stackrel{\otimes[\rho_{|H}]}{\to} R_{\mathbb{C}}(H) \stackrel{\epsilon_{\mathbb{C}}}{\to} \mathbb{C} \ .
\end{array}$$

**Theorem 4.1** $\mathrm{index}_\rho([E] \cap \mathbf{ch}_H^X(A)) = \langle \mathbf{ch}_X^H(b(\{E\})), A \rangle_\rho$

*Proof.* Let $M$ be a cocompact free even-dimensional $Z_G(H)$-manifold equipped with a invariant Riemannian metric and a Dirac operator $D$ associated to a $Z_G(H)$-equivariant Dirac bundle $F \to M$. Furthermore, let $f = (f_1, f_2) : M \to EG \times X^H$ be a $Z_G(H)$-equivariant continuous map. We form $[D] \in K_0^{Z_G(H)}(M)$ represented by the Kasparov module $(L^2(M,F), \mathcal{F})$. Then $f_*[D] \in K_0^{Z_G(H)}(EG \times X^H)$.

Note that $K_0^{Z_G(H)}(EG \times X^H)_{\mathbb{C}}$ is spanned by elements arising in this form. This can be seen as follows. First observe that every class in $K_0(EG \times_{Z_G(H)} X^H)$ can be represented in the form $\bar{f}_*[\bar{D}]$, where $\bar{f} : \bar{N} \to EG \times_{Z_G(H)} X^H$ is a map from a closed *Spin*$^c$-manifold, and $\bar{D}$ is the *Spin*$^c$-Dirac operator on $\bar{N}$. A proof of this result is given in [BHS]. We now consider the pull-back

$$\begin{array}{ccc} N & \xrightarrow{f} & EG \times X^H \\ \downarrow & & \downarrow \\ \bar{N} & \xrightarrow{\bar{f}} & EG \times_{Z_G(H)} X^H \end{array} .$$

The manifold $N$ carries a $Z_G(H)$-invariant *Spin*$^c$-structure with associated Dirac operator $D$. The class $f_*[D]$ corresponds to $\bar{f}_*[\bar{D}]$ under the isomorphism $K_0^{Z_G(H)}(EG \times X^H) \cong K_0^{Z_G(H)}(EG \times X^H)$.

Let $\phi \in \hat{H}$ be a finite-dimensional representation. It gives rise to an element $[\phi] \in K_0^H(*)_{\mathbb{C}}$ under the natural identification $R_{\mathbb{C}}(H) \cong K_0^H(*)_{\mathbb{C}}$. Let $T : K_0^{Z_G(H)}(EG \times X^H) \otimes K_0^H(*)_{\mathbb{C}} \to K_0^G(X)$ be the composition $m_* \circ \mathrm{Ind}_{Z_G(H)\times H}^G(\mathrm{pr}_2)_* \circ \mathrm{Ind}_{Z_G(H)\times H}^G \circ mult$, which is part of the definition of $\mathbf{ch}_H^X$.

We first study $\mathrm{index}_\rho([E] \cap T(f_*[D] \otimes [\phi]))$. We have $mult \circ f_*([D] \otimes [\phi]) = f_* \circ mult([D] \otimes [\phi])$, and $mult([D] \otimes [\phi]) \in K_0^{Z_G(H)\times H}(M)$ is represented by the Kasparov module $(L^2(M,F) \otimes V_\phi, \mathcal{F} \otimes \mathrm{id})$. Furthermore, $\mathrm{Ind}_{Z_G(H)\times H}^G \circ f_* \circ mult([D] \otimes [\phi]) = \mathrm{Ind}_{Z_G(H)\times H}^G(f_*)\mathrm{Ind}_{Z_G(H)\times H}^G(mult([D] \otimes [\phi]))$. Explicitly, $\mathrm{Ind}_{Z_G(H)\times H}^G(mult([D] \otimes [\phi]))$ is represented by a Kasparov module which is constructed in the following way. Consider the exact sequence

$$0 \to K \to Z_G(H) \times H \to G ,$$

where $K = Z_G(H) \cap H = Z_H(H)$. We identify $K\backslash Z_G(H) \times H$ with the subgroup $Z_G(H)H \subseteq G$.

Note that we consider $M$ as a $Z_G(H) \times H$-manifold via the action of the first factor. The $Z_G(H)H$-manifold $\hat{M} := K\backslash M$ carries an induced equivariant Dirac bundle $\hat{F}$. We further consider the flat $Z_G(H)H$-equivariant bundle $\hat{V}_\phi := V_\phi \times_K M$ over $\hat{M}$. The twisted bundle $\hat{F} \otimes \hat{V}_\phi$ is a $Z_G(H)H$-equivariant Dirac bundle. We consider the cocompact proper $G$-manifold $\tilde{M} := G \times_{Z_G(H)H} \hat{M}$. The $Z_G(H)H$-equivariant Dirac bundle $\hat{F} \otimes \hat{V}_\phi$ induces a $G$-equivariant Dirac bundle $\tilde{F}_\phi \to \tilde{M}$ in a natural way with associated operator $\tilde{D}_\phi$. Then $\mathrm{Ind}_{Z_G(H)\times H}^G(mult([D] \otimes [\phi]))$ is represented by $[\tilde{D}_\phi]$. The map $\mathrm{Ind}_{Z_G(H)\times H}^G(f_*)$ is induced by the $G$-map $\tilde{f} : \tilde{M} \to G \times_{Z_G(H)H} (K\backslash EG \times X^H)$ given by $\tilde{f}([g, Km]) :=$

$[g,(Kf_1(m),f_2(m))]$. It is now clear that $T(f_*[D]\otimes[\phi])$ is represented by $h_*[\tilde D_\phi]$, where $h:\tilde M\to X$ is given by $h([g,Km])=gf_2(m)$.

It follows from the associativity of the Kasparov product that

$$\mathrm{index}_\rho([E]\cap T(f_*[D]\otimes[\phi]))=\mathrm{index}_\rho([E]\cap h_*[\tilde D_\phi])=\mathrm{index}_\rho([h^*E]\cap[\tilde D_\phi])\ .$$

By Theorem 2.2 and Proposition 2.6 we obtain $\mathrm{index}_\rho([h^*E]\cap[\tilde D_\phi])=\mathrm{index}(\bar D_{\phi,h^*E,\rho})$, where $\tilde D_{\phi,h^*E}$ is the $G$-invariant Dirac operator associated to $\tilde F\otimes h^*E$, and $\bar D_{\phi,h^*E,\rho}$ is the operator on the orbifold $\bar M:=G\backslash\tilde M$ induced by $\tilde D_{\phi,h^*E}$ and the twist $\rho$. Restriction from $\tilde M$ to the submanifold $\{1\}\times\hat M$ provides an isomorphism

$$\Big(C^\infty(\tilde M,\tilde F_\phi\otimes h^*E)\otimes V_\rho\Big)^G\cong\Big(C^\infty(\hat M,\hat F\otimes\hat V_\phi\otimes\bar f_2^*E_{|X^H})\otimes V_\rho\Big)^{Z_G(H)\times H}\ ,$$

where $\bar f_2:\hat M\to X^H$ is induced by $f_2$. Since the action of $H$ on the latter spaces is implemented by the action on the fibres of $V_\phi\otimes\bar f_2^*E_{|X^H}\otimes V_\rho$ we further obtain

$$\Big(C^\infty(\tilde M,\tilde F_\phi\otimes h^*E)\otimes V_\rho\Big)^G=C^\infty(\hat M,\hat F\otimes(V_\phi\otimes\bar f_2^*E_{|X^H}\otimes V_\rho)^H)^{K\backslash Z_G(H)}\ .$$

In the present situation we have $\bar M=Z_G(H)\backslash M=(K\backslash Z_G(H))\backslash\hat M$, i.e. the orbifold is smooth, and it carries the Dirac bundle $\bar F$ with associated Dirac operator $\bar D$. We define the $(K\backslash Z_G(H))$-equivariant bundle $E_{\phi\otimes\rho}:=(V_\phi\otimes E_{|X^H}\otimes V_\rho)^H$ over $X^H$. Furthermore, we consider the quotient $\overline{\bar f_2^*E_{\phi\otimes\rho}}:=(K\backslash Z_G(H))\backslash\bar f_2^*E_{\phi\otimes\rho}$ over $\bar M$. The identifications above show that $\mathrm{index}_\rho(\bar D_{\phi,h^*E})=\mathrm{index}(\bar D_{\overline{\bar f_2^*E_{\phi\otimes\rho}}})$, i.e. it is the index of a twisted Dirac operator. Writing the index of the twisted Dirac operator in terms of Chern characters we obtain

$$\mathrm{index}_\rho([E]\cap T(f_*[D]\otimes[\phi]))=\langle\mathbf{ch}(\overline{\bar f_2^*E_{\phi\otimes\rho}}),\mathbf{ch}([\bar D])\rangle\ .$$

Note that $\overline{\bar f_2^*E_{\phi\otimes\rho}}=\bar f^*\overline{\mathrm{pr}_2^*E_{\phi\otimes\rho}}$, where $\mathrm{pr}_2:EG\times X^H\to X^H$, $\bar f:\bar M\to EG\times_{Z_G(H)}X^H$ is induced by $f$, and $\overline{\mathrm{pr}_2^*E_{\phi\otimes\rho}}:=Z_G(H)\backslash\mathrm{pr}_2^*E_{\phi\otimes\rho}$. We conclude that

$$\langle\mathbf{ch}(\overline{\bar f_2^*E_{\phi\otimes\rho}}),\mathbf{ch}([\bar D])\rangle=\langle\mathbf{ch}(\overline{\mathrm{pr}_2^*E_{\phi\otimes\rho}}),\mathbf{ch}(\bar f_*[\bar D])\rangle\ .$$

The right-hand side can now be written as

$$\big\langle\epsilon_{\mathbb C}\left(\mathbf{ch}_X^H(b([E]))\otimes[\phi]\otimes\rho\right),(\mathrm{pr}_2)_*\mathbf{ch}(\bar f_*[\bar D])\big\rangle=\langle\mathbf{ch}_X^H(E),(\mathrm{pr}_2)_*\mathbf{ch}(\bar f_*[\bar D])\otimes[\phi]\rangle_\rho\ .$$

Note that $\mathbf{ch}_H^X((\mathrm{pr}_2)_*\mathbf{ch}(\bar f_*[\bar D])\otimes[\phi])=T(f_*[D]\otimes[\phi])$. Therefore we have shown

$$\mathrm{index}_\rho([E]\cap\mathbf{ch}_H^X((\mathrm{pr}_2)_*\mathbf{ch}(\bar f_*[\bar D])\otimes[\phi])))=\langle\mathbf{ch}_X^H(b([E])),\mathbf{ch}(\bar f_*[\bar D])\otimes[\phi]\rangle_\rho\ .$$

Since the classes $(\mathrm{pr}_2)_*\mathbf{ch}(\bar f_*[\bar D])\otimes[\phi]$ for varying data $M,F,f,\phi$ span $H_{ev}(Z_G(H)\backslash X^H,\mathbb C)\otimes R_{\mathbb C}(H)$ the theorem follows. □

## 4.2 Decomposition

**Lemma 4.2** *Let $X$ be a finite proper $G$-CW-complex. If $x \in K_0^G(X)_{\mathbb{C}}$ and* $\mathrm{index}([E] \cap x) = 0$ *for all $G$-equivariant complex vector bundles $E$ on $X$, then $x = 0$.*

*Proof.*Because of the isomorphism (7) it suffices to show that if $A \in H_{ev}(Z_G(C)\backslash X^C, \mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\mathrm{gen}(C))$ and $\mathrm{index}([E] \cap \mathbf{ch}_C^X(A)) = 0$ for all $E$, then $A = 0$. By Theorem 4.1 we have $\mathrm{index}([E] \cap \mathbf{ch}_C^X(A)) = \langle \mathbf{ch}_H^C(b(\{E\})), A\rangle$. Using the surjectivity of $b$ and of the isomorphism (5), and the fact that the pairing

$$\langle .,. \rangle : \big(H^{ev}(Z_G(C)\backslash X^C, \mathbb{C}) \otimes \mathbb{C}(\mathrm{gen}(C))\big)^{W_G(C)} \otimes \big(H_{ev}(Z_G(C)\backslash X^C, \mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\mathrm{gen}(C))\big)$$

is nondegenerate we see that $\langle \mathbf{ch}_H^C(b(\{E\})), A\rangle = 0$ for all $E$ indeed implies $A = 0$. □

Let now $M$ be an even-dimensional proper cocompact $G$-manifold equipped with a $G$-invariant Riemannian metric $g^M$ and a $G$-equivariant Dirac bundle $F$ with associated Dirac operator $D$. Let $[D]_{\mathbb{C}} \in K_0^G(M)_{\mathbb{C}}$ be the equivariant $K$-homology class of $D$.

The $G$-space $M$ has the $G$-homotopy type of a finite proper $G$-CW-complex. In particular, we have the isomorphism (7)

$$\oplus_{(C)\in C\mathcal{F}Cyc(G)} \mathbf{ch}_C^M(1 \otimes r^*) : \bigoplus_{(C)\in C\mathcal{F}Cyc(G)} H_{ev}(Z_G(C)\backslash M^C, \mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\mathrm{gen}(C)) \to K_0^G(M)_{\mathbb{C}} \,.$$

Therefore, there exist uniquely determined classes $[D](C) \in H_{ev}(Z_G(C)\backslash M^C, \mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\mathrm{gen}(C))$ such that

$$\sum_{(C)\in C\mathcal{F}Cyc(G)} \mathbf{ch}_C^M(1 \otimes r^*)([D](C)) = [D]_{\mathbb{C}} \,.$$

**Theorem 4.3** *We have the equality*

$$[D](C) = [\hat{U}] \,,$$

*where $[\hat{U}]$ is given by* $\mathrm{gen}(C) \ni g \to [\hat{U}(g)] \in H_{ev}(Z_G(C)\backslash M^C, \mathbb{C})$*, and $\hat{U}(g)$ was defined in (4).*

*Proof.* Let $E$ be any $G$-equivariant complex vector bundle over $M$. Then we have

$$\mathrm{index}([E] \cap [D]_{\mathbb{C}}) = \sum_{(C)\in C\mathcal{F}Cyc(G)} \langle \mathbf{ch}_M^C(b(\{E\})), [D](C)\rangle \,.$$

Using the definition of $\epsilon_{\mathbb{C}}$ and Proposition 3.2 we can write out the summands of right-hand side as follows

$$\langle \mathbf{ch}^C_M(b(\{E\})), [D](C)\rangle = \frac{1}{|C|} \sum_{g\in \mathrm{gen}(C)} \langle [\mathbf{ch}(g,E)], [D](C)(g)\rangle \ .$$

On the other hand the index formula Theorem 2.7 gives

$$\mathrm{index}([E]\cap [D]_{\mathbb{C}}) = \sum_{(C)\in C\mathcal{F}Cyc(G)} \frac{1}{|C|} \sum_{g\in \mathrm{gen}(C)} \langle [\mathbf{ch}(g,E)], [\hat{U}](g)\rangle \ .$$

Varying $E$ and using Lemma 4.2 we conclude $[D](C) = [\hat{U}]$. □

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