The topology of T-duality for T^n-bundles

Ulrich Bunke[*], Phillip Rumpf[†] and Thomas Schick [‡]

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Contents

[*]Mathematisches Institut, Georg-August-Universität Göttingen, bunke@uni-math.gwdg.de

[†]Fakultät für Mathematik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany, p_rump01@math.uni-muenster.de

[‡]Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstr. 3, 37073 Göttingen, Germany, schick@uni-math.gwdg.de

1 Introduction

1.1 String theory is a part of mathematical quantum physics. Its ultimate goal is the construction of quantum theories modeling the basic structures of our universe. More specifically, a string theory should associate a quantum field theory to a target consisting of a manifold equipped with further geometric structures like metrics, complex structures, vector bundles with connections, etc. A schematic picture is

$$\text{target} \xrightarrow{\text{string theory}} \text{quantum field theory} \quad .$$

The target is thought of to encode fundamental properties of the universe. Actually there are several types of string theories, the most important ones for the present paper are called of type IIA and IIB (see [18, Ch. 10]).

1.2 T-duality is a relation between two string theories on the level of quantum field theories to the effect that two different targets can very well lead to the same quantum field theory. The simplest example is the duality of bosonic string theories on the circles of radius R and R^{-1} (see [18, Ch. 8]). A relevant problem is to understand the factorization of the T-duality given on the level of quantum theory through T-duality on the level of targets. Schematically it is the problem of understanding the dottet arrow in

$$\begin{tikzcd} \text{target} \arrow[r, "\text{string theory, e.g. IIA}"] \arrow[d, dotted, "\text{target level } T\text{-duality}"] & \text{quantum field theory} \arrow[d, "\text{quantum level } T\text{-duality}"] \\ \text{target} \arrow[r, "\text{string theory, e.g IIB}"] & \text{quantum field theory} \end{tikzcd} \quad .$$

The problem starts with the question of existence, and even of the meaning of such an arrow.

1.3 T-duality on the target level is an intensively studied object in physics as well as in mathematics. We are not qualified to review the extensive relevant literature here, but let us mention mirror symmetry as one prominent aspect, mainly studied in algebraic geometry (see e.g. [20]).

1.4 In general, the target of a string theory is a manifold equipped with further geometric structures which in physics play the role of low-energy effective fields. The problem of topological T-duality can be understood schematically as the question of studying the dotted arrow in the following diagram.

$$\begin{tikzcd} \text{target} \arrow[r, "\text{forget geometry}"] \arrow[d, "\text{target level } T\text{-duality}"] & \text{underlying topological space} \arrow[d, dotted, "\text{topological } T\text{-duality}"] \\ \text{target} \arrow[r, "\text{forget geometry}"] & \text{underlying topological space} \end{tikzcd} .$$

1.5 At this level one faces the following natural problems.

(1) How can one characterize the topological T-dual of a topological space? It is not a priori clear that this is possible at all.

(2) If one understands the characterization of T-duals on the topological level, then one wonders if a given space admits a T-dual.

(3) Given a satisfactory characterization of topological T-duals one asks for a classification of T-duals of a given space.

As long as string theory is not part of rigorous mathematics the answer to the first question has to be found by physical reasoning and is part of the construction of mathematical models. Once an answer has been proposed the remainig two questions can be studied rigorously by methods of algebraic topology.

This is the philosophy of the present paper. For a certain class of spaces to be explained below we propose a mathematical characterization of topological T-duals. On this basis we then present a thourough and rigorous study of the existence and classification problems.

1.6 The expression "space" has to be understood in a somewhat generalized sense since we consider targets with additional non-trivial B-field background. There are several possibilities to model these backgrounds mathematically. In the present paper we use an axiomatic approach going under the notion of a twist, see A.1.

1.7 Topological T-duality in the presence of non-trivial B-field backgrounds has been studied mainly in the case of T^n-principal bundles ([3], [4], [5], [1], [2], [14], [15], [16]).

Our proposal for the characterizations of T-duals in terms of T-duality triples is strongly based on the analysis made in these papers.

1.8 The quantum field theory level T-duality predicts transformation rules for the low-energy effective fields which are objects of classical differential geometry like metrics and connections on the T^n-bundle, but also more exotic objects like a connective structure and a curving of the B-field background (these notions are explained in the framework of gerbes e.g. in [12]). These transformation rules are known as Buscher rules [9], [10].

1.9 The Buscher rules provide local rules for the behaviour of the geometric objects under T-duality on the target level. The underlying spaces of the targets (being principal bundles on manifolds) are locally isomorphic. Therefore, topological T-duality is really interesting only on the global level. The idea for setting up a chacterization of a topological T-dual comes from the desire to realize the Buscher transformation rules globally. The analysis of this transition from geometry to topology has been started in the case of circle bundles, e.g. [2] and continued including the higher dimensional case with [3],[5], [1], without stating a precise mathematical definition of topological T-duality there.

1.10 Currently, such a precise mathematical definition of topological T-duality has to be given in an ad-hoc manner. For T^n-bundles with twists we know three possibilities:

(1) A definition in the framework of non-commutative geometry can be extracted from the works [14], [15], [16] and will be explained in 2.26.

(2) The homotopy theoretic definition used in the present paper is based on the notion of a T-duality triple (see Definition 2.10).

(3) Following an idea of T. Pantev, in a forthcoming paper [8] we propose a definition of topological T-duality for T^n-bundles with twists using Pontrjagin duality for topological group stacks.

Surprisingly, all three definitions eventually lead to equivalent theories of topological T-duality for T^n-bundles with twists (the equivalence of (1) and (2) is shown in [19], and the equivalence of (2) and (3) is shown in [8]). This provides strong evidence for the fact that these definitions for topological T-duality correctly reflect the T-duality on the target or even quantum theory level.

1.11 If two spaces (with twist, i.e. B-field background) are in T-duality then this has strong consequences on certain of their topological invariants. For example, there are distinguished isomorphisms (called T-duality isomorphisms, see Definition 2.25) between their twisted cohomology groups and twisted K-groups. The existence of these T-duality isomorphisms has already been observed in [1], [14] and their follow-ups. The desire for a T-duality isomorphism actually was one of our main guiding principle which led to the introduction of the notion of a T-duality triple and therefore our mathematical definition of topological T-duality.

1.12 Having understood T-duality on the level of underlying topological spaces one can now lift back to the geometric level. We hope that the topological classification results (and their natural generalizations to topological stacks in order to include non-free T^n-actions) will find applications to mirror symmetry in algebraic geometry and string theory.

2 Topological T-duality via T-duality triples

2.1 In this Section we propose a mathematical set-up for topological T-duality of total spaces of T^n-bundles with twists and give detailed statements of our classification results. We will also shed some light on the relation with other pictures in the literature.

2.2 In the present paper we will use elements of the homotopy classification theory of principal fibre bundles [13, Ch 4]. Therefore, spaces in the present paper are always assumed to be Hausdorff and paracompact.

2.3 Let us fix a base space B and $n \in \mathbb{N}$. By $T^n := \underbrace{U(1) \times \cdots \times U(1)}_{n\text{-factors}}$ we denote the n-torus. The fundamental notion of the theory is that of a pair.

Definition 2.1 *A pair (E, h) over B consists of a principal T^n-bundle $E \to B$ and a cohomology class $h \in H^3(E, \mathbb{Z})$. An isomorphism of pairs $\phi\colon (E, h) \to (E', h')$ is an* *isomorphism*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

of T^n-principal bundles such that $\phi^* h' = h$. We let $P(B)$ denote the set of isomorphism classes of pairs over B.

We can extend P to a functor

$$P : \{\text{Spaces}\}^{op} \to \{\text{Sets}\} \ .$$

Let $f : B' \to B$ be a continuous map and $(E, h) \in P(B)$. Then we define $(E', h') := P(f)(E, H)$ as the pull-back of (E, h). More precisely, the T^n-bundle $E' \to B'$ is defined by the pull-back diagram

$$\begin{tikzcd} E' \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd} \ ,$$

and $h' := F^* h$.

2.4 The study of topological T-duality started with the case of circle bundles, i.e. $n = 1$. Guided by the experience obtained in [3], [1], [2], and [14], a mathematical definition of topological T-duality for pairs in the case $n = 1$ was given in [6]. In the latter paper T-duality appears in two flavors.

On the one hand, T-duality is a relation (see [6, Def. 2.9]) which may or may not be satisfied by two pairs (E, h) and $(\hat{E}, \hat{h})$ over B. The relation has a cohomological characterization. We will not recall the details of the definition here since it will be equivalent to Definition 2.10 in terms of T-duality triples (reduced to the case $n = 1$).

On the other hand we construct in [6] a T-duality transformation, a natural automorphisms of functors of order two

$$T \colon P \to P \ , \tag{2.2}$$

which assigns to each pair (E, h) a specific T-dual $(\hat{E}, \hat{h}) := T(E, h)$.

The existence of such a transformation is a special property of the case $n = 1$. It has already been observed in [3], [14], and [6] that such a transformation can not exist for general higher dimensional torus bundles. The first reason is that for $n \geq 2$ not every pair admits a T-dual which implies that T in (2.2) could at most be partially defined. An additional obstruction (to a partially defined transformation) is the non-uniqueness of T-duals.

2.5 In order to describe topological T-duality in the higher-dimensional ($n > 1$) case we introduce the notion of a T-duality triple. To this end we must categorify the third integral cohomology using the notion of twists.

There are various models for twists, some of them are reviewed in A.1. The reader not familiar with the concept of twists and twisted cohomology theories is advised to consult this appendix. The results of the present paper are independent of the choice of the model. Therefore, let us once and for all fix a model for twists.

Let us recall the essential properties of twists used in the constructions below. First of all we have a transformation

$$\{\text{category of twists over } B\}/_{\text{isomorphism}} \xrightarrow{\cong} H^3(B, \mathbb{Z})$$

which is natural in B. For a twist $\mathcal{H}$ we let $[\mathcal{H}]$ denote the cohomology class corresponding to the isomorphism class of $\mathcal{H}$. Furthermore, given isomorphic twists $\mathcal{H}, \mathcal{H}'$, the set $\mathtt{Hom}_{\mathrm{Twists}}(\mathcal{H}, \mathcal{H}')$ is a torsor over $H^2(B, \mathbb{Z})$, and this structure is again compatible with the functoriality in B. In this paper we frequently identify the based set of automorphisms $\mathtt{Hom}_{\mathrm{Twists}}(\mathcal{H}, \mathcal{H})$ with $H^2(B, \mathbb{Z})$.

For a twist $\mathcal{H}$ over B we will use the schematic notation

$$\mathcal{H} \dashrightarrow B$$

which aquires real sense if one realizes twists as gerbes or bundles of compact operators over B.

2.6 We fix an integer $n \geq 1$ and a connected base space B with a base point $b \in B$. A T^n-principal bundle $\pi : F \to B$ is classified by an n-tuple of Chern classes $c_1, \ldots, c_n \in$

$H^2(B,\mathbb{Z})$. Let $\hat{\pi} : \hat{F} \to B$ be a second T^n-principal bundle with Chern classes $\hat{c}_1, \dots, \hat{c}_n \in H^2(B,\mathbb{Z})$.

Let $\mathcal{H}$ be a twist on F such that its characteristic class lies in the second filtration step of the Leray-Serre spectral sequence filtration, i.e. satisfies $[\mathcal{H}] \in \mathcal{F}^2H^3(F,\mathbb{Z})$ (see A.2 for notation). Furthermore we assume that its leading part fulfills

$$[\mathcal{H}]^{2,1} = [\sum_{i=1}^{n} y_i \otimes \hat{c}_i] \in {}^{\pi}E_\infty^{2,1} \ , \tag{2.3}$$

where y_i are generators of the cohomology of the fibre $U(1)^n$ of F, compare again Appendix A.2. Similarly, let $\hat{\mathcal{H}}$ be a twist on $\hat{F}$ such that $[\hat{\mathcal{H}}] \in \mathcal{F}^2H^3(\hat{F},\mathbb{Z})$ and (with similar notation)

$$[\hat{\mathcal{H}}]^{2,1} = [\sum_{i=1}^{n} \hat{y}_i \otimes c_i] \in {}^{\hat{\pi}}E_\infty^{2,1} \ . \tag{2.4}$$

We assume that we have an isomorphism of twists $u : \hat{p}^*\hat{\mathcal{H}} \to p^*\mathcal{H}$. as indicated in the diagram

$$\begin{tikzcd}
 & p^*\mathcal{H} \arrow[dl] \arrow[dr, dotted] & & \hat{p}^*\hat{\mathcal{H}} \arrow[ll, "u"'] \arrow[dl, dotted] \arrow[dr] & \\
\mathcal{H} \arrow[dr, dotted] & & F \times_B \hat{F} \arrow[dl, "p"] \arrow[dr, "\hat{p}"] \arrow[dd, "r"] & & \hat{\mathcal{H}} \arrow[dl, dotted] \\
 & F \arrow[dr, "\pi"] & & \hat{F} \arrow[dl, "\hat{\pi}"] & \\
 & & B & &
\end{tikzcd} \quad . \tag{2.5}$$

We require that this isomorphism satisfies the condition $\mathcal{P}(u)$ which we now describe. Let F_b and $\hat{F}_b$ denote the fibers of F and $\hat{F}$ over $b \in B$ and consider the induced diagram

$$\begin{tikzcd}
 & F_b \times \hat{F}_b \arrow[dl, "p_b"'] \arrow[dr, "\hat{p}_b"] & \\
F_b & & \hat{F}_b
\end{tikzcd} \quad .$$

The assumptions on $\mathcal{H}$ and $\hat{\mathcal{H}}$ imply the existence of isomorphisms $v : \mathcal{H}_{|F_b} \xrightarrow{\sim} 0$ and $\hat{v} : 0 \xrightarrow{\sim} \hat{\mathcal{H}}_{|\hat{F}_b}$. We now consider the composition

$$u(b) := (0 \stackrel{\hat{p}_b^*\hat{v}}{\to} \hat{p}_b^*\hat{\mathcal{H}}_{|\hat{F}_b} \stackrel{u_{|F_b \times \hat{F}_b}}{\to} p_b^*\mathcal{H}_{|F_b} \stackrel{p_b^*v}{\to} 0) \in H^2(F_b \times \hat{F}_b,\mathbb{Z}) \ . \tag{2.6}$$

The condition $\mathcal{P}(u)$ requires that

$$[u(b)] = [\sum_{i=1}^{n} y_i \cup \hat{y}_i] \in H^2(F_b \times \hat{F}_b, \mathbb{Z})/(\mathtt{im}(p_b^*) + \mathtt{im}(\hat{p}_b^*)). \tag{2.7}$$

The class $[u(b)]$ in this quotient is well-defined independent of the choice of v and $\hat{v}$.

Definition 2.8 *An n-dimensional T-duality over B triple is a triple*

$$((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$$

consisting of T^n-bundles $\pi : F \to B$, $\hat{\pi} : \hat{F} \to B$, twists $\mathcal{H} \dashrightarrow F$, $\hat{\mathcal{H}} \dashrightarrow \hat{F}$ satisfying (2.3) and (2.4), respectively, and an isomorphism $u : \hat{p}^*\hat{\mathcal{H}} \xrightarrow{\sim} p^*\mathcal{H}$ (for notation see (2.5)) which satisfies condition $\mathcal{P}(u)$.

Definition 2.9 *We will say that the triple $((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ extends the pair $(F, [\mathcal{H}])$ and connects the two pairs $(F, [\mathcal{H}])$ and $(\hat{F}, [\hat{\mathcal{H}}])$.*

2.7 We can now define our notion of topological T-duality based on T-duality triples.

Definition 2.10 *Two pairs (F, h) and $(\hat{F}, \hat{h})$ over B are in T-duality if here is a T-duality triple connecting them.*

The main results of the present paper concern the following problems:

(1) classification of isomorphism classes of T-duality triples over B

(2) classification of T-duality triples which connect two given pairs

(3) existence and classification of T-duality triples extending a given pair

2.8 There is a natural notion of an isomorphism of T-duality triples. Its details will be spelled out in Definition 4.5. If $f : B \to B'$ is a continuous map, and $x := ((F', \mathcal{H}'), (\hat{F}', \hat{\mathcal{H}}'), u')$ is a T-duality triple over B', then one defines a T-duality triple $((F, H), (\hat{F}, \hat{H}), u) = f^*x$ over B in a canonical way. First of all the underlying T^n-bundles are given by the pull-back diagrams

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d] & F' \arrow[d] \\ B \arrow[r, "f"] & B' \end{tikzcd} , \quad \begin{tikzcd} \hat{F} \arrow[r, "\hat{\phi}"] \arrow[d] & \hat{F}' \arrow[d] \\ B \arrow[r, "f"] & B' \end{tikzcd} .$$

Then we define the twists $\mathcal{H} := \phi^*\mathcal{H}'$ and $\hat{\mathcal{H}} := \hat{\phi}^*\hat{\mathcal{H}}'$. Finally we consider the induced map $\psi := (\phi, \hat{\phi}) : F \times_B \hat{F} \to F' \times_{B'} \hat{F}'$ and define u as the composition

$$\hat{p}^*\hat{\mathcal{H}} \cong \psi^*(\hat{p}')^*\hat{\mathcal{H}}' \stackrel{\psi^*u'}{\to} \psi^*(p')^*\mathcal{H}' \cong p^*\mathcal{H}$$

of natural isomorphisms and the pull-back of u' via ψ.

Definition 2.11 *We define the functor*

$$\mathtt{Triple}_n : \{\mathit{spaces}\}^{op} \to \{\mathit{sets}\}$$

which associates to a space B the set of isomorphism classes $\mathtt{Triple}_n(B)$ of n-dimensional T-duality triples over B.

2.9 In Lemma 7.1 we will observe that the functor $\mathtt{Triple}_n$ is homotopy invariant. In general, given a contravariant homotopy invariant functor from spaces to sets one asks whether it can be represented by a classifying space. If this is the case, then the functor can be studied by applying methods of algebraic topology to its classifying space. Our study of the functor $\mathtt{Triple}_n$ follows this philosophy.

2.10 In the following we describe a space $\mathbf{R}_n$ which will turn out to be a classifying space of the functor $\mathtt{Triple}_n$ by Theorem 2.14.

Consider the product of two copies of the Eilenberg-MacLane space $K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ with canonical generators $x_1, \dots, x_n$ and $\hat{x}_1, \dots, \hat{x}_n$ of the second integral cohomology. We consider the class $q := \sum_{i=1}^n x_i \cup \hat{x}_i$ as a map $q : K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \to K(\mathbb{Z}, 4)$.

Definition 2.12 *Let $\mathbf{R}_n$ be the homotopy fiber of q.*

We consider the two components of the map $(\mathbf{c}, \hat{\mathbf{c}}) : \mathbf{R}_n \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ as the classifying maps of two T^n-principal bundles $\pi_n : \mathbf{F}_n \to \mathbf{R}_n$ and $\hat{\pi}_n : \hat{\mathbf{F}}_n \to \mathbf{R}_n$. By a calculation of the cohomology of $\mathbf{F}_n$, $\hat{\mathbf{F}}_n$ and $\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n$ we show the following Theorem.

Theorem 2.13 (Theorem 4.6) *There exists a unique isomorphism class of n-dimensional T-duality triples $[x_{n,univ}] = [(\mathbf{F}_n, \mathcal{H}_n), (\hat{\mathbf{F}}_n, \hat{\mathcal{H}}_n), \mathbf{u}_n] \in \mathtt{Triple}_n(\mathbf{R}_n)$ with underlying T^n-bundles isomorphic to $\mathbf{F}_n$ and $\hat{\mathbf{F}}_n$.*

Let P_n denote the set-valued functor classified by $\mathbf{R}_n$. This functor associates to B the set $P_n(B)$ of homotopy classes $[f]$ of maps $f : B \to \mathbf{R}_n$. The universal triple $[x_{n,univ}]$ induces a natural transformation of functors $\Psi_B : P_n \to \mathtt{Triple}_n(B)$ by

$$\Psi_B([f]) := \mathtt{Triple}_n(f)[x_{n,univ}] = f^*[x_{n,univ}] \ .$$

The following theorem characterizes $\mathbf{R}_n$ as a classifying space of the functor $\mathtt{Triple}_n$.

Theorem 2.14 (Theorem 7.24) *The natural transformation Ψ is an isomorphism of functors.*

2.11 In order to prove Theorem 2.14 we must investigate the fine structure of the functor $\mathtt{Triple}_n$. Of particular importance is the following action of $H^3(B, \mathbb{Z})$ on $\mathtt{Triple}_n(B)$ (see 7.3). Let $x := ((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ represent a class $[x] \in \mathtt{Triple}_n(B)$, and let $\alpha \in H^3(B, \mathbb{Z})$. We choose a twist $\mathcal{V}$ in the class α and set $x + \mathcal{V} := ((F, \mathcal{H} \otimes \pi^*\mathcal{V}), (\hat{F}, \hat{\mathcal{H}} \otimes \hat{\pi}^*\mathcal{V}), u \otimes r^*\mathtt{id}_{\mathcal{V}})$ (see (2.5) for the definition of r). Then we define $[x] + \alpha := [x + \mathcal{V}]$.

We now consider the set $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ of isomorphism classes of n-dimensional T-duality triples over fixed T^n-bundles F and $\hat{F}$ (see 7.2). The group $H^3(B, \mathbb{Z})$ acts naturally on $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ by the same construction as above.

Proposition 2.15 (Proposition 7.4) $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ *is an $H^3(B, \mathbb{Z})$-torsor.*

2.12 In terms of the classifying spaces, fixing F and $\hat{F}$ corresponds to fixing classifying maps $(c,\hat{c}) : B \to K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2)$. The set $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ then corresponds to the set of homotopy classes of lifts in the diagram

$$\begin{tikzcd} & \mathbf{R}_n \arrow[d, "(\mathbf{c},\hat{\mathbf{c}})"] \\ B \arrow[ur, dotted, "f"] \arrow[r, "(c,\hat{c})"'] & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) \end{tikzcd} .$$

Since the homotopy fiber of $(\mathbf{c},\hat{\mathbf{c}})$ has the homotopy type of a $K(\mathbb{Z},3)$-space it is clear by obstruction theory that $H^3(B,\mathbb{Z})$ acts freely and transitively on the set of such lifts. In combination with Proposition 2.15 this leads to the key step in the proof that $\mathbf{R}_n$ is the correct classifying space.

2.13 Let now ψ and $\hat{\psi}$ be bundle automorphisms of F and $\hat{F}$. We can realize ψ and $\hat{\psi}$ as right multiplication by maps $\psi,\hat{\psi} : B \to T^n \cong K(\mathbb{Z}^n,1)$. In this way the homotopy classes of ψ and $\hat{\psi}$ can be considered as classes $[\psi],[\hat{\psi}] \in H^1(B,\mathbb{Z}^n)$. Let $x := ((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ be an n-dimensional T-duality triple. Then we form the triple $x^{(\psi,\hat{\psi})} := ((F,\psi^*\mathcal{H}),(\hat{F},\hat{\psi}^*\hat{\mathcal{H}}),(\psi,\hat{\psi})^*u)$. We introduce the notation $\hat{c} \cup [\psi] := \sum_{i=1}^n \hat{c}_i \cup [\psi]_i \in H^3(B,\mathbb{Z})$, where $\hat{c}_1,\dots,\hat{c}_n$ are the components of the Chern class of $\hat{F}$, and $[\psi]_1,\dots,[\psi]_n$ are the components of $[\psi]$. We define $c \cup [\hat{\psi}]$ similarly. Then we show:

Proposition 2.16 (Proposition 7.31) *In* $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ *we have*

$$[x^{(\psi,\hat{\psi})}] = [x] + \hat{c} \cup [\psi] + c \cup [\hat{\psi}] \ .$$

There is a natural forgetful map

$$\Psi : \mathtt{Triple}_n^{(F,\hat{F})}(B) \to \mathtt{Triple}_n(B) \ .$$

Recall the definition (2.5) of the map r and note that $\mathtt{im}({}^r d_2^{2,1}) \subseteq H^3(B,\mathbb{Z})$ (see A.2 for notation) is exactly the subgroup of elements which can be written in the form $c \cup a + \hat{c} \cup b$ for $a,b \in H^1(B,\mathbb{Z}^n)$. Proposition 7.31 immediately implies:

Corollary 2.17 *If* $\alpha \in \mathtt{im}({}^r d_2^{2,1}) \subseteq H^3(B,\mathbb{Z})$*, then we have* $\Psi([x]+\alpha) = \Psi([x])$*.*

2.14 Let $(e_1,\dots,e_n,\hat{e}_1,\dots,\hat{e}_n)$ be the standard basis of $\mathbb{Z}^{2n}$. Let $O(n,n,\mathbb{Z})\subset GL(2n,\mathbb{Z})$ be the subgroup of transformations which fix the quadratic form $q:\mathbb{Z}^{2n}\to\mathbb{Z}$ with $q(\sum_{i=1}^n a_ie_i+b_i\hat{e}_i):=\sum_{i=1}^n a_ib_i$.

Proposition 2.18 (Lemma 4.1) *The group $O(n,n,\mathbb{Z})$ acts by homotopy equivalences on $\mathbf{R}_n$. We have an induced action of $O(n,n,\mathbb{Z})$ on the functor $\mathtt{Triple}_n$ by automorphisms.*

In the literature this group is sometimes called the *T-duality group.*

2.15 Recall the definition 2.1 of the functor $B\mapsto P(B)$ which associates to a space B the set of isomorphism classes of n-dimensional pairs over B. We will write $\tilde{P}_{(0)}:=P$ since this functor appears at the lowest level of a tower of functors $\tilde{P}_{(0)}\leftarrow\tilde{P}_{(1)}\leftarrow\dots$ (see 5.4). In the notation for these functors we will not indicate the dimension n of the torus T^n explicitly.

The functor $\tilde{P}_{(0)}$ is homotopy invariant (the proof of [6, Lemma 2.2] extends from the case $n=1$ to arbitrary $n\geq 1$). Generalizing again the approach of [6] from the case $n=1$ to general $n\geq 1$ we construct a classifying space $\tilde{\mathbf{R}}_n(0)$ for the functor $\tilde{P}_{(0)}$ as follows. Let $U^n\to K(\mathbb{Z}^n,2)$ be the universal T^n-bundle. Then we define

$$\tilde{\mathbf{R}}_n(0):=U^n\times_{T^n}\mathtt{Map}(T^n,K(\mathbb{Z},3))\ .$$

The natural map $\tilde{\mathbf{R}}_n(0)\to K(\mathbb{Z}^n,2)$ classifies a T^n-principal bundle $\tilde{\mathbf{F}}_n(0)\to\tilde{\mathbf{R}}_n(0)$ which admits a natural map $\tilde{\mathbf{F}}_n(0)\to K(\mathbb{Z},3)$. We interpret the homotopy class of this map as a class $\tilde{\mathbf{h}}(0)\in H^3(\tilde{\mathbf{F}}_n(0),\mathbb{Z})$. The isomorphism class of the universal pair $[\tilde{\mathbf{F}}_n(0),\tilde{\mathbf{h}}(0)]\in\tilde{P}_{(0)}(\tilde{\mathbf{R}}_n(0))$ induces a natural transformation of functors $\tilde{v}_B:[B,\tilde{\mathbf{R}}_n(0)]\to\tilde{P}_{(0)}(B)$ (see Lemma 5.1) which turns out to be an isomorphism.

2.16 In Section 5 we introduce the one-connected cover $\tilde{\mathbf{R}}_n\to\tilde{\mathbf{R}}_n(0)$. It is the universal covering of a certain connected component of $\tilde{\mathbf{R}}_n(0)$. The first entry of the universal triple $((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathcal{H}}_n),u_n)$ over $\mathbf{R}_n$ gives rise to a classifying map

$$f(0):\mathbf{R}_n\to\tilde{\mathbf{R}}_n(0)\ .$$

We shall see (Lemma 5.4) that $f(0)$ has a factorization

$$\begin{tikzcd} & \tilde{\mathbf{R}}_n \arrow[d] \\ \mathbf{R}_n \arrow[ur, dotted, "f"] \arrow[r, "f(0)"] & \tilde{\mathbf{R}}_n(0) \end{tikzcd} \quad . \tag{2.19}$$

Note that the factorization f is not unique.

Theorem 2.20 (Theorem 5.4) *The map $f : \mathbf{R}_n \to \tilde{\mathbf{R}}_n$ is a weak homotopy equivalence.*

2.17 There are two natural transformations of functors

$$\begin{tikzcd} & \mathtt{Triple}_n \arrow[dl, "s"'] \arrow[dr, "\hat{s}"] & \\ P & & P \end{tikzcd} \quad ,$$

where

$$s((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u) := (F,[\mathcal{H}]) \ , \quad \hat{s}((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u) := (\hat{F},[\hat{\mathcal{H}}]) \ .$$

The problem of the existence and the classification of T-duals of a pair $(F,h) \in P(B)$ is essentially a question about the fibre $s^{-1}(F,h) \subseteq \mathtt{Triple}_n(B)$. The transformation s is realized on the level of classifiying spaces by the map

$$\tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(0)$$

in (2.19). This allows to translate questions about the fibres of s to homotopy theory.

2.18 Consider a pair (F,h) over a space B. The representatives of elements of $s^{-1}(F,h)$ will be called extensions of (F,h).

Definition 2.21 *An extension of (F,h) to an n-dimensional T-duality triple is an n-dimensional T-duality triple $((F,\mathcal{H}),(\hat{F},\mathcal{H}),u)$ over B such that $[\mathcal{H}] = h$.*

The difference between the notions of an extension of (F,h) and an element in the fibre $s^{-1}(F,h)$ is seen on the level of the notion of an isomorphism of extensions (see Definition 7.33). Roughly speaking, an isomorphism of extensions of (F,h) is an isomorphism of triples such that the underlying bundle isomorphism of F is the identity.

Definition 2.22 *We let* $\mathtt{Ext}(F,h)$ *denote the set of isomorphism classes of extensions of* (F,h) *to* n*-dimensional* T*-duality triples.*

We have a natural surjective map

$$\mathtt{Ext}(F,h) \to s^{-1}(F,h)$$

which in general may not be injective.

2.19 We then consider the following two problems.

(1) Under which conditions does (F,h) admit an extension, i.e. is the set $\mathtt{Ext}(F,h)$ non-empty?

(2) Describe the set $\mathtt{Ext}(F,h)$.

Answers to these questions settle the problem of existence and classification of T-duals of (F,h) in the following sense.

(1) The pair (F,h) admits a T-dual if and only if $\mathtt{Ext}(F,h)$ is not empty.

(2) The set of T-duals of (F,h) can be written as $\hat{s}(\mathtt{Ext}(F,h)) \subseteq P(B)$.

2.20 As a consequence of Theorem 2.20 we derive the following answer to the first question.

Theorem 2.23 (Theorem 5.7) *The pair* (F,h) *admits an extension to a* T*-duality triple* $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ *if and only if* $h \in \mathcal{F}^2H^3(F,\mathbb{Z})$*.*

In particular, the condition $h \in \mathcal{F}^2H^3(F,\mathbb{Z})$ is a necessary and sufficient condition for the existence of a T-dual to (F,h). If we write out the leading part of h as $h^{2,1} = [\sum_{i=1}^n y_i \otimes \hat{c}_i] \in {}^\pi E_\infty^{2,1}$, then we can read off some information about the Chern classes $\hat{c}_1,\dots,\hat{c}_n$ of the T-dual bundle $\hat{F}$. In fact we have ${}^\pi E_\infty^{2,1} = {}^\pi E_2^{2,1}/\mathtt{im}({}^\pi d_2^{0,2})$, and ${}^\pi d_2^{0,2}(\sum_{i<j} A_{i,j} y_i \cup y_j) = \sum_{i,j}(A_{i,j} - A_{j,i}) y_j \otimes c_i$, where we set $A_{i,j} := 0$ for $i \geq j$.

It follows that the Chern classes $\hat{c}_i$ of the dual bundle $\hat{F}$ are determined by the pair (F, h) up to a change $\hat{c}_i \mapsto \hat{c}_i + \sum_{j=1}^n B_{i,j} c_j$ for some antisymmetric matrix $B \in \mathtt{Mat}(n, n, \mathbb{Z})$. Of course, the classes $\hat{c}_i$ are completely determined by the choice of an extension of (F, h).

We fix a pair (F, h). If $n \geq 2$, then even the topology of the T-dual bundle $\hat{F}$ may depend on the choice of the extension of the pair (F, h) to a T-duality triple. We have already demonstrated this by an example in [6, Section 4.4].

The discussion above gives a description of the topological invariants of a[1] T-dual of (F, h) in terms of relations between the Chern classes $c, \hat{c}$ and the H^3-classes $[\mathcal{H}]$ and $[\hat{\mathcal{H}}]$. Our results improve the results of [15][2].

2.21 Let $x := ((F, \mathcal{H}), (\hat{F}, \mathcal{H}), u)$ represent a class $\{x\} \in \mathtt{Ext}(F, h)$. Then we let $c(x), \hat{c}(x)$ denote the Chern classes of $F, \hat{F}$. Note that the group $\ker(\pi^*) \subseteq H^3(B, \mathbb{Z})$ acts on $\mathtt{Ext}(F, h)$. Furthermore we have a homomorphism $C : H^1(B, \mathbb{Z}^n) \to \ker(\pi^*)$ given by $C(a) := \sum_{i=1}^n c_i \cup a_i$, where a_i denotes the components of a.

The first two assertions of the following theorem have already been discussed above.

Theorem 2.24 *(1) The set $\mathtt{Ext}(F, h)$ is non-empty if and only if $h \in \mathcal{F}^2 H^3(F, \mathbb{Z})$.*

(2) If $\{x\} \in \mathtt{Ext}(F, h)$, then for any antisymmetric matrix $B \in \mathtt{Mat}(n, n, \mathbb{Z})$ there exists $\{x'\} \in \mathtt{Ext}(F, h)$ such that $\hat{c}_i(x') = \hat{c}_i(x) + \sum_{j=1}^n B_{i,j} c_j(x)$. Vice versa, if $\{x'\} \in \mathtt{Ext}(F, h)$, then $\hat{c}(x')$ is of this form.

(3) If $\{x\} \in \mathtt{Ext}(F, h)$, then $\{\{x'\} \in \mathtt{Ext}(F, h) | \hat{c}(x') = \hat{c}(x)\}$ is an orbit under the effective action of $\ker(\pi^*)/\mathtt{im}(C)$.

We prove this theorem in 7.29. Note that in terms of the Leray-Serre spectral sequence of $\pi : F \to B$ we can identify $\ker(\pi^*)/\mathtt{im}(C) \cong \mathtt{im}({}^\pi d_3^{0,2})$.

[1] At the moment of writing this paper is was not clear that the notions of a T-dual used in the present paper coincides with that of [15]. Meanwhile results in this direction have been obtained in [19].

[2] The calculation in [15] is not correct since the inclusion $X_j \to X$ in (4) of [15] does not exist in general.

2.22 Our work on T-duality was inspired by [3]. In [3] the authors study smooth T^n-bundles equipped with real valued cohomology classes $h_{\mathbb{R}}$. Guided by the principles which we explained in 1.9, in [3] the T-dual pair is constructed geometrically using differential forms. Furthermore it was observed in [3] that a condition similar to $h_{\mathbb{R}} \in \mathcal{F}^2 H^3(F, \mathbb{R})$ is necessary and sufficient for the construction of a T-dual to work. The problem of non-uniqueness of the construction was not addressed in that paper. In the present paper we provide a precise counterpart of [3] including the full information of integral cohomology.

2.23 An interesting feature of topological T-duality is the T-duality isomorphism in twisted cohomology theories. In [6, Definition 3.1], compare 6.2, we introduced axioms (in particular the concept of admissibility) for a twisted cohomology theory so that one can define for two pairs in T-duality a T-duality transformation which is then proved to be an isomorphism. Examples of twisted cohomology theories satisfying the admissibility axioms are twisted K-theory and twisted real (periodic) cohomology. See A.1 for a definition of twisted K-theory.

The existence of a T-duality isomorphism has been previously observed in [3], [1], [2], [14] and was our main guiding principle for the definition of the T-duality relation.

2.24 Let us assume that $((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ is a T-duality triple. Let $(X, \mathcal{H}) \mapsto h(X, \mathcal{H})$ be a twisted cohomology theory, where the notation suggests that the cohomology groups depend on two entries in a functorial way, namely the space X and the twist $\mathcal{H}$. The following definition uses the notation of (2.5).

Definition 2.25 *The T-duality transformation $T : h(F, \mathcal{H}) \to h(\hat{F}, \hat{\mathcal{H}})$ is defined by*

$$T := \hat{p}_! \circ u^* \circ p^* \ .$$

Note that T shifts degrees by $-n$. Furthermore, it is linear over $h(B)$.

Theorem 2.26 (Theorem 6.3) *If the twisted cohomology theory $h(\dots, \dots)$ is T-admissible then the T-duality transformation is an isomorphism.*

2.25 A T-duality isomorphism for twisted K-theory and twisted periodic de Rham cohomology was also obtained in [3]. In contrast to this paper, we take torsion in the third cohomology into account. We refer to [6, Section 4.3] for an explicit example which shows that the torsion part plays a significant role.

2.26 The approach of [14], [15], and [16] to T-duality uses ideas from noncommutative topology. The class $h \in H^3(F, \mathbb{Z})$ is interpreted as the Dixmier-Douady class of a unique isomorphism class of a stable continuous trace algebra $\mathcal{A} := \mathcal{A}(F, h)$ with spectrum F. Equivalently, the twist $\mathcal{H} \dashrightarrow F$ is a bundle of C^*-algebras with fibre the compact operators on a separable infinite-dimensional complex Hilbert space such that $[\mathcal{H}] = h$ (see A.1). Then we can write $\mathcal{A}(F, h) \cong C_0(F, \mathcal{H})$ (we assume for simplicity that B (and hence F) is locally compact).

The authors study the question of lifting the T^n-action on F to an $\mathbb{R}^n$-action on $\mathcal{A}$ such that the Mackey invariant is trivial. In this case the crossed product $\hat{\mathcal{A}} := \mathcal{A} \rtimes \mathbb{R}^n$ is again a continuous trace algebra with a spectrum $\hat{F}$ which is a T^n-principal bundle over B. Let $\hat{h} \in H^3(\hat{F}, \mathbb{Z})$ denote the Dixmier-Douady class of $\hat{\mathcal{A}}$. From the point of view of [14], [15], [16] the pair $(\hat{F}, \hat{h})$ is the T-dual of (F, h).

There is an obvious similarity of the following notions and their role in the theory of topological T-duality.

- $\mathbb{R}^n$-action on $\mathcal{A}(F, h)$ lifting the T^n-action on F with trivial Mackey obstruction
- Extension of (F, h) to a T-duality triple.

The equivalence of the two approaches is established in [19].

In the approach of [14], the T-duality isomorphism for twisted K-theory is equivalent to an isomorphism $K(\mathcal{A}) \cong K(\hat{\mathcal{A}})$. In fact, this isomorphism is Connes' Thom isomorphism for crossed products with $\mathbb{R}^n$. .

Using the approach via noncommutative topology, the natural two problems are to decide under which conditions the required $\mathbb{R}^n$-action on $\mathcal{A}$ exists, and to study the set of choices for such an action. A satisfactory picture can be obtained in the cases $n = 1$ and $n = 2$. The case $n = 1$ is easy and has been reviewed in [6]. The main results of [14] deal with the case $n = 2$. The necessary and sufficient condition for the existence of the $\mathbb{R}^n$ action is again that $h \in \mathcal{F}^2H^3(F, \mathbb{Z})$. It is then claimed in [14], that the action is unique. This is not always true. In fact, it follows from the diagram given in [14], Theorem 4.3.3, and the observation that d_2'' (we use the notation of [14]) factors over $p_! : H^2(F, \mathbb{Z}) \to H^0(B, \mathbb{Z})$, that the group $H^0(B, \mathbb{Z})/\mathtt{im}(p_!)$ acts freely on the set of $\mathbb{R}^n$-actions with trivial Mackey invariant lifting the T^n-action on F.

2.27 Let us mention a very interesting aspect of [14] which goes beyond the theory covered by the present paper. As explained above the necessary and sufficient condition for the existence of a T-dual of (F, h) is $h \in \mathcal{F}^2H^3(F, \mathbb{Z})$. The lift of the T^n action on F to an $\mathbb{R}^n$-action on $\mathcal{A}(F, h)$ exists if $h \in \mathcal{F}^1H^3(F, \mathbb{Z})$, but under this weaker condition one might encounter a non-trivial Mackey obstruction. In this case the crossed product $\hat{\mathcal{A}} := \mathcal{A} \rtimes \mathbb{R}^n$ is not the algebra of sections of a bundle of compact operators on a T^n-principal bundle over B. It has been observed in [14] (case $n = 2$) that one can interpret $\hat{\mathcal{A}}$ as an algebra of sections of a bundle of noncommutative tori over B. In other words, the T-dual of (F, h) can be realized as a bundle of noncommutative tori. For further discussion of this phenomenon see also [15] and [16].

2.28 In [5] the point of view of [14, 15] is generalized even further by considering torus bundles with a completely arbitrary H-flux differential 3-form with integral periods[3]. It is then argued that the resulting dual should be a bundle of non-associative non-commutative tori.

2.29 Assume that $F = B \times T^n$ is the trivial T^n-bundle, and that we consider the trivial twist $h = 0$. Then the T-dual bundle is again the trivial bundle, $\hat{F} = B \times T^n$, and the dual twist vanishes: $\hat{h} = 0$. In this situation the T-duality transformation $T : K(B \times T^n) \to K(B \times T^n)$ is a K-theory version of the Fourier-Mukai transformation (see e.g. [17]).

Note that the algebraic geometric analog is more precise. In this case F and $\hat{F}$ are bundles of dual abelian varieties. On $F \times_B \hat{F}$ one has the so-called Poincaré sheaf $\mathcal{P}$. Its first Chern

[3]In the present paper's language [4] considers pairs (F, h) without any condition on $h \in H^3(F, \mathbb{Z})$

class c_1 (considered as an automorphism of the trivial twist) satisfies the condition $\mathcal{P}(c_1)$. The Fourier-Mukai transformation is a functor $T : D^b(F) \to D^b(\hat{F})$ between bounded derived categories of coherent sheaves given on objects by $T(X) = R\hat{p}_*(\mathcal{P} \overset{L}{\otimes} p^*X)$. Thus the T-duality transformation considered in the present paper is a coarsification of the Fourier-Mukai transformation since it takes in a certain sense only the isomorphism classes of objects into account. The tensor product with the Poincaré sheaf plays the role of an automorphism of the trivial twist.

A bundle of abelian varieties has a section. Therefore this case corresponds to the case of trivial T^n-bundles in the present paper. Non-trivial bundles can be interpreted as bundles of torsors. In this case a good analog of the Poincaré bundle such that $\mathcal{P}(c_1)$ (see 2.7) is satisfied may not exist. In the topological situation we must replace the Poincaré bundle by an isomorphism u of non-trivial twists in order to satisfy $\mathcal{P}(u)$, and to have a T-duality *isomorphism*. In algebraic geometry a similar observation is known (see e.g [11]), where twists are represented by Azumaya algebras.

3 The space $\mathbf{R}_n$

3.1 If G is an abelian group and $k \in \mathbb{N}$, then we consider the homotopy type $K(G,k)$ of the Eilenberg MacLane space. It is characterized by $\pi_i(K(G,k)) \cong 0$ for $i \neq k$, and $\pi_k(K(G,k)) = G$. We denote a CW-complex of this homotopy type by the same symbol.

The Eilenberg-MacLane space $K(G,k)$ classifies the cohomology functor $H^k(\dots,G)$. In fact, there is a universal class $z \in H^k(K(G,k),G)$ such that $f \mapsto f^*(z)$ induces a natural isomorphism $[B,K(G,k)] \to H^k(B,G)$, where $[B,K(G,k)]$ denotes homotopy classes of maps.

Occasionally, we will interpret $K(\mathbb{Z},2)$ also as the classifying space of $T^1 := U(1)$. An explicit model is U/T^1, where U is the unitary group of a separable infinite dimensional Hilbert space with the strong topology. The bundle $U \to U/T^1$ is the universal T^1-principal bundle. Note further that $K(\mathbb{Z}^n,2) \cong K(\mathbb{Z},2)^n$ has the homotopy type of BT^n, and this space carries an universal T^n-bundle $U^n \to K(\mathbb{Z},2)^n$.

3.2 We consider the product of Eilenberg-MacLane space $K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$. Let $x_i,\hat{x}_i$, $i=1,\dots,n$ be the canonical generators of $H^2(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2),\mathbb{Z})$. Let $q: K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)\to K(\mathbb{Z},4)$ be the map classifying

$$x_1\cup\hat{x}_1+\dots+x_n\cup\hat{x}_n\in H^4(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2),\mathbb{Z})\ .$$

Definition 3.1 *We define the homotopy type $\mathbf{R}_n$ by the homotopy pull-back diagram*

$$\begin{array}{ccc}\mathbf{R}_n & \stackrel{(\mathbf{c},\hat{\mathbf{c}})}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)\\ \downarrow & & q\downarrow\\ * & \to & K(\mathbb{Z},4)\end{array}\ .$$

In other words, $\mathbf{R}_n$ is defined as the homotopy fiber of q.

3.3 For later use we determine the homotopy groups of $\mathbf{R}_n$.

Lemma 3.2 *The homotopy groups of $\mathbf{R}_n$ are given by*

i	0	1	2	3	≥ 4
$\pi_i(\mathbf{R}_n)$	$*$	0	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

Proof. The homotopy fiber of $(\mathbf{c},\hat{\mathbf{c}})$ is homotopy equivalent to the one of $*\to K(\mathbb{Z},4)$, i.e to $K(\mathbb{Z},3)$. The assertion follows immediately from the long exact sequence of homotopy groups. □

3.4 Write $\mathbf{c}=(\mathbf{c}_1,\dots,\mathbf{c}_n)$ and $\hat{\mathbf{c}}=(\hat{\mathbf{c}}_1,\dots,\hat{\mathbf{c}}_n)$, i.e. we let $\mathbf{c}_i$ and $\hat{\mathbf{c}}_i$ denote the components of $\mathbf{c}$ or $\hat{\mathbf{c}}$, respectively.

Lemma 3.3 *We have*

i	0	1	2	3	4
$H^i(\mathbf{R}_n,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$

.

Here $H^2(\mathbf{R}_n,\mathbb{Z})$ is freely generated by the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$, and $H^4(\mathbf{R}_n,\mathbb{Z})$ is generated by all possible products of the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$ subject to one relation

$$0=\mathbf{c}_1\cup\hat{\mathbf{c}}_1+\cdots+\mathbf{c}_n\cup\hat{\mathbf{c}}_n\ .$$

Proof. Recall from the proof of Lemma 3.2 that the homotopy fiber of $\mathbf{R}_n\to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$ is a $K(\mathbb{Z},3)$. The relevant part of the second page of the Leray-Serre spectral sequence ${}^{(\mathbf{c},\hat{\mathbf{c}})}E_2^{p,q}\cong H^p(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2),H^q(K(\mathbb{Z},3),\mathbb{Z}))$ therefore becomes

3	$\mathbb{Z}$	0	$*$	0	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{\frac{2n(2n+1)}{2}}$
q/p	0	1	2	3	4

.

We read off that $H^2(\mathbf{R}_n,\mathbb{Z})\cong\mathbb{Z}^{2n}$ is generated by the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$. The group ${}^{(\mathbf{c},\hat{\mathbf{c}})}E_2^{4,0}\cong H^4(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2),\mathbb{Z})$ is freely generated by all possible products of the components of $\mathbf{c}$ and $\hat{\mathbf{c}}$.

Let $z_3\in H^3(K(\mathbb{Z},3),\mathbb{Z})\cong{}^{(\mathbf{c},\hat{\mathbf{c}})}E_2^{0,3}$ be the canonical generator. It also generates the group ${}^tE_2^{0,3}$ of the Leray-Serre spectral sequence of the homotopy fibration $t:*\to K(\mathbb{Z},4)$. A part of its second page is

3	$\mathbb{Z}$	0	0	0	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
q/p	0	1	2	3	4

.

Since $H^3(*,\mathbb{Z})\cong 0\cong H^4(*,\mathbb{Z})$ we conclude that ${}^td_2^{0,3}(z_3)=z_4\in H^4(K(\mathbb{Z},4),\mathbb{Z})\cong{}^tE_2^{4,0}$ is the generator. Now by construction $q^*z_4=x_1\cup\hat{x}_1+\cdots+x_n\cup\hat{x}_n$, and by naturality of the spectral sequences ${}^{(\mathbf{c},\hat{\mathbf{c}})}d_2^{0,3}(z)=q^*z_4$. This implies the assertion about $H^4(\mathbf{R}_n,\mathbb{Z})$.□

3.5 Recall that we consider $K(\mathbb{Z}^n,2)\cong BT^n$ (see 3.1).

Definition 3.4 *We define $\pi_n : \mathbf{F}_n \to \mathbf{R}_n$ to be the T^n-bundle which is classified by $\mathbf{c} : \mathbf{R}_n \to K(\mathbb{Z}^n, 2)$.*

Let $U \to K(\mathbb{Z}, 2)$ be the universal T^1-bundle. The n-fold product $U^n \to K(\mathbb{Z}, 2)^n \cong K(\mathbb{Z}^n, 2)$ is the universal T^n-bundle. By definition we get a pull-back diagram

$$\begin{array}{ccc} \mathbf{F}_n & \to & U^n \\ \pi_n \downarrow & & \downarrow \\ \mathbf{R}_n & \stackrel{\mathbf{c}}{\to} & K(\mathbb{Z}, 2)^n \end{array} .$$

3.6 In the following we use the spectral sequence notation introduced in A.2

Lemma 3.5 *We have*

i	0	1	2	3
$H^i(\mathbf{F}_n, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^n$	$\mathbb{Z}$

Here the group $H^2(\mathbf{F}_n, \mathbb{Z})$ is freely generated by the components of $\pi_n^* \hat{\mathbf{c}}$. In particular, restriction to the fiber of $\mathbf{F}_n \to \mathbf{R}_n$ induces the zero homomorphism on $H^2(\mathbf{F}_n, \mathbb{Z})$. Furthermore, $H^3(\mathbf{F}_n, \mathbb{Z})$ is generated by a class $\mathbf{h}_n \in \mathcal{F}^2 H^3(\mathbf{F}_n, \mathbb{Z})$ which is characterized by $[\mathbf{h}_n]^{2,1} = \sum_{i=1}^n [y_i \otimes \hat{\mathbf{c}}_i] \in {}^{\pi_n}E_\infty^{2,1}$.

Proof. We write out the second page ${}^{\pi_n}E_2^{p,q}$.

3	$\mathbb{Z}^{\frac{n(n-1)(n-2)}{6}}$				
2	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	0	$\mathbb{Z}^{n^2(n-1)}$	0	
1	$\mathbb{Z}^n$	0	$\mathbb{Z}^{2n^2}$	0	
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$
q/p	0	1	2	3	4

.

We know that ${}^{\pi_n}d_2^{0,1}(y_i) = \mathbf{c}_i \in {}^{\pi_n}E_2^{2,0} \cong H^2(\mathbf{R}_n, \mathbb{Z})$. It follows that ${}^{\pi_n}d_2^{0,1}$ is an isomorphism onto the subgroup of ${}^{\pi_n}E_2^{0,2}$ generated by $\mathbf{c}$ so that ${}^{\pi_n}E_3^{0,1} \cong 0$ and ${}^{\pi_n}E_3^{2,0}$ is freely generated by the components of $\hat{\mathbf{c}}$. We see already that $H^1(\mathbf{F}_n, \mathbb{Z}) \cong 0$.

The group ${}^{\pi_n}E_2^{0,2}$ is freely generated by all products $y_i \cup y_j$, $i < j$. We now use the multiplicativity of the Leray-Serre spectral sequence in order see that ${}^{\pi_n}d_2^{0,2}(y_i \cup y_j) =$

$y_j \otimes \mathbf{c}_i - y_i \otimes \mathbf{c}_j$. We conclude that ${}^{\pi_n}d_2^{0,2}$ is injective. This implies that ${}^{\pi_n}E_3^{0,2} \cong 0$, and it follows that $H^2(\mathbf{F}_n, \mathbb{Z})$ is freely generated by the components of $\pi_n^*\hat{\mathbf{c}}$.

We have ${}^{\pi_n}E_2^{4,0} \cong H^4(\mathbf{R}_n, \mathbb{Z})$ and ${}^{\pi_n}d_2^{1,2}(y_i \otimes \mathbf{c}_j) = \mathbf{c}_i \cup \mathbf{c}_j$ and ${}^{\pi_n}d_2^{1,2}(y_i \otimes \hat{\mathbf{c}}_j) = \mathbf{c}_i \cup \hat{\mathbf{c}}_j$.

In order to calculate $\ker({}^{\pi_n}d_2^{1,2})$ recall the relation $\mathbf{c}_1 \cup \hat{\mathbf{c}}_1 + \cdots + \mathbf{c}_n \cup \hat{\mathbf{c}}_n = 0$. Let $h := y_1 \otimes \hat{\mathbf{c}}_1 + \cdots + y_n \cup \hat{\mathbf{c}}_n$. Then we have ${}^{\pi_n}d_2^{1,2}(h) = 0$. We claim that $\ker({}^{\pi_n}d_2^{1,2}) \cong \mathbb{Z}h \oplus \mathtt{im}({}^{\pi_n}d_2^{0,2})$. Let $t := \sum_{i,j=1}^n a_{i,j} y_i \otimes \mathbf{c}_j + b_{i,j} y_i \otimes \hat{\mathbf{c}}_j$ for $a_{i,j}, b_{i,j} \in \mathbb{Z}$ and assume that ${}^{\pi_n}d_2^{1,2}(t) = 0$. Then $\sum_{i,j=1}^n a_{i,j}\mathbf{c}_i \cup \mathbf{c}_j + b_{i,j}\mathbf{c}_i \cup \hat{\mathbf{c}}_j = 0$. This implies that $a_{i,j} + a_{j,i} = 0$, $b_{i,j} = 0$ for $i \neq j$, and that there exists $b \in \mathbb{Z}$ such that $b_{i,i} = b$ for all $i = 1, \dots, n$. But then we can write $t = \sum_{i<j} a_{i,j}{}^{\pi_n}d_2^{0,2}(y_j \cup y_i) + bh$. It follows that ${}^{\pi_n}E_3^{2,1} \cong \mathbb{Z}$ is generated by the class of h.

The group ${}^{\pi_n}E_2^{3,0}$ is freely generated by the products $y_i \cup y_j \cup y_k$, $i < j < k$. Furthermore, ${}^{\pi_n}d_2^{0,3}(y_i \cup y_j \cup y_k) = y_j \cup y_k \otimes \mathbf{c}_i - y_i \cup y_k \otimes \mathbf{c}_j + y_i \cup y_j \otimes \mathbf{c}_k$. We thus calculate that ${}^{\pi_n}d_2^{0,3}$ is injective. We conclude that $H^3(\mathbf{F}_n, \mathbb{Z}) \cong \mathcal{F}^2H^3(\mathbf{F}_n, \mathbb{Z})$ is generated by the class $\mathbf{h}_n$ represented by $h \in {}^{\pi_n}E_2^{1,2}$. $\square$

3.7 In the proof of Lemma 3.2 we have found a homotopy Cartesian square

$$\begin{array}{ccc} K(\mathbb{Z},3) & \stackrel{i}{\to} & \mathbf{R}_n \\ \downarrow & & (\mathbf{c},\hat{\mathbf{c}}) \downarrow \\ * & \to & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) \end{array} .$$

We pull this square back along the map $\psi : K(\mathbb{Z}^n, 2) \cong U^n \times K(\mathbb{Z}^n, 2) \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ and obtain a cube of homotopy Cartesian squares

$$\begin{array}{ccccccc} i^*\mathbf{F}_n & & & \stackrel{I}{\longrightarrow} & & & \mathbf{F}_n \\ & \mathtt{pr} \searrow & & & & \pi_n \swarrow & \\ & & K(\mathbb{Z},3) & \stackrel{i}{\to} & \mathbf{R}_n & & \\ \lambda \downarrow & & \downarrow & & (\mathbf{c},\hat{\mathbf{c}}) \downarrow & & \kappa \downarrow \\ & & * & \to & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2) & & \\ & \nearrow & & & & \psi \nwarrow & \\ K(\mathbb{Z},1)^n & & & \longrightarrow & & & K(\mathbb{Z}^n,2) \end{array} .$$

Lemma 3.6 *We have* $I^*\mathbf{h}_n = \pm \mathtt{pr}^* z_3$*, where* $z_3 \in H^3(K(\mathbb{Z},3),\mathbb{Z})$ *is the canonical generator.*

Proof. The fiber of κ is equivalent to $K(\mathbb{Z},3)$. The second page ${}^{\kappa}E_2$ of the corresponding Leray-Serre spectral sequence has the form

3	$\mathbb{Z}$	0	$*$	$*$	$*$
2	0	0	0	0	$*$
1	0	0	0	0	$*$
0	$\mathbb{Z}$	0	$\mathbb{Z}^n$	0	$\mathbb{Z}^{\frac{n(n+1)}{2}}$
q/p	0	1	2	3	4

.

We know that $H^3(\mathbf{F}_n,\mathbb{Z}) \cong \mathbb{Z}$ is freely generated by $\mathbf{h}_n$. We see that ${}^{\kappa}d_2^{0,3} = 0$ and ${}^{\kappa}E_2^{0,3} \cong H^3(\mathbf{R}_n,\mathbb{Z})$ is generated by $\mathbf{h}_n$. On the other hand the group ${}^{\kappa}E_2^{0,3} \cong H^3(K(\mathbb{Z},3),\mathbb{Z})$ is freely generated by z_3. Therefore $\mathbf{h}_n = \pm z_3$.

Note that $i^*\mathbf{F}_n \cong K(\mathbb{Z},3) \times K(\mathbb{Z},1)^n$ so that the Leray-Serre spectral sequence ${}^{\lambda}E$ of λ degenerates. Let $I^* : {}^{\kappa}E_2 \to {}^{\lambda}E_2$ be the induced map of the second pages and note that ${}^{\lambda}E_2$ has the form

3	$\mathbb{Z}$	$\mathbb{Z}^n$	$*$	$*$	$*$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^n$	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{\frac{n(n-1)(n-2)}{6}}$	$\mathbb{Z}^{\frac{n(n-1)(n-2)(n-3)}{24}}$
q/p	0	1	2	3	4

,

where ${}^{\lambda}E_2^{0,3}$ is freely generated by $z_3 \in H^3(K(\mathbb{Z},3),\mathbb{Z})$. The map I induces an equivalence of the fibers of λ and κ. In particular, it induces an isomorphism $I^* : {}^{\kappa}E_2^{0,3} \to {}^{\lambda}E_2^{0,3}$ identifying the generators above. This implies that $I^*\mathbf{h}_n = \pm \mathtt{pr}^* z_3$. $\square$

4 The T-duality group and the universal triple

4.1 Let $(e_1, \ldots, e_n, \hat{e}_1, \ldots, \hat{e}_n)$ be the standard basis of $\mathbb{Z}^{2n}$. Let $G_n \subset GL(2n, \mathbb{Z})$ be the subgroup of transformations which fix the form $q : \mathbb{Z}^{2n} \to \mathbb{Z}$ given by $q(\sum_{i=1}^n a_i e_i + b_i \hat{e}_i) := \sum_{i=1}^n a_i b_i$. Usually denoted $O(n, n, \mathbb{Z})$, G_n will here be called the group of T-duality transformations.

4.2 Each $g \in G_n$ induces an equivalence $g : K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$.

Lemma 4.1 *There exist a unique homotopy class of lift $\tilde{g}$ in the diagram*

$$\begin{array}{ccc} \mathbf{R}_n & \xrightarrow{\tilde{g}} & \mathbf{R}_n \\ \downarrow{\scriptstyle (\mathbf{c},\hat{\mathbf{c}})} & & \downarrow{\scriptstyle (\mathbf{c},\hat{\mathbf{c}})} \\ K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) & \xrightarrow{g} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} .$$

Proof. We apply obstruction theory to the problem of existence and classification of lifts $\tilde{g}$. In fact, the obstruction is the class $(\mathbf{c}, \hat{\mathbf{c}})^* g^* (\sum_{i=1}^n x_i \cup \hat{x}_i) \in H^4(\mathbf{R}_n, \mathbb{Z})$ which vanishes since g preserves q. Therefore a lifts $\tilde{g}$ exist. The set of homotopy classes of lifts is a torsor over $H^3(\mathbf{R}_n, \mathbb{Z})$. Since $H^3(\mathbf{R}_n, \mathbb{Z}) \cong \{0\}$, the lift $\tilde{g}$ is unique. $\square$

The correspondence $G_n \ni g \mapsto \tilde{g}$ induces a homotopy action of G_n on $\mathbf{R}_n$.

4.3 We let $t \in G_n$ be the transformation given by $t(e_i) = \hat{e}_i$ and $t(\hat{e}_i) = e_i$.

Definition 4.2 *The universal T-duality is the lift $T := \tilde{t} : \mathbf{R}_n \to \mathbf{R}_n$ of t according to 4.1.*

Note that $T \circ T = \mathtt{id}_{\mathbf{R}_n}$ since $t^2 = 1 \in G_n$.

4.4

Definition 4.3 *The universal dual T^n-bundle is defined by the pull-back*

$$\begin{array}{ccc}\hat{\mathbf{F}}_n & \stackrel{\tilde{T}}{\to} & \mathbf{F}_n\\ \hat{\pi}_n\downarrow & & \pi_n\downarrow\\ \mathbf{R}_n & \stackrel{T}{\to} & \mathbf{R}_n\end{array}.$$

Furthermore, we define $\hat{\mathbf{h}}_n := \tilde{T}^*\mathbf{h}_n \in H^3(\hat{\mathbf{F}}_n, \mathbb{Z})$.

4.5 We consider the pull-back diagram

$$\begin{array}{ccccc} & & \mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n & & \\ & \mathbf{p}_n \swarrow & & \hat{\mathbf{p}}_n \searrow & \\ \mathbf{F}_n & & \mathbf{r}_n\downarrow & & \hat{\mathbf{F}}_n \\ & \pi_n \searrow & & \hat{\pi}_n \swarrow & \\ & & \mathbf{R}_n & & \end{array}.$$

Lemma 4.4 *We have*

i	0	1	2	3
$H^i(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}$

Moreover, $\pi_n^*\mathbf{h}_n = \hat{\pi}_n^*\hat{\mathbf{h}}_n$, and this element generates $H^3(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z})$.

Proof. We use the Leray-Serre spectral sequence ${}^{\mathbf{r}_n}E$. The relevant part of its second page has the form

3	$\mathbb{Z}^{\frac{2n(2n-1)(2n-2)}{6}}$				
2	$\mathbb{Z}^{\frac{2n(2n-1)}{2}}$	0	$\mathbb{Z}^{2n^2(2n-1)}$	0	
1	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{4n^2}$	0	
0	$\mathbb{Z}$	0	$\mathbb{Z}^{2n}$	0	$\mathbb{Z}^{n(2n+1)-1}$
q/p	0	1	2	3	4

Now ${}^rE_2^{0,1}$ is freely generated by $y_i, \hat{y}_i$, and ${}^rE_2^{2,0}$ is freely generated by $\mathbf{c}_i, \hat{\mathbf{c}}_i$. We know that ${}^rd_2^{0,1}(y_i) = \mathbf{c}_i$ and ${}^rd_2^{0,1}(\hat{y}_i) = \hat{\mathbf{c}}_i$. We conclude that ${}^rd_2^{0,1}$ is injective and $H^1(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z}) \cong 0$.

The group ${}^{\mathbf{r}_n}E_2^{0,2}$ is freely generated by the products $y_i \cup y_j$, $\hat{y}_i \cup \hat{y}_j$, $i<j$ and $y_i \cup \hat{y}_j$. We have ${}^{\mathbf{r}_n}d_2^{0,2}(y_i \cup y_j) = y_j \otimes \mathbf{c}_i - y_i \otimes \mathbf{c}_j$, ${}^{\mathbf{r}_n}d_2^{0,2}(\hat{y}_i \cup \hat{y}_j) = \hat{y}_j \otimes \hat{\mathbf{c}}_i - \hat{y}_i \otimes \hat{\mathbf{c}}_j$, and ${}^{\mathbf{r}_n}d_2^{0,2}(y_i \cup \hat{y}_j) = \hat{y}_j \otimes \mathbf{c}_i - y_i \otimes \hat{\mathbf{c}}_j$. It follows that ${}^{r}d_2^{0,2}$ is injective.

In a similar way we see that ${}^{\mathbf{r}_n}d_2^{0,3}$ is injective.

We now calculate ${}^{\mathbf{r}_n}E_3^{2,1}$. We claim that this group is freely generated by one class which can be represented by $\sum_{i=1}^n y_i \otimes \hat{\mathbf{c}}_i \in {}^{\mathbf{r}_n}E_2^{2,1}$ or alternatively by $\sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i \in {}^{\mathbf{r}_n}E_2^{2,1}$. In view of the construction of $\mathbf{h}_n$ and $\hat{\mathbf{h}}_n$, and by naturality of the Leray-Serre spectral sequence this would imply the assertion of the lemma about the third cohomology.

The group ${}^{\mathbf{r}_n}E_2^{4,0}$ is generated by the products $\mathbf{c}_i \cup \mathbf{c}_j, \hat{\mathbf{c}}_i \cup \hat{\mathbf{c}}_j$, $i \le j$, and all products $\mathbf{c}_i \cup \hat{\mathbf{c}}_j$ subject to one relation $\sum_{i=1}^n \mathbf{c}_i \cup \hat{\mathbf{c}}_i = 0$. Let $t = \sum_{i,j} a_{i,j} y_i \otimes \mathbf{c}_j + \sum_{i,j} b_{i,j} y_i \otimes \hat{\mathbf{c}}_j + \sum_{i,j} c_{i,j} \hat{y}_i \otimes \mathbf{c}_j + \sum_{i,j} d_{i,j} \hat{y}_i \otimes \hat{\mathbf{c}}_j$ and assume that ${}^{\mathbf{r}_n}d_2^{2,1}(t) = 0$. Then we have $\sum_{i,j} a_{i,j} \mathbf{c}_i \otimes \mathbf{c}_j + \sum_{i,j} b_{i,j} \mathbf{c}_i \otimes \hat{\mathbf{c}}_j + \sum_{i,j} c_{i,j} \hat{\mathbf{c}}_i \otimes \mathbf{c}_j + \sum_{i,j} d_{i,j} \hat{\mathbf{c}}_i \otimes \hat{\mathbf{c}}_j = 0$. This implies that $a_{i,j} + a_{j,i} = 0$, $d_{i,j} + d_{j,i} = 0$, for all i,j, $b_{i,j} + c_{i,j} = 0$ for all $i \neq j$, and that there exists a unique $e \in \mathbb{Z}$ such that $b_{i,i} + c_{i,i} = e$ for all $i = 1, \dots, n$. We can now write $t = \sum_{i<j} a_{i,j} {}^{\mathbf{r}_n}d_2^{0,2}(y_j \cup y_i) + \sum_{i<j} d_{i,j} {}^{\mathbf{r}_n}d_2^{0,2}(\hat{y}_j \cup \hat{y}_i) - \sum_{i \neq j} b_{i,j} {}^{\mathbf{r}_n}d_2^{0,2}(y_i \cup \hat{y}_j) - \sum_{i=1}^n b_{i,i} {}^{\mathbf{r}_n}d_2^{0,2}(y_i \cup \hat{y}_i) + e \sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i$. This already shows that ${}^{\mathbf{r}_n}E_3^{2,1}$ is freely generated by the class of $\sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i$. Finally note that $\sum_{i=1}^n {}^{\mathbf{r}_n}d_2^{0,2}(y_i \cup \hat{y}_i) = \sum_{i=1}^n \hat{y}_i \otimes \mathbf{c}_i - \sum_{i=1}^n y_i \otimes \hat{\mathbf{c}}_i$. This finishes the proof of the claim. □

4.6 Let $\mathcal{H}_n \in T(\mathbf{F}_n)$ be a twist with isomorphism class $[\mathcal{H}_n] = \mathbf{h}_n \in H^3(\mathbf{F}_n, \mathbb{Z})$. Set further $\hat{\mathcal{H}}_n := \tilde{T}^* \mathcal{H}_n \in T(\hat{\mathbf{F}}_n)$, where $\tilde{T}$ was defined in 4.3. Then $[\hat{\mathcal{H}}_n] = \hat{\mathbf{h}}_n$.

Recall the definition of $\mathbf{p}_n$ and $\hat{\mathbf{p}}_n$ in 4.5. Since $\mathbf{p}_n^* \mathbf{h}_n = \hat{\mathbf{p}}_n^* \hat{\mathbf{h}}_n$ by Lemma 4.4, we conclude that there exists an isomorphism of twists $\mathbf{u}_n : \hat{\mathbf{p}}_n^* \hat{\mathcal{H}}_n \to \mathbf{p}_n^* \mathcal{H}_n$. Since $H^2(\mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n, \mathbb{Z}) \cong 0$ by Lemma 4.4 this isomorphism is unique.

4.7 The notion of a T-duality triple over a space B was introduced in Definition 2.8. Here we clarify the notion of an isomorphism between such triples $x := ((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ and $x' := ((F', \mathcal{H}'), (\hat{F}', \hat{\mathcal{H}}'), u')$.

Definition 4.5 *We say that x and x' are isomorphic if there exist underlying bundle isomorphisms*

$$\begin{array}{ccc} F & \stackrel{\psi}{\to} & F' \\ \downarrow & & \downarrow \\ B & = & B \end{array} \quad , \quad \begin{array}{ccc} \hat{F} & \stackrel{\hat{\psi}}{\to} & \hat{F}' \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

and isomorphisms of twists

$$v : \psi^*\mathcal{H}' \to \mathcal{H} \ , \quad \hat{v} : \hat{\psi}^*\hat{\mathcal{H}}' \to \hat{\mathcal{H}}$$

such that the composition

$$\hat{p}^*\hat{\mathcal{H}} \stackrel{\hat{p}^*\hat{v}^{-1}}{\to} \hat{p}^*\hat{\psi}^*\hat{\mathcal{H}}' \cong (\psi,\hat{\psi})^*(\hat{p}')^*\hat{\mathcal{H}}' \stackrel{(\psi,\hat{\psi})^*u'}{\to} (\psi,\hat{\psi})^*(p')^*\mathcal{H}' \cong p^*\psi^*\mathcal{H}' \stackrel{p^*v}{\to} p^*\mathcal{H}$$

is equal to $u : \hat{p}^*\hat{\mathcal{H}} \to p^*\mathcal{H}$. Here $(\psi,\hat{\psi}) : F \times_B \hat{F} \to F' \times_B \hat{F}'$ is the induced map (and the other notation is as in 2.5).

4.8

Theorem 4.6 $((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathcal{H}}_n),\mathbf{u}_n)$ *represents the unique isomorphism class of T-duality triples over $(\mathbf{F}_n,\hat{\mathbf{F}}_n)$.*

Proof. We must verify that $((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathcal{H}}_n),\mathbf{u}_n)$ is a T-duality triple, because uniqueness is clear from the conditions on T-duality triples of 2.5, from uniqueness of $[\mathcal{H}_n]$ and $[\hat{\mathcal{H}}_n]$ by Lemma 3.5 and the uniqueness of $\mathbf{u}_n$ by Lemma 4.4.

The classes $[\mathcal{H}]$ and $[\hat{\mathcal{H}}]$ have the required properties by construction. It remains to show that condition $\mathcal{P}(u)$ is satisfied (see 2.5). We fix a base point $* \in \mathbf{R}_n$. Note that by Lemma 5.4 and Proposition 5.5 (which are independent of the result to be proved) we have a canonical equivalence $\mathbf{R}_1 \cong \tilde{\mathbf{R}}_1$. In [6] we studied in detail the topology of $\mathbf{R}_1$ and the associated T-duality. The idea of the proof is to reduce the present task to the case $n = 1$.

We consider the n-fold product $F \to B$ of the T^1-bundle $\mathbf{F}_1 \to \mathbf{R}_1$ i.e. we set $F := \mathbf{F}_1^n$ and $B := \mathbf{R}_1^n$. Let $p_i : B \to \mathbf{R}_1$ denote the projections. Let $(z,\hat{z}) : B \to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$ be the map whose components classify $z_i := p_i^* x$ and $\hat{z}_i := p_i^* \hat{x}$, and where $x, \hat{x} \in H^2(\mathbf{R}_1, \mathbb{Z})$ are the canonical generators.

We now apply obstruction theory to the lifting problem

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & f \nearrow & (\mathbf{c}, \hat{\mathbf{c}}) \downarrow \\ B & \stackrel{(z,\hat{z})}{\to} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} .$$

It follows from $z_i \cup \hat{z}_i = 0$ that $\sum_{i=1}^n z_i \cup \hat{z}_i = 0$ so that this diagram admits a lift f. Since $H^3(B, \mathbb{Z}) \cong 0$ the homotopy class (of lifts) of the lift f is in fact uniquely determined.

We therefore have a pull-back diagram of principal T^n-bundles

$$\begin{array}{ccc} F & \stackrel{\tilde{f}}{\to} & \mathbf{F}_n \\ \pi \downarrow & & \pi_n \downarrow \\ B & \stackrel{f}{\to} & \mathbf{R}_n \end{array} .$$

We define $h := \tilde{f}^* \mathbf{h}_n$. In this way we obtain a pair (F, h) over B. We further have natural projections $\mathtt{pr}_i : F \to \mathbf{F}_1$, $i = 1, \ldots, n$. Using the characterizations of $\mathbf{h}_n$ and $\mathbf{h}_1$ in Lemma 3.5, the naturality of Leray-Serre spectral sequences, and $H^3(B, \mathbb{Z}) \cong 0$, we see that

$$h = \sum_{i=1}^n \mathtt{pr}_i^* \mathbf{h}_1 . \tag{4.7}$$

Let $T_B : B \to B$ be the product of the T-duality transformations $T : \mathbf{R}_1 \to \mathbf{R}_1$ on each factor. Since T is a homotopy equivalence, there is a unique homotopy classes of lifts α in

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & \alpha \nearrow & T \downarrow \\ B & \stackrel{f \circ T_B}{\to} & \mathbf{R}_n \end{array} .$$

Since the composition $B \stackrel{\alpha}{\to} \mathbf{R}_n \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ coincides with $(z, \hat{z})$ we have $\alpha = f$, and the following diagram commutes upto homotopy

$$\begin{array}{ccc} B & \stackrel{f}{\to} & \mathbf{R}_n \\ T_B \downarrow & & T \downarrow \\ B & \stackrel{f}{\to} & \mathbf{R}_n \end{array} .$$

This shows that we have a pull-back diagram

$$\begin{array}{ccc} \hat{F} & \xrightarrow{\hat{f}} & \hat{\mathbf{F}}_n \\ \hat{\pi}\downarrow & & \hat{\pi}_n\downarrow \\ B & \xrightarrow{f} & \mathbf{R}_n \end{array},$$

where $\hat{F} \to B$ is the n-fold product of $\hat{\mathbf{F}}_1 \to \mathbf{R}_1$.

We get the commutative diagram

$$\begin{array}{ccccc} F_b \times \hat{F}_b & \to & F \times_B \hat{F} & \xrightarrow{\tilde{f} \times_B \hat{f}} & \mathbf{F}_n \times_{\mathbf{R}_n} \hat{\mathbf{F}}_n \\ \downarrow & & r\downarrow & & \mathbf{r}_n\downarrow \\ \{b\} & \to & B & \xrightarrow{f} & \mathbf{R}_n \end{array},$$

where we assume that $f(b) = *$.

It follows from Lemma 4.4 (in the case $n = 1$) and the Künneth formula that $H^2(F \times_B \hat{F}, \mathbb{Z}) \cong 0$. Therefore $(\tilde{f} \times_B \hat{f})^* u : (\tilde{f} \times_B \hat{f})^* \hat{\mathbf{p}}_n^* \hat{\mathcal{H}} \to (\tilde{f} \times_B \hat{f})^* \mathbf{p}_n^* \mathcal{H}$ is the unique isomorphism.

Let $\mathcal{V} \in T(\mathbf{F}_1)$ and $\hat{\mathcal{V}} \in T(\hat{\mathbf{F}}_1)$ be twists in the classes $\mathbf{h}_1$ and $\hat{\mathbf{h}}_1$. Then by (4.7) there exists isomorphisms of twists $\kappa : \tilde{f}^* \mathcal{H} \to \sum_{i=1}^n \mathtt{pr}_i^* \mathcal{V}$ and $\hat{\kappa} : \hat{f}^* \hat{\mathcal{H}} \to \sum_{i=1}^n \mathtt{pr}_i^* \hat{\mathcal{V}}$. The choice of these isomorphisms is not unique. However, their restrictions to the fiber over $\{b\} \to B$ is unique. This follows from the structure of $H^2(F, \mathbb{Z})$ (and of $H^2(\hat{F}, \mathbb{Z})$) implied by Lemma 3.5 in the case $n = 1$ and the Künneth formula.

At this moment we fix some choices of κ and $\hat{\kappa}$.

Let $q_i : F \times_B \hat{F} \to \mathbf{F}_1 \times_{\mathbf{R}_1} \hat{\mathbf{F}}_1$ be the projection onto the ith component. Note that $\tilde{f} \circ p = \mathbf{p}_n \circ (\tilde{f} \times_B \hat{f})$, $\hat{f} \circ \hat{p} = \hat{\mathbf{p}}_n \circ (\tilde{f} \times_B \hat{f})$, $\mathbf{p}_1 \circ q_i = \mathtt{pr}_i \circ r$ and $\hat{\mathbf{p}}_1 \circ q_i = \mathtt{pr}_i \circ \hat{p}$. We now have fixed isomorphisms

$$p^*\kappa\colon (\tilde{f} \times_B \hat{f})^* \mathbf{p}_n^* \mathcal{H} \cong \sum_{i=1}^n q_i^* \mathbf{p}_1^* \mathcal{V}\,, \qquad \hat{p}^*\hat{\kappa} : (\tilde{f} \times_B \hat{f})^* \hat{\mathbf{p}}_n^* \hat{\mathcal{H}} \cong \sum_{i=1}^n q_i^* \hat{\mathbf{p}}_1^* \hat{\mathcal{V}}.$$

Note that there is a unique isomorphism $\psi : \hat{\mathbf{p}}_1^* \hat{\mathcal{V}} \to \mathbf{p}_1^* \mathcal{V}$. This induces another isomorphism

$$\Phi : (\tilde{f} \times_B \hat{f})^* \hat{\mathbf{p}}_n^* \hat{\mathcal{H}} \overset{\hat{p}^*\hat{\kappa}}{\cong} \sum_{i=1}^n q_i^* \hat{\mathbf{p}}_1^* \hat{\mathcal{V}} \overset{\sum_{i=1}^n q_i^*\psi}{\cong} \sum_{i=1}^n q_i^* \mathbf{p}_1^* \mathcal{V} \overset{(p^*\kappa)^{-1}}{\cong} (\tilde{f} \times_B \hat{f})^* \mathbf{p}_n^* \mathcal{H}\,.$$

It follows that $(\tilde{f} \times_B \hat{f})^* u = \Phi$. We can now restrict Φ to the fiber $F_b \times \hat{F}_b$.

It was shown in [6, 3.2.4] that the restriction of ψ to the fiber $T^1 \times \hat{T}^1$ is classified by a generator of $H^2(T^1 \times \hat{T}^1, \mathbb{Z})$, namely by $y \cup \hat{y}$ in the canonical basis of $H^1(T^1 \times \hat{T}^1, \mathbb{Z})$. If we restrict the whole composition defining Φ to the fiber the we see that $[(\tilde{f}, \hat{f})^* u(b)] = [\sum_{i=1}^n y_i \cup \hat{y}_i] \in H^2(F_b \times \hat{F}_b, \mathbb{Z}) / (\mathtt{im}(p_b^*) + \mathtt{im}(\hat{p}_b^*))$, as required.

□

5 Pairs and triples

5.1 Recall the construction of the classifying space $\tilde{\mathbf{R}}_n(0)$ of pairs 2.15. The goal of the present section is the identification of $\mathbf{R}_n$ with a one-connected covering $\tilde{\mathbf{R}}_n(0)$.

We start with the universal T^n-bundle $U^n \to K(\mathbb{Z}^n, 2)$ and the T^n-space $\mathtt{Map}(T^n, K(\mathbb{Z}, 3))$, where T^n acts by reparametrization. In a first step we form the associated bundle

$$p : \tilde{\mathbf{R}}_n(0) := U^n \times_{T^n} \mathtt{Map}(T^n, K(\mathbb{Z}, 3)) \to K(\mathbb{Z}^n, 2) \ .$$

We define a T^n-bundle via pull-back

$$\begin{array}{ccc} \tilde{\mathbf{F}}_n(0) & \to & U^n \\ \downarrow & & \downarrow \\ \tilde{\mathbf{R}}_n(0) & \xrightarrow{p} & K(\mathbb{Z}^n, 2) \end{array} \ .$$

There is a canonical map

$$\tilde{\mathbf{h}}_n(0) \colon \tilde{\mathbf{F}}_n(0) \to K(\mathbb{Z}, 3) \ .$$

It is given by $\tilde{\mathbf{h}}_n(0)([v, \phi], u) := \phi(s)$, where $s \in T^n$ is the unique element such that $sv = u$. Here $u, v \in U^n$, $\phi \in \mathtt{Map}(T^n, K(\mathbb{Z}, 3))$, $[v, \phi] \in \tilde{\mathbf{R}}_n(0)$, and $([v, \phi], u) \in \tilde{\mathbf{F}}_n(0)$.

5.2 Recall that a pair (F,h) over a space B consists of a T^n-bundle $F \to B$ and a class $h \in H^3(F,\mathbb{Z})$. An isomorphism between pairs (F,h) and (F',h') is given by a diagram

$$\begin{array}{ccc} F & \stackrel{\Phi}{\to} & F' \\ \downarrow & & \downarrow \\ B & = & B \end{array},$$

where Φ is a T^n-bundle isomorphism such that $\Phi^*h' = h$.

Given a map $f : B' \to B$ of spaces we can form the pull-back

$$\begin{array}{ccc} F' & \stackrel{\tilde{f}}{\to} & F \\ \downarrow & & \downarrow \\ B' & \stackrel{f}{\to} & B \end{array}.$$

We define the pair $f^*(F,h) := (F', \tilde{f}^*h)$ over B'. Pull-back preserves isomorphism classes of pairs.

5.3 Let $\tilde{P}_{(0)}$ be the contravariant set-valued functor which associates to each space B the set $\tilde{P}_{(0)}(B)$ of isomorphism classes of pairs.

Lemma 5.1 *The space $\tilde{\mathbf{R}}_n(0)$ is a classifying space for $\tilde{P}_{(0)}$. More precisely, the pair $[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{h}}_n(0)] \in P_{(0)}(\tilde{\mathbf{R}}_n(0))$ induces a natural isomorphism $\tilde{v}_B : [B, \tilde{\mathbf{R}}_n(0)] \to \tilde{P}_{(0)}(B)$ such that $\tilde{v}_B(f) = f^*[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{h}}_n(0)]$.*

Proof. This is completely analogous to the proof of [6, Proposition 2.6]. We therefore refrain from repeating the proof here. □

5.4 By Lemma 5.1 we have an isomorphism $\pi_0(\tilde{\mathbf{R}}_n(0)) \cong \tilde{P}_{(0)}(*)$. Furthermore, note that $P_{(0)}(*) \cong H^3(T^n,\mathbb{Z})$ canonically. We define $\tilde{\mathbf{R}}_n(1) \subset \tilde{\mathbf{R}}_n(0)$ to be the component which corresponds to $0 \in H^3(T^n,\mathbb{Z})$. Restricting the pair $[\tilde{\mathbf{F}}_n(0), \tilde{\mathbf{H}}_n(0)]$ gives the pair $[\tilde{\mathbf{F}}_n(1), \tilde{\mathbf{h}}_n(1)]$ over $\tilde{\mathbf{R}}_n(1)$. We let $\tilde{P}_{(1)}$ be the functor classified by $\tilde{\mathbf{R}}_n(1)$. Observe that $\tilde{P}_{(1)}(B) \subset \tilde{P}_{(0)}(B)$ is the set of isomorphism classes of pairs $[F,h]$ such that the restriction of h to the fibers of F vanishes.

5.5 By Lemma 5.1 we have $\tilde{P}_{(0)}(S^1) \cong H^3(S^1 \times T^n)$. By the Künneth formula $H^3(S^1 \times T^n, \mathbb{Z}) \cong H^3(T^n, \mathbb{Z}) \oplus H^2(T^n, \mathbb{Z})$, and $\pi_1(\tilde{\mathbf{R}}_n(1)) \cong \tilde{P}_{(1)}(S^1 \times T^n) \cong H^2(T^n, \mathbb{Z})$ corresponds to the second summand. On can check that this bijection is a group homomorphism.

We consider the isomorphism $\phi : \pi_1(\tilde{\mathbf{R}}_n(1)) \xrightarrow{\sim} H^2(T^n, \mathbb{Z})$ as a cohomology class $\phi \in H^1(\tilde{\mathbf{R}}_n(1), H^2(T^n, \mathbb{Z})))$, i.e. as a homotopy class of maps $\phi : \tilde{\mathbf{R}}_n(1) \to K(H^2(T^n, \mathbb{Z}), 1)$. We define $\tilde{\mathbf{R}}_n$ as the homotopy pullback

$$\begin{array}{ccc} \tilde{\mathbf{R}}_n & \to & \tilde{\mathbf{R}}_n(1) \\ \downarrow & & \phi \downarrow \\ * & \to & K(H^2(T^n, \mathbb{Z}), 1) \end{array} .$$

Furthermore, we consider the pull-back

$$\begin{array}{ccc} \tilde{\mathbf{F}}_n & \to & \tilde{\mathbf{F}}_n(1) \\ \downarrow & & \downarrow \\ \tilde{\mathbf{R}}_n & \to & \tilde{\mathbf{R}}_n(1) \end{array} ,$$

and we let $\tilde{\mathbf{h}}_n \in H^3(\tilde{\mathbf{F}}_n, \mathbb{Z})$ be the pullback of $\tilde{\mathbf{h}}_n(1)$. Note that by construction and naturality, $\tilde{\mathbf{h}}_n$ pulls back to zero on the fiber of $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. Since $\tilde{\mathbf{R}}_n$ is simply connected, it then even belongs to the second step $\mathcal{F}^2 H^3(\tilde{\mathbf{F}}_n, \mathbb{Z})$ of the Leray-Serre filtration associated with $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$.

5.6

Lemma 5.2 *The homotopy groups of $\tilde{\mathbf{R}}_n$ are given by*

i	0	1	2	3	≥ 4
$\pi_i(\tilde{\mathbf{R}}_n)$	$*$	0	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

Proof. By construction, $\tilde{\mathbf{R}}_n$ is connected and simply connected. The homotopy fiber of $\tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$ is equivalent to the homotopy fiber of $* \to K(H^2(T^n, \mathbb{Z}), 1)$, i.e. to $K(H^2(T^n, \mathbb{Z}), 0)$. Hence this map induces an isomorphism $\pi_i(\tilde{\mathbf{R}}_n) \cong \pi_i(\tilde{\mathbf{R}}_n(1))$ for $i \geq 2$.

Note that $K(\mathbb{Z}, k)$ is an h-space for each k. Hence have an equivalence

$$\mathtt{Map}(T^1, K(\mathbb{Z}, k)) \simeq K(\mathbb{Z}, k) \times \Omega K(\mathbb{Z}, k-1) \simeq K(\mathbb{Z}, k) \times K(\mathbb{Z}, k-1) .$$

We use the exponential law to write $\mathtt{Map}(T^n, K(\mathbb{Z},3))$ as an iterated mapping space, and we obtain in this way an equivalence

$$\mathtt{Map}(T^n, K(\mathbb{Z},3)) \simeq K(\mathbb{Z},3) \times K(\mathbb{Z},2)^n \times K(\mathbb{Z},1)^{\frac{n(n-1)}{2}} \times K(\mathbb{Z},0)^{\frac{n(n-1)(n-2)}{6}} .$$

The long exact sequence of homotopy groups for

$$\mathtt{Map}(T^n, K(\mathbb{Z},3)) \to \tilde{\mathbf{R}}_n(0) \to K(\mathbb{Z},2)^n$$

and the fact that $\pi_3(K(\mathbb{Z},2)^n) \cong 0 \cong \pi_1(K(\mathbb{Z},2)^n)$ and $\pi_2(K(\mathbb{Z},2)^n) \cong \mathbb{Z}^n$ yields the exact sequence

$$0 \to \mathbb{Z}^n \to \pi_2(\tilde{\mathbf{R}}_n(1)) \to \mathbb{Z}^n \xrightarrow{\delta} \mathbb{Z}^{\frac{n(n-1)}{2}} \xrightarrow{\alpha} \pi_1(\tilde{\mathbf{R}}_n(1)) \to 0 .$$

Furthermore, we observe that $\pi_3(\tilde{\mathbf{R}}(1)) \cong \mathbb{Z}$ and $\pi_i(\tilde{\mathbf{R}}_n(1)) = 0$ for $i \geq 4$.

We have seen in 5.5 that $\pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^2(T^2,\mathbb{Z}) \cong \mathbb{Z}^{\frac{n(n-1)}{2}}$. We conclude that α must be surjective. Consequently it is injective and $\delta = 0$.

We therefore have an exact sequence

$$0 \to \mathbb{Z}^n \to \pi_2(\tilde{\mathbf{R}}_n(1)) \to \mathbb{Z}^n \to 0 ,$$

and this implies that $\pi_2(\tilde{\mathbf{R}}_n(1)) \cong \pi_2(\tilde{\mathbf{R}}_n) \cong \mathbb{Z}^{2n}$. □

5.7 Since the fiber of $p\colon \tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$ is equivalent to $K(H^2(T^n,\mathbb{Z}),0) \cong \pi_1(\tilde{\mathbf{R}}_n(1))$ we can consider this map as the universal covering of $\tilde{\mathbf{R}}_n(1)$. We now consider the problem of existence and classification of lifts in the diagram

$$\begin{array}{ccccc} & & \tilde{\mathbf{R}}_n & \to & * \\ & \tilde{f} \nearrow & p\downarrow & & \downarrow \\ B & \xrightarrow{f} & \tilde{\mathbf{R}}_n(1) & \xrightarrow{\phi} & K(H^2(T^n,\mathbb{Z}),1) \end{array} .$$

It follows from the construction of $\tilde{\mathbf{R}}_n$ that a lift $\tilde{f}$ exists if and only if $\phi \circ f$ is homotopic to a constant map. The lift itself depends on the choice of an explicit homotopy. If a lift exists, the set of homotopy classes of lifts is a torsor over $H^0(B, H^2(T^n,\mathbb{Z}))$.

5.8 The classification of homotopy classes $\tilde{f}$ (considered just as maps, not as lifts) lifting a homotopy class f is more subtle. In order to study this problem we assume that B is path connected and equipped with a base point $b \in B$. Let $\tilde{f}_0$ be a lift of f and consider $x \in \pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^0(B, H^2(T^n, \mathbb{Z}))$. Then we consider the lift $\tilde{f}_1 = x\tilde{f}_0$, i.e. the composition of $\tilde{f}_0$ with the deck transformation associated to x.

Assume that $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic. Let $H : I \times B \to \tilde{\mathbf{R}}_n$ be a homotopy. Then $p \circ H : S^1 \times B \to \tilde{\mathbf{R}}_n(1)$ can be considered as a map $h : B \to \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1))$. We have the following diagram

$$\begin{array}{ccc} \{b\} & \stackrel{x}{\to} & \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1)) \\ \downarrow & h \nearrow & \mathtt{ev}_1 \downarrow \\ B & \stackrel{f}{\to} & \tilde{\mathbf{R}}_n(1) \end{array} , \qquad (5.3)$$

where $\mathtt{ev}_1 : \mathtt{Map}(S^1, \tilde{\mathbf{R}}_n(1)) \to \tilde{\mathbf{R}}_n(1)$ is the evaluation at $1 \in S^1$.

Vice versa, if $x \in \pi_1(\tilde{\mathbf{R}}_n(1))$ is such that the diagram above admits a lift h, then $\tilde{f}_0$ and $x\tilde{f}_0$ are homotopic.

5.9 The existence problem for a lift h can be studied using obstruction theory. The fiber of the map $\mathtt{ev}_1$ is $\Omega\tilde{\mathbf{R}}_n(1))$. In the proof of Lemma 5.2 we have seen that the homotopy groups of $\tilde{\mathbf{R}}_n(1)$ are given by

i	0	1	2	3	≥ 4
$\pi_i(\tilde{\mathbf{R}}_n(1))$	$*$	$\mathbb{Z}^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

It follows that the homotopy groups of $\Omega\tilde{\mathbf{R}}_n(1)$ are given by

i	0	1	2	≥ 3
$\pi_i(\Omega\tilde{\mathbf{R}}_n)$	$Z^{\frac{n(n-1)}{2}}$	$\mathbb{Z}^{2n}$	$\mathbb{Z}$	0

.

We therefore have obstructions in $H^1(B, \mathbb{Z}^{\frac{n(n-1)}{2}})$, $H^2(B, \mathbb{Z}^{2n})$ and $H^3(B, \mathbb{Z})$, and a general discussion seems to be complicated.

5.10 Let $\tilde{P}_n$ be the set-valued functor classified by $\tilde{\mathbf{R}}_n$. For each space B we have a natural transformation $p_B : \tilde{P}_n(B) \to \tilde{P}_{(1)}(B)$ induced by composition with the map $p : \tilde{\mathbf{R}}_n \to \tilde{\mathbf{R}}_n(1)$.

We conclude from Subsection 5.7 that the fibers of p_B are homogeneous spaces over $H^0(B, H^2(T^n, \mathbb{Z}))$.

Consider a class $[F, h] \in \tilde{P}_{(1)}(B)$. By Subsection 5.7, it belongs to the image of p_B if and only the restriction $[F, h]_{|B^{(1)}}$ to a 1-skeleton $B^{(1)} \subset B$ is trivial. Since every principal torus bundle on a 1-dimensional complex is trivial, this is equivalent to the condition that the restriction of h to $F_{|B^{(1)}}$ is trivial, or equivalently $h \in \mathcal{F}^2 H^3(F, \mathbb{Z})$. Let us fix a map $f : B \to \tilde{\mathbf{R}}_n(1)$ representing a pair $[F, h]$ with this property. If we choose a homotopy from $\phi \circ f$ to the constant map, then we distinguish an element in the fiber $p_B^{-1}([F, h])$.

5.11 In Section 3 we have introduced a pair $(\mathbf{F}_n, \mathbf{h}_n)$ over $\mathbf{R}_n$ such that $\mathbf{h}_n \in \mathcal{F}^2 H^3(\mathbf{F}_n, \mathbb{Z})$. This isomorphism class of pairs gives rise to a classifying map $f(1) : \mathbf{R}_n \to \tilde{\mathbf{R}}_n(1)$.

Lemma 5.4 *The set of homotopy classes of maps f which are lifts of $f(1)$ in the diagram*

$$\begin{array}{ccc} & & \tilde{\mathbf{R}}_n \\ & f \nearrow & p \downarrow \\ \mathbf{R}_n & \stackrel{f(1)}{\to} & \tilde{\mathbf{R}}_n(1) \end{array}$$

is a torsor over $H^2(T^n, \mathbb{Z})$.

Proof. Since $\mathbf{R}_n$ is simply connected we know the existence of lifts. Let now $x \in \pi_1(\tilde{\mathbf{R}}_n(1))$. We choose a base point $b \in \mathbf{R}_n$ and identify $\pi_1(\tilde{\mathbf{R}}_n(1)) \cong H^2(\mathbf{F}_{n,b}, \mathbb{Z})$, where $\mathbf{F}_{n,b}$ denotes the fiber of $\mathbf{F}_n$ over b. In particular we view $x \in H^2(\mathbf{F}_{n,b}, \mathbb{Z})$.

We must show that the existence of a lift h in the diagram 5.3 (with B replaced by $\mathbf{R}_n$ and f replaced by $f(1)$) implies that $x = 0$. Assume that a lift h exists, adjoint to a map $H \colon S^1 \times \mathbf{R}_n \to \tilde{\mathbf{R}}_n(1)$. This corresponds to a T^n-bundle $F \to S^1 \times \mathbf{R}_n$ and a class $h \in H^3(F, \mathbb{Z})$. Let $\mathtt{pr} : S^1 \times \mathbf{R}_n \to \mathbf{R}_n$ be the projection. Since $\mathbf{R}_n$ is simply connected $\mathtt{pr}$ induces an isomorphism in second cohomology and the bundle F is the pull-back via $\mathtt{pr}$ of a T^n-bundle from $\mathbf{R}_n$. Since H restricts to f on $\{1\} \times \mathbf{R}_n$ and the corresponding T^n-bundle is $\mathbf{F}_n$, necessarily $F \cong \mathtt{pr}^* \mathbf{F}_n = S^1 \times \mathbf{F}_n$. By the Künneth formula $H^3(F, \mathbb{Z}) \cong H^3(\mathbf{F}_n, \mathbb{Z}) \oplus H^2(\mathbf{F}_n, \mathbb{Z})$, with corresponding decomposition $h = \mathbf{h}_n \oplus u$. By the definition of h and the calculation of $\pi_1(\tilde{\mathbf{R}}_n(1))$ in Subsection 5.5, the restriction of u

to $\mathbf{F}_{n,b}$ is x. Since the restriction $H^2(\mathbf{F}_n,\mathbb{Z}) \to H^2(\mathbf{F}_{n,b},\mathbb{Z})$ is trivial by the description of $H^2(\mathbf{F}_n,\mathbb{Z})$ given in Lemma 3.5 it follows that $x=0$. □

5.12 We now fix one choice of f in Lemma 5.4.

Proposition 5.5 *The map $f : \mathbf{R}_n \to \tilde{\mathbf{R}}_n$ is a weak homotopy equivalence.*

Proof. By Lemma 3.2 and Lemma 5.2 it suffices to show that f induces isomorphisms on π_2 and π_3.

Note that $H^2(\tilde{\mathbf{R}}_n,\mathbb{Z}) \cong \mathtt{Hom}_{\mathbb{Z}}(\pi_2(\tilde{\mathbf{R}}_n),\mathbb{Z}) \cong \mathbb{Z}^{2n}$. We have a natural map $x : \tilde{\mathbf{R}}_n \to K(\mathbb{Z}^n,2)$ which classifies the T^n-principal bundle $\tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. We study the components $x_i \in H^2(\tilde{\mathbf{R}}_n,\mathbb{Z})$.

We consider the second page of the Leray-Serre spectral sequence ${}^{\tilde{p}}E_2^{p,q}$ of the fibration $\tilde{p} : \tilde{\mathbf{F}}_n \to \tilde{\mathbf{R}}_n$. Since $\mathbf{h}_n \in \mathcal{F}^2H^3(\tilde{\mathbf{F}}_n,\mathbb{Z})$, there are elements $\hat{x}_1,\dots,\hat{x}_n \in H^2(\tilde{\mathbf{R}}_n,\mathbb{Z})$ and $a_{i,j} \in \mathbb{Z}$ such that $\tilde{\mathbf{h}}_n^{2,1}$ is represented by $\tilde{h} := \sum_{i,j} a_{i,j} y_i \otimes x_j + \sum_i y_i \otimes \hat{x}_i \in {}^{\tilde{p}}E_2^{2,1}$.

Under the pull-back induced by f the spectral sequence ${}^{\tilde{p}}E$ is mapped to the spectral sequence ${}^{\pi_n}E$ considered in 3.6. In particular $[\tilde{h}] \in {}^{\pi_n}E_\infty^{2,1}$ is mapped to $[\sum_{i=1}^n y_i \otimes \hat{\mathbf{c}}_i] \in {}^{\pi_n} \in E_\infty^{2,1}$. Using that $f^*x_i = \mathbf{c}_i$ we conclude that $f^*\hat{x}_i = \hat{\mathbf{c}}_i + \sum_{i,j} a_{i,j}\mathbf{c}_j$. It follows that $f^* : H^2(\tilde{\mathbf{R}}_n,\mathbb{Z}) \to H^2(\mathbf{R}_n,\mathbb{Z})$ maps $(x_1,\dots,x_n,\hat{x}_1,\dots,\hat{x}_n)$ to a basis and is therefore surjective and injective. This implies that $f_* : \pi_2(\mathbf{R}_n) \to \pi_2(\tilde{\mathbf{R}}_n)$ is an isomorphism.

It now suffices to show that $f : \pi_3(\mathbf{R}_n) \to \pi_3(\tilde{\mathbf{R}}_n)$ is surjective. A generator $g \in \pi_3(\tilde{\mathbf{R}}_n)$ is represented by a map $g : S^3 \to \tilde{\mathbf{R}}_n$. The corresponding pair is the trivial torus bundle $\mathtt{pr}_1 : S^3 \times T^n \to S^3$ with the cohomology class of the form $h = \mathtt{pr}_1^*z$ for some $z \in H^3(S^3,\mathbb{Z})$ which is a generator.

It suffices to show that the isomorphism class of pairs $[S^3 \times T^n, \mathtt{pr}_1^*z]$ is the pull back of $[\mathbf{F}_n,\mathbf{h}_n]$ on $\mathbf{R}_n$. We consider the composition $g : S^3 \to K(\mathbb{Z},3) \xrightarrow{i} \mathbf{R}_n$, where the map i was defined in 3.7 and the first map realizes a generator of $\pi_3(K(\mathbb{Z},3))$. It then follows immediately from Lemma 3.6 that $g^*[\mathbf{F}_n,\mathbf{h}_n] = [S^3 \times T^n, \pm\mathtt{pr}_1^*z]$. Choosing the opposite generator of $\pi_3(K(\mathbb{Z},3))$, if necessary, the assertion follows. $\square$

5.13

Corollary 5.6 *The functors P_n classified by $\mathbf{R}_n$ and $\tilde{P}_n$ classified by $\tilde{\mathbf{R}}_n$ are isomorphic. The group $H^2(T^n,\mathbb{Z})$ acts freely on the set of such isomorphisms, and it acts transitively if we fix the composition with $p_*\colon \tilde{P}_n \to \tilde{P}_{(1)}$.*

5.14 We consider a pair (F,h) over a space B. Recall the notion of an extension of a pair to a T-duality triple 2.21.

Theorem 5.7 *The pair (F,h) admits an extension to a T-duality triple $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ if and only if $h \in \mathcal{F}^2H^3(F,\mathbb{Z})$.*

Proof. The condition is necessary by definition of a triple. We show that it is also sufficient. Let $g : \tilde{\mathbf{R}}_n \to \mathbf{R}_n$ be a homotopy inverse of a choice of f in Proposition 5.5. Then $g^*((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathbf{H}}_n),\mathbf{u}_n)$ is a T-duality triple over $\tilde{\mathbf{R}}_n$ such that $g^*(\mathbf{F}_n,[\mathcal{H}_n])$ is isomorphic to $(\tilde{\mathbf{F}}_n,\tilde{\mathbf{h}}_n)$. Therefore we can consider $g^*((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathcal{H}}_n),\mathbf{u}_n)$ as an extension of $(\tilde{\mathbf{F}}_n,\tilde{\mathbf{h}}_n)$ to a T-duality triple.

Let now (F,h) be a pair over B with classifying map $\phi : B \to \tilde{\mathbf{R}}_n(0)$. Every lift Ψ in the diagram

$$\begin{array}{ccc} & & \tilde{\mathbf{R}}_n \\ & \Psi\nearrow & \downarrow \\ B & \stackrel{\psi}{\to} & \tilde{\mathbf{R}}_n(0) \end{array}$$

yields an extension $\Psi^*g^*((\mathbf{F}_n,\mathcal{H}_n),(\hat{\mathbf{F}}_n,\hat{\mathbf{H}}_n),\mathbf{u}_n)$ of (F,h) to a T-duality triple.

The lifting problem can be decomposed into two stages

$$\begin{array}{ccc} B & \stackrel{\Psi}{\to} & \tilde{\mathbf{R}}_n \\ \| & & p\downarrow \\ B & \stackrel{\Psi_1}{\to} & \tilde{\mathbf{R}}_n(1) \\ \| & & \downarrow \\ B & \stackrel{\psi}{\to} & \tilde{\mathbf{R}}_n(0) \end{array}$$

The lift Ψ_1 exists since $h \in \mathcal{F}^1 H^3(F,\mathbb{Z})$. Since p is the universal covering map, the stronger condition $h \in \mathcal{F}^2 H^3(F,\mathbb{Z})$ implies the existence of the lift Ψ in the second stage.

□

6 T-duality transformations in twisted cohomology

6.1 In [6], Section 3.1, we have introduced a set of axioms which describe the basic properties of twisted cohomology theories. Below we will only use the axioms. Let $h(\dots,\dots)$ be a twisted cohomology theory.

6.2 Let $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ be a T-duality triple over a space B. It gives rise to the diagram

$$\begin{array}{ccccc} & & F\times_B \hat{F} & & \\ & p\swarrow & & \searrow \hat{p} & \\ F & & r\downarrow & & \hat{F} \\ & \pi\searrow & & \swarrow\hat{\pi} & \\ & & B & & \end{array} \quad . \tag{6.1}$$

Recall the Definition 2.25 of the T-duality transformation

$$T := \hat{p}_! \circ u^* \circ p^* : h(F,\mathcal{H}) \to h(\hat{F},\hat{\mathcal{H}}) \ .$$

6.3 There is a unique 1-dimensional T-duality triple over a point. Recall the following definition from [6].

Definition 6.2 *The twisted cohomology theory h is called T-admissible, if the T-duality transformation associated with the unique one-dimensional T-duality triple over a point is an isomorphism (of degree -1).*

6.4

Theorem 6.3 *If $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ is a T-duality triple over a finite dimensional CW-complexes B, and if h is a T-admissible twisted cohomology theory, then the T-duality transformation 2.25 is an isomorphism.*

Proof. Exactly as in [6], Proof of Theorem 3.13, one uses induction on the dimension of the base space, the Mayer-Vietoris exact sequence and the 5-lemma to reduce this to the case $B=*$. For $B=*$ one observes that the n-dimensional T-duality transformation is an iterated 1-dimensional T-duality transformation which is an isomorphism by the definition of T-admissibility. □

7 Classification of T-duality triples and extensions

7.1 Recall from 2.8 the definition of the functor $B \to \mathtt{Triple}_n(B)$ which associates to a space the set of isomorphism classes (see Definition 4.5) of n-dimensional T-duality triples.

Lemma 7.1 *The functor $B \mapsto \mathtt{Triple}_n(B)$ is homotopy invariant.*

Proof. Let $x=((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ be a triple over B. Let $H:[0,1]\times B\to B'$ be a map. Let f_0,f_1 denote the evaluations of H at the endpoints of the interval. We must show that $f_0^*x\cong f_1^*x$. Let $\mathtt{pr}:[0,1]\times B\to B$ denote the projection. In the first step we choose bundle isomorphisms which extend the identity at 0

$$\begin{array}{ccc} H^*F & \stackrel{\psi}{\leftarrow} & \mathtt{pr}^*f_0^*F \\ \downarrow & & \downarrow \\ [0,1]\times B & = & [0,1]\times B \end{array}, \quad \begin{array}{ccc} H^*\hat{F} & \stackrel{\hat{\psi}}{\leftarrow} & \mathtt{pr}^*f_0^*\hat{F} \\ \downarrow & & \downarrow \\ [0,1]\times B & = & [0,1]\times B \end{array}.$$

Evaluation at $\{1\}\times B$ gives isomorphisms $\psi_1 : f_0^*F \to f_1^*F$ and $\hat{\psi}_1 : f_0^*\hat{F} \to f_1^*\hat{F}$.

Let $\Phi_0 : f_0^*F \to F$, $\hat{\Phi}_0 : f_0^*\hat{F} \to \hat{F}$, $\Phi_1 : f_1^*F \to F$, $\hat{\Phi}_1 : f_1^*\hat{F} \to \hat{F}$, $\gamma : H^*F \to F$, and $\hat{\gamma} : H^*\hat{F} \to \hat{F}$ denote the induced maps. Let $J_i : f_0^*F \to \mathtt{pr}^*f_0^*F$ and $\hat{J}_i : f_0^*\hat{F} \to \mathtt{pr}^*f_0^*\hat{F}$ be the canonical embeddings over $B \to \{i\}\times B \subset [0,1]\times B$, $i = 0,1$. Since $\gamma\circ\psi\circ J_0 = \Phi_0$ and $\hat{\gamma}\circ\hat{\psi}\circ\hat{J}_0 = \hat{\Phi}_0$ we have canonical isomorphisms of twists

$$J_0^*\psi^*\gamma^*\mathcal{H} \cong \Phi_0^*\mathcal{H}, \quad \hat{J}_0^*\hat{\psi}^*\hat{\gamma}^*\hat{\mathcal{H}} \cong \hat{\Phi}_0^*\hat{\mathcal{H}} \ .$$

These uniquely extend to isomorphisms of twists

$$w : \psi^*\gamma^*\mathcal{H} \to \mathtt{pr}^*\Phi_0^*\mathcal{H}, \quad \hat{w} : \hat{\psi}^*\hat{\gamma}^*\hat{\mathcal{H}} \to \mathtt{pr}^*\hat{\Phi}_0^*\hat{\mathcal{H}} \ .$$

We now restrict to $\{1\}\times B$ and obtain isomorphisms of twist $v : \psi_1^*\Phi_1^*\mathcal{H} \cong J_1^*\psi^*\gamma^*\mathcal{H} \stackrel{J_1^*w}{\to} J_1^*\mathtt{pr}^*\Phi_0^*\mathcal{H} \cong \Phi_0^*\mathcal{H}$ and $\hat{v} : \hat{\psi}_1^*\hat{\Phi}_1^*\hat{\mathcal{H}} \cong \hat{J}_1^*\hat{\psi}^*\hat{\gamma}^*\hat{\mathcal{H}} \stackrel{\hat{J}_1^*\hat{w}}{\to} \hat{J}_1^*\mathtt{pr}^*\hat{\Phi}_0^*\hat{\mathcal{H}} \cong \hat{\Phi}_0^*\hat{\mathcal{H}}$. We have the following diagram of isomorphisms of twists over $I\times(f_0^*F\times_B f_0^*\hat{F})$:

$$\begin{array}{ccc} \hat{p}^*\mathtt{pr}^*\hat{\Phi}_0^*\hat{\mathcal{H}} & \stackrel{\mathtt{pr}^*(\Phi_0,\hat{\Phi}_0)^*u}{\to} & p^*\mathtt{pr}^*\Phi_0^*\mathcal{H} \\ \hat{p}^*\hat{w}\uparrow & & p^*w\uparrow \\ \hat{p}^*\hat{\psi}^*\hat{\gamma}^*\hat{\mathcal{H}} & \stackrel{(\gamma\circ\psi,\hat{\gamma}\circ\hat{\psi})^*u}{\to} & p^*\psi^*\gamma^*\mathcal{H} \end{array}.$$

Here $p : I\times(f_0^*F\times_B f_0^*\hat{F}) \to I\times f_0^*F$ and $\hat{p} : I\times(f_0^*F\times_B f_0^*\hat{F}) \to I\times f_0^*\hat{F}$ are the projections. This diagram commutes after restriction to $\{0\}\times B$. Hence it commutes, and so does its restriction to $\{1\}\times B$. The latter restriction gives

$$\begin{array}{ccc} \hat{p}_1^*\hat{\Phi}_0^*\hat{\mathcal{H}} & \stackrel{(\Phi_0,\hat{\Phi}_0)^*u}{\to} & p_1^*\Phi_0^*\mathcal{H} \\ \hat{p}_1^*v\uparrow & & \hat{p}_1^*v\uparrow \\ \hat{p}_1^*\hat{\psi}_1^*\hat{\Phi}_1^*\hat{\mathcal{H}} & \stackrel{(\psi,\hat{\psi}_1)^*(\Phi_1,\hat{\Phi}_1)^*u}{\to} & p_1^*\psi_1^*\Phi_1^*\mathcal{H} \end{array}$$

with the projections $p_1 : f_0^*F\times_B f_0^*\hat{F} \to f_0^*F$ and $\hat{p}_1 : f_0^*F\times_B f_0^*\hat{F} \to f_0^*\hat{F}$.

This shows that the bundle isomorphisms $\psi_1, \hat{\psi}_1$ and the isomorphisms of twists $v, \hat{v}$ form an isomorphism of triples $f_0^*x \cong f_1^*x$. $\square$

7.2 Let us fix the T^n-bundles F and $\hat{F}$ over B. Then we can consider triples of the form $x := ((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$, $x' := ((F, \mathcal{H}'), (\hat{F}, \hat{\mathcal{H}}'), u')$.

Definition 7.2 *We say that x is isomorphic to x' over $(F, \hat{F})$ if there exists an isomorphism of triples $x \cong x'$ such that the underlying bundle isomorphisms are the identity maps (see 4.5).*

Let $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ denote the set of isomorphism classes of triples over $(F, \hat{F})$.

7.3 We have a natural action of $H^3(B, \mathbb{Z})$ on $\mathtt{Triple}_n^{(F,\hat{F})}(B)$. Let $\delta \in H^3(B, \mathbb{Z})$ and choose a twist $\mathcal{V}$ in the corresponding isomorphism class. If $x := ((F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u)$ represents $[x] \in \mathtt{Triple}_n^{(F,\hat{F})}(B)$, then we define the triple

$$x + \mathcal{V} := ((F, \mathcal{H} \otimes \mathcal{V}), (\hat{F}, \hat{\mathcal{H}} \otimes \mathcal{V}), u \otimes r^*\mathtt{id}_{\mathcal{V}}) \ , \tag{7.3}$$

and $[x] + \delta := [x + \mathcal{V}]$, where $r \colon F \times_B \hat{F} \to B$ is the projection.

Proposition 7.4 $H^3(B, \mathbb{Z})$ *acts freely and transitively on the set* $\mathtt{Triple}_n^{(F,\hat{F})}(B)$*.*

Proof. We prove Proposition 7.4 in several steps. First recall the following principle, which we will employ several times. Let a group G act on a set S. If τ is an equivalence relation on S preserved by G, then G acts freely and transitively on S if and only if the two following conditions hold:

(1) G acts transitively on the set S/τ of equivalence classes

(2) the isotropy group of one (and hence every) equivalence class acts freely and transitively on this equivalence class.

7.4 Since F and $\hat{F}$ are fixed at the moment, we use the notation $[\mathcal{H}, \hat{\mathcal{H}}, u]$ for the class $[(F, \mathcal{H}), (\hat{F}, \hat{\mathcal{H}}), u]$. Let us first introduce an equivalence relation τ_1 on the set $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ by defining that $[\mathcal{H}, \hat{\mathcal{H}}, u]$ is equivalent to $[\mathcal{H}', \hat{\mathcal{H}}', u']$ if and only if $\mathcal{H} \cong \mathcal{H}'$ and $\hat{\mathcal{H}} \cong \hat{\mathcal{H}}'$. The action of $H^3(B, \mathbb{Z})$ preserves this equivalence relation.

Lemma 7.5 *The action of $H^3(B,\mathbb{Z})$ on the set of equivalence classes $\mathtt{Triple}_n^{(F,\hat{F})}(B)/\tau_1$ is transitive with stabilizer $\ker(\pi^*)\cap\ker(\hat{\pi}^*)$.*

Proof. The assertion about the stabilizer is an immediate consequence of the definitions. We now verify transitivity of the action on the set of equivalence classes. If $[\mathcal{H},\hat{\mathcal{H}},u]\in\mathtt{Triple}_n^{(F,\hat{F})}(B)$, then by the definition of a T-duality triple, the leading part $[\mathcal{H}]^{2,1}=[\sum_{i=1}^n y_i\otimes\hat{c}_i]\in{}^{\pi}E_\infty^{2,1}$ of $[\mathcal{H}]$ is determined by $\hat{F}$, and similar $[\hat{\mathcal{H}}]^{2,1}$ is determined by F. Let us consider a second class $[\mathcal{H}',\hat{\mathcal{H}}',u']\in\mathtt{Triple}_n^{(F,\hat{F})}(B)$. It follows from the structure of the spectral sequences that there are classes $\delta,\hat{\delta}\in H^3(B,\mathbb{Z})$ such that $[\mathcal{H}']=[\mathcal{H}]+\pi^*\delta$ and $[\hat{\mathcal{H}}']=[\hat{\mathcal{H}}]+\hat{\pi}^*\hat{\delta}$.

Lemma 7.6 *We have $\delta-\hat{\delta}\in\ker(\pi^*)+\ker(\hat{\pi}^*)$.*

Proof. We assume without loss of generality that B is connected and choose a base point $b\in B$. Because of the presence of the isomorphisms u and u' of the pullbacks of the twists to $F\times_B\hat{F}$ we know that $r^*(\delta-\hat{\delta})=0$ (with r as in (7.3)). We choose twists $\mathcal{V}$ and $\hat{\mathcal{V}}$ representing δ and $\hat{\delta}$, respectively. For our question, we can replace $\mathcal{H}'$ by $\mathcal{H}\otimes\pi^*\mathcal{V}$ and $\hat{\mathcal{H}}'$ by $\hat{\mathcal{H}}\otimes\hat{\pi}^*\hat{\mathcal{V}}$. Thus, with the choice of an appropriate isomorphism of twists $w:r^*\mathcal{V}\to r^*\hat{\mathcal{V}}$ we can write $u'=u\otimes w$. Note that we have canonical isomorphisms $\mathcal{V}_{|\{b\}}\cong 0$, $\hat{\mathcal{V}}_{|\{b\}}\cong 0$. Therefore we can consider

$$(w(b):0\stackrel{\sim}{\to}r^*\mathcal{V}_{|F_b\times\hat{F}_b}\stackrel{w_{|F_b\times\hat{F}_b}}{\to}r^*\hat{\mathcal{V}}_{|F_b\times\hat{F}_b}\stackrel{\sim}{\to}0)\in H^2(F_b\times\hat{F}_b,\mathbb{Z})\ .$$

We know by Lemma A.1 that ${}^rd_2^{0,2}(w(b))=0$ and ${}^rd_3^{0,2}(w(b))=\delta-\hat{\delta}+\mathtt{im}({}^rd_2^{1,1})$. Let $W:=H^2(F_b\times\hat{F}_b,\mathbb{Z})/(\mathtt{im}(p_b^*)+\mathtt{im}(\hat{p}_b^*))$, and let $[\dots]$ denote for the moment classes in this quotient. Note that by 2.6 we have well-defined classes $[u(b)],[u'(b)]\in W$. Since u and u' satisfy the condition $\mathcal{P}$ we have $[u(b)]=[u'(b)]$, and it follows that $[w(b)]=[u'(b)]-[u(b)]=0$. Accordingly, we choose a decomposition $w(b)=p_b^*(d)+\hat{p}_b^*(\hat{d})$ for some $d\in H^2(F_b,\mathbb{Z})$ and $\hat{d}\in H^2(\hat{F}_b,\mathbb{Z})$. Then

$$0={}^rd_2^{0,2}(w(b))={}^rd_2^{0,2}(p_b^*(d))+{}^rd_2^{0,2}(\hat{p}_b^*(\hat{d})).$$

Since $\mathtt{im}({}^rd_2^{0,2}p_b^*)\cap\mathtt{im}({}^rd_2^{0,2}\hat{p}_b^*)=0$, ${}^{\pi}d_2^{0,2}(d)=0$ and ${}^{\hat{\pi}}d_2^{0,2}(\hat{d})=0$. Therefore

$$\delta-\hat{\delta}+\mathtt{im}({}^rd_2^{1,1})={}^rd_3^{0,2}(w(b))={}^{\pi}d_3^{0,2}(d)+{}^{\hat{\pi}}d_3^{0,2}(\hat{d})+\mathtt{im}({}^rd_2^{1,1}).$$

Finally we use that $\mathtt{im}({}^r d_2^{1,1}) = \mathtt{im}({}^\pi d_2^{1,1}) + \mathtt{im}({}^{\hat\pi} d_2^{1,1})$ since $H^1(F_b \times \hat F_b, \mathbb{Z}) \cong H^1(F_b, \mathbb{Z}) \oplus H^1(\hat F_b, \mathbb{Z})$. Using the relation between the images of ${}^\pi d_*^{*,*}, {}^{\hat\pi} d_*^{*,*}$ and the kernels of π^* and $\hat\pi^*$ we obtain the assertion. □

Write now $\delta - \hat\delta = -a + b$ with $a \in \ker(\pi^*)$ and $b \in \ker(\hat\pi^*)$. Then $e := \delta + a = \hat\delta + b$, and $[\mathcal{H}'] = [\mathcal{H}] + \pi^*\delta = [\mathcal{H}] + \pi^* e$, $[\hat{\mathcal{H}}'] = [\hat{\mathcal{H}}] + \hat\pi^*\hat\delta = [\hat{\mathcal{H}}] + \hat\pi^* e$. It follows that $[\mathcal{H}', \hat{\mathcal{H}}', u'] \sim_{\tau_1} [\mathcal{H}, \hat{\mathcal{H}}, u] + e$. This proves the transitivity statement of Lemma 7.5. □

7.5 Fix now an equivalence class for τ_1. In fact, we can fix the corresponding twists $\mathcal{H}$ and $\hat{\mathcal{H}}$. The isomorphism classes over $(F, \hat F)$ in the given τ_1-equivalence class can be represented in the form $[u] := [\mathcal{H}, \hat{\mathcal{H}}, u]$ for varying u. By $\mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})$ we denote the set of these isomorphism classes.

We introduce an equivalence relation τ_2 on $\mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})$. Let $i\colon F_b \times \hat F_b \hookrightarrow F \times_B \hat F$ be the inclusion of the fiber over the base point $b \in B$. We declare $[u], [u'] \in \mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})$ to be τ_2-equivalent if and only if the class $w \in H^2(F \times_B \hat F, \mathbb{Z})$ which is determined by $u + w = u'$ satisfies

$$i^* w \in i^*(p^* H^2(F, \mathbb{Z}) + \hat p^* H^2(\hat F, \mathbb{Z})) \ . \tag{7.7}$$

Note that $[u_1] = [u_2] \in \mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})$ if and only if there are $a \in H^2(F, \mathbb{Z})$, $\hat a \in H^2(\hat F, \mathbb{Z})$ (representing automorphisms of $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively) such that $u_1 = p^* a + \hat p^* \hat a + u_2$. It follows that, although the class $w \in H^2(F \times_B \hat F, \mathbb{Z})$ is not determined by the classes $[u], [u'] \in \mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})$, the condition (7.7) is well defined, and it defines an equivalence relation.

Lemma 7.8 *The subgroup* $\ker(\pi^*) \cap \ker(\hat\pi^*) \subset H^3(B, \mathbb{Z})$ *preserves* τ_2*, acts transitively on* $\mathtt{Triple}(\mathcal{H}, \hat{\mathcal{H}})/\tau_2$*, and the isotropy subgroup of each* τ_2*-equivalence class is*

$$\mathtt{im}({}^\pi d_2^{1,1}) \cap \mathtt{im}({}^{\hat\pi} d_2^{1,1}) \subset \ker(\pi^*) \cap \ker(\hat\pi^*) \subset H^3(B, \mathbb{Z}).$$

Proof. Choose $\delta \in \ker(\pi^*) \cap \ker(\hat\pi^*) \subset H^3(B, \mathbb{Z})$ and a twist $\mathcal{V}$ representing δ. In order to describe the action of δ on $\mathtt{Triple}(\mathcal{H}, \mathcal{H}')$, we choose trivializations $w\colon \pi^*\mathcal{V} \xrightarrow{\sim} 0$ and

$\hat{w}\colon \hat{\pi}^*\mathcal{V} \xrightarrow{\sim} 0$. If $[u] \in \mathtt{Triple}(\mathcal{H}, \mathcal{H}')$, then $[u]+\delta$ is represented by $[u \otimes \hat{p}^*\hat{w} \circ p^*w^{-1}]$. We introduce the cohomology class $v := \hat{p}^*\hat{w} \circ p^*w^{-1} \in H^2(F \times_B \hat{F}, \mathbb{Z})$.

Assume that $[u] \sim_{\tau_2} [u']$, i.e. the class $w \in H^2(F \times_B \hat{F}, \mathbb{Z})$ defined by $u' = u + w$ satisfies (7.7). Then we have $[u]+\delta \sim_{\tau_2} [u']+\delta$, since $[u]+\delta = [u+v]$, $[u']+\delta = [u'+v]$, and $u'+v = u+v+w$. This shows that the action of $\ker(\pi^*) \cap \ker(\hat{\pi}^*)$ preserves τ_2.

We now calculate the stabilizer of the τ_2-equivalence classes. We choose $\delta \in \ker(\pi^*) \cap \ker(\hat{\pi}^*)$ fixing $[u]$, and continue to use the above notation. The restriction of $\pi^*\mathcal{V}$ to the fiber F_b is canonically trivial. We can therefore consider $x := -w_{|F_b} \in H^2(F_b, \mathbb{Z})$. Similarly we have $\hat{x} := \hat{w}_{|\hat{F}_b} \in H^2(\hat{F}, \mathbb{Z})$. We can now write $i^*v = p_b^*(x) + \hat{p}_b^*(\hat{x})$. By Lemma A.1, ${}^{\pi}d_2^{0,2}(x) = 0$, ${}^{\hat{\pi}}d_2^{0,2}(\hat{x}) = 0$, ${}^{\pi}d_3^{0,2}(x) = -[\delta] \in H^3(B,\mathbb{Z})/\mathtt{im}({}^{\pi}d_2^{1,1})$, and ${}^{\hat{\pi}}d_3^{0,2}(\hat{x}) = [\delta] \in H^3(B,\mathbb{Z})/\mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$. Note that x is well-defined modulo restrictions of classes in $H^2(F,\mathbb{Z})$ to F_b. Similarly, $\hat{x}$ is well-defined modulo restrictions of $H^2(\hat{F},\mathbb{Z})$ to $\hat{F}_b$. Therefore the condition that ${}^{\pi}d_3^{0,2}(x) = 0$ *and* ${}^{\hat{\pi}}d_3^{0,2}(\hat{x}) = 0$ is independent of choices. Furthermore, it is equivalent to $\delta \in \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$. This holds if and only if x is a restriction of a class in $H^2(F,\mathbb{Z})$ to F_b, and $\hat{x}$ is the restriction of a class in $H^2(\hat{F},\mathbb{Z})$ to $\hat{F}_b$.

We see that $\delta \in \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$ implies that $i^*v \in i^*p^*H^2(F,\mathbb{Z}) + i^*\hat{p}^*H^2(\hat{F},\mathbb{Z})$, and therefore $[u]+\delta \sim_{\tau_2} [u]$. Vice versa, if $i^*v \in i^*p^*H^2(F,\mathbb{Z}) + i^*\hat{p}^*H^2(\hat{F},\mathbb{Z})$, then we write $i^*v = i^*p^*y + i^*\hat{p}^*\hat{y}$ for $y \in H^2(F,\mathbb{Z})$ and $\hat{y} \in H^2(\hat{F},\mathbb{Z})$. We then have $p_b^*(x) + \hat{p}_b^*(\hat{x}) = p_b^*(y_{|F_b}) + \hat{p}_b^*(\hat{y}_{|\hat{F}_b})$. This implies that $x = y_{|F_b}$ and $\hat{x} = \hat{y}_{|\hat{F}_b}$. We now see that ${}^{\pi}d_3^{0,2}(x) = 0$ *and* ${}^{\hat{\pi}}d_3^{0,2}(\hat{x}) = 0$, and therefore $\delta \in \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$.

It remains to prove transitivity. Choose a second class $[u'] \in \mathtt{Triple}(\mathcal{H}, \mathcal{H}')$. Then we have $u' = u + h$ with $h \in H^2(F \times_B \hat{F}, \mathbb{Z})$ such that the $c := i^*h$ is of the form $c = p_b^*(a) + \hat{p}_b(\hat{a})$ for $(a, \hat{a}) \in H^2(F_b,\mathbb{Z}) \oplus H^2(\hat{F}_b,\mathbb{Z})$. Since c is in the image of i^* we have $0 = {}^{r}d_2^{0,2}(c) = {}^{r}d_2^{0,2}(p_b^*(a)) + {}^{r}d_2^{0,2}(\hat{p}_b^*(\hat{a}))$. Consequently, ${}^{\pi}d_2^{0,2}a = 0$ and ${}^{\hat{\pi}}d_2^{0,2}(\hat{a}) = 0$. Set $\delta := {}^{\hat{\pi}}d_3^{0,2}(\hat{a})$. Then $\delta = -{}^{\pi}d_3^{0,2}(a)$. It follows that $\delta \in \ker(\pi^*) \cap \ker(\hat{\pi}^*)$. By Lemma A.1 and similarly to the above considerations $[u]+\delta \sim_{\tau_2} [u']$. □

7.6 We now fix a class $[u_0] \in \mathtt{Triple}(\mathcal{H},\hat{\mathcal{H}})$ and let

$$U := \{[u] \in \mathtt{Triple}(\mathcal{H},\hat{\mathcal{H}}) | [u] \sim_{\tau_2} [u_0]\}$$

denote the corresponding τ_2-equivalence class.

Lemma 7.9 *The subgroup $\ker(i^*) \subset H^2(F \times_B \hat{F},\mathbb{Z})$ acts transitively on U, and the stabilizer of each element is given by*

$$p^* H^2(F,\mathbb{Z}) \cap \ker(i^*) + \hat{p}^* H^2(\hat{F},\mathbb{Z}) \cap \ker(i^*)$$

Alternatively, if

$$H_{\tau_2} := (i^*)^{-1} i^* \left(p^* H^2(F,\mathbb{Z}) + \hat{p}^* H^2(\hat{F},\mathbb{Z})\right) \subset H^2(F \times_B \hat{F},\mathbb{Z}) \ ,$$

then H_{τ_2} acts transitively on U with point stabilizers $p^* H^2(F,\mathbb{Z}) + \hat{p}^* H^2(\hat{F},\mathbb{Z})$.

Proof. Let $[u] \in U$ and $w \in \ker(i^*)$. Then w obviously satisfies (7.7), and therefore we have $[u+h] \sim_{\tau_2} [u]$. Hence $\ker(i^*)$ acts on U.

Let now $[u],[u'] \in U$. Then $u' = u+w$ and w satisfies (7.7). Let us write $i^*w = i^*p^*x + i^*\hat{p}^*\hat{x}$ for $x \in H^2(F,\mathbb{Z})$ and $\hat{x} \in H^2(\hat{F},\mathbb{Z})$. We define $w' := w - p^*x - \hat{p}^*\hat{x}$. Then we have $[u+w] = [u+w']$ on the one hand, and $i^*w' = 0$ on the other. It follows that $[u'] = [u+w']$. This shows that $\ker(i^*)$ acts transitively on U.

Assume that $w \in \ker(i^*)$ and $[u+w] = [u]$. Then we have $w = p^*x + \hat{p}^*\hat{x}$ for some $x \in H^2(F,\mathbb{Z})$ and $\hat{x} \in H^2(\hat{F},\mathbb{Z})$. From $0 = i^*w = i^*(p^*x + \hat{p}^*\hat{x})$ we conclude that $i^*p^*x = 0$ and $i^*\hat{p}^*\hat{x} = 0$. This shows the assertion about the point stabilizers.

In order to deduce the second assertion of the lemma we shall observe that the natural map $\ker(i^*) \to H_{\tau_2}$ induces an isomorphism

$$\frac{\ker(i^*)}{p^* H^2(F,\mathbb{Z}) \cap \ker(i^*) + \hat{p}^* H^2(\hat{F},\mathbb{Z}) \cap \ker(i^*)} \stackrel{\sim}{\to} \frac{H_{\tau_2}}{p^* H^2(F,\mathbb{Z}) + \hat{p}^* H^2(\hat{F},\mathbb{Z})} . \tag{7.10}$$

Injectivity is clear since $p^* H^2(F,\mathbb{Z}) \cap \ker(i^*) + \hat{p}^* H^2(\hat{F},\mathbb{Z}) \cap \ker(i^*) = (p^* H^2(F,\mathbb{Z}) + \hat{p}^* H^2(\hat{F},\mathbb{Z})) \cap \ker(i^*)$. In order to show surjectivity we consider $w \in H_{\tau_2}$. We write $i^*w = i^*p^*x + i^*\hat{p}^*\hat{x}$ for some $x \in H^2(F,\mathbb{Z})$ and $\hat{x} \in H^2(\hat{F},\mathbb{Z})$. Then $w' := w - p^*x - \hat{p}^*x$ represents the same class as w on the right hand side of (7.10), and it satisfies $i^*w' = 0$.$\Box$

7.7 We define a map

$$\mu\colon \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1}) \to H_{\tau_2}/(p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z})) \tag{7.11}$$

in the following way. Represent $a \in \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1}) \subset \ker(\pi^*) \cap \ker(\hat{\pi}^*) \subset H^3(B,\mathbb{Z})$ by a map $f\colon B \to K(\mathbb{Z},3)$. Choose a homotopy $H\colon [0,1]\times F \to K(\mathbb{Z},3)$ between the constant map and $f\circ\pi$, and a homotopy $\hat{H}\colon [1,2]\times\hat{F} \to K(\mathbb{Z},3)$ between $f\circ\hat{\pi}$ and the constant map. Note that the homotopy classes of these homotopies can be modified by concatenation with homotopies from the constant map to the constant map, i.e. by elements of $H^3(\Sigma F,\mathbb{Z}) \cong H^2(F,\mathbb{Z})$ or of $H^2(\hat{F},\mathbb{Z})$, respectively.

We can now pull back H and $\hat{H}$ to $[0,1]\times F\times_B\hat{F}$ and $[1,2]\times F\times_B\hat{F}$. By construction, these two pull-backs coincide at $\{1\}\times F\times_B\hat{F}$, and therefore they can be concatenated to a map $[0,2]\times F\times_B\hat{F} \to K(\mathbb{Z},3)$. This concatenation is constant at both ends of the interval $[0,2]$ and thus factors over the suspension $\Sigma((F\times_B\hat{F})_+)$. Therefore it represents an element in $H^3(\Sigma((F\times_B\hat{F})_+),\mathbb{Z}) \cong H^2(F\times_B\hat{F},\mathbb{Z})$. Its class $\mu(a) \in H^2(F\times_B\hat{F},\mathbb{Z})/(p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z}))$ is independent of the choice of the homotopies H and $\hat{H}$. We still have to check that $\mu(a)$ belongs to the subspace $H_{\tau_2}/(p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z}))$. This will follow from a universal construction of μ for such classes and is therefore postponed until we discuss this universal construction, compare (7.21).

Observe that this secondary operation actually makes sense for all elements in $\ker(\pi^*)\cap\ker(\hat{\pi}^*)$, provided that we allow arbitrary values in $H^2(F\times_B\hat{F},\mathbb{Z})/(p^*H^2(F,\mathbb{Z})+\hat{p}^*H^2(\hat{F},\mathbb{Z}))$.

7.8 Note that the groups $\mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$ and $H_{\tau_2}/(p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z}))$ act on U.

Lemma 7.12 *The map* $\mu\colon \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1}) \to H_{\tau_2}/(p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z}))$ *is compatible with the action of both groups on* U*.*

Proof. Choose $[u]\in U$ and $a \in \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$. As in 7.7 we represent a by a map $f: B \to K(\mathbb{Z},3)$. We choose a twist $\mathcal{K}$ on $K(\mathbb{Z},3)$ such that $[\mathcal{K}] \in H^3(K(\mathbb{Z},3),\mathbb{Z})$ is the canonical generator. The homotopies H and $\hat{H}$ (see 7.7) give rise to isomorphisms of twists $w: 0 \xrightarrow{\sim} \pi^*f^*\mathcal{K}$ and $\hat{w}: \hat{\pi}^*f^*\mathcal{K} \xrightarrow{\sim} 0$. We consider $\hat{w}\circ w^{-1} \in H^2(F\times\hat{F},\mathbb{Z})$

and observe that $[u] + a = [u + \hat{w} \circ w^{-1}]$. The Lemma now follows since by construction $\hat{w} \circ w^{-1}$ represents $\mu(a)$. □

7.9

Lemma 7.13 *The map μ is an isomorphism.*

By a combination of Lemma 7.13 with Lemma 7.12 and Lemma 7.9 we obtain:

Corollary 7.14 $\mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1})$ *acts freely and transitively on U.*

Together with Lemma 7.5 and Lemma 7.8, this finishes the proof of Proposition 7.4. □

7.10 We now prove Lemma 7.13. We have an isomorphism

$$\ker(i^*)/r^*H^2(B,\mathbb{Z}) \cong \ker({}^{r}d_2^{1,1}) \ . \tag{7.15}$$

Let $i_b \colon F_b \to F$ and $\hat{i}_b \colon \hat{F}_b \to \hat{F}$ be the inclusions of the fibers over the basepoint. Using isomorphisms similar to (7.15) for π and $\hat{\pi}$ we see that (7.15) induces an isomorphism

$$\frac{\ker(i^*)}{p^*(\ker(i_b^*)) + \hat{p}^*(\ker(\hat{i}_b)^*)} \xrightarrow{\cong} \frac{\ker({}^{r}d_2^{1,1})}{\ker({}^{\pi}d_2^{1,1}) \oplus \ker({}^{\hat{\pi}}d_2^{1,1})} \ .$$

To prove Lemma 7.13, we now consider the diagram

$$\begin{array}{ccc} \frac{\ker({}^{r}d_2^{1,1})}{\ker({}^{\pi}d_2^{1,1}) \oplus \ker({}^{\hat{\pi}}d_2^{1,1})} & \stackrel{{}^{\pi}d_2^{1,1}\circ \mathtt{pr}_1}{\longrightarrow} & \mathtt{im}({}^{\pi}d_2^{1,1}) \cap \mathtt{im}({}^{\hat{\pi}}d_2^{1,1}) \\ \cong\uparrow & & \mu\downarrow \\ \frac{\ker(i^*)}{p^*(\ker(i_b^*)) + \hat{p}^*(\ker(\hat{i}_b^*))} & \stackrel{\cong\alpha}{\longrightarrow} & \frac{H_{\tau_2}}{p^*H^2(F,\mathbb{Z}) + \hat{p}^*H^2(\hat{F},\mathbb{Z})} \end{array} \ . \tag{7.16}$$

Here, the upper horizontal map is induced by the projection $\mathtt{pr}_1 : {}^{r}E_2^{1,1} \cong {}^{\pi}E_2^{1,1} \oplus {}^{\hat{\pi}}E_2^{1,1} \to {}^{\pi}E_2^{1,1}$ composed with the differential. Since ${}^{r}d_2^{1,1} = {}^{\pi}d_2^{1,1} + {}^{\hat{\pi}}d_2^{1,1}$, on $\ker({}^{r}d_2^{1,1})$ this map indeed maps to the intersection of the two images and is an isomorphism. The map α is the isomorphism (7.10), where we note that $p^* \ker(i_b^*) = p^*H^2(F,\mathbb{Z}) \cap \ker(i^*)$ and $\hat{p}^* \ker(\hat{i}_b^*) = \hat{p}^*H^2(\hat{F},\mathbb{Z}) \cap \ker(i^*)$. Consequently, all maps in (7.16) apart from μ are isomorphisms, and Lemma 7.13 follows immediately from the following lemma.

Lemma 7.17 *The diagram* (7.16) *commutes (up to sign).*

Note that this reduces the proof of Proposition 7.4 to a question about the cohomology of fiber bundles, more precisely about a precise description of the secondary operation μ.

7.11 To study commutativity of the diagram, we can deal with one element in the lower left corner at a time. Since all the constructions are natural, it therefore suffices to study a universal situation and prove the assertion there. To do this, we first have to identify such a universal situation.

Given is a base space B together with two T^n-bundles $F, \hat{F}$, classified by their first Chern classes $c_1, \dots, c_n, \hat{c}_1, \dots, \hat{c}_n \in H^2(B, \mathbb{Z})$. Moreover, we have classes $a_1, \dots, a_n, \hat{a}_1, \dots, \hat{a}_n \in H^1(B, \mathbb{Z})$ such that

$$ {}^r d_2^{1,1}(\sum_i (a_i \otimes y_i - \hat{a}_i \otimes \hat{y}_i)) = \sum_i (a_i \cup c_i - \hat{a}_i \cup \hat{c}_i) = 0 \in H^3(B, \mathbb{Z}) \ . $$

7.12 We now describe the universal case for this kind of data. Let $t \colon K \to K(\mathbb{Z}, 2)^{2n} \times K(\mathbb{Z}, 1)^{2n}$ be the homotopy fiber of the map $K(\mathbb{Z}, 2)^{2n} \times K(\mathbb{Z}, 1)^{2n} \to K(\mathbb{Z}, 3)$ classified by $\sum_i x_i \cup a_i - \hat{x}_i \cup \hat{a}_i$, where x_i, $\hat{x}_i$, a_i, $\hat{a}_i$ are the canonical generators of the cohomology of the factors of the source space. The universal T^n-bundles classified by $\mathbf{c}_i$ and $\hat{\mathbf{c}}_i$ pull back to bundles $\pi^u \colon F^u \to K$ and $\hat{\pi}^u \colon \hat{F}^u \to K$, and $r^u \colon F^u \times_K \hat{F}^u \to K$ is the fiber product.

A straightforward calculation with the Leray-Serre spectral sequence of t shows that we have isomorphisms

$$ \begin{aligned} t^* : H^1(K(\mathbb{Z}, 1)^{2n}, \mathbb{Z}) &\xrightarrow{\sim} H^1(K, \mathbb{Z}) \\ t^* : H^2(K(\mathbb{Z}, 2)^{2n}, \mathbb{Z}) \oplus H^2(K(\mathbb{Z}, 1)^{2n}, \mathbb{Z}) &\xrightarrow{\sim} H^2(K, \mathbb{Z}) \\ \bar{t}^* : H^3(K(\mathbb{Z}, 2)^{2n} \times K(\mathbb{Z}, 1)^{2n}, \mathbb{Z}) / (\sum_i (a_i \cup x_i - \hat{a}_i \cup \hat{x}_i)\mathbb{Z}) &\xrightarrow{\sim} H^3(K, \mathbb{Z}) \end{aligned} \tag{7.18} $$

Continuing with the Leray-Serre spectral sequences of r^u, π^u and $\hat{\pi}^u$, we see that for $k = 2, 3$

$$ H^k(F^u \times_K \hat{F}^u, \mathbb{Z}) = \ker(i^*) \ , \quad H^k(F^u, \mathbb{Z}) = \ker(i_b^*) \ , \quad H^k(\hat{F}^u, \mathbb{Z}) = \ker(\hat{i_b}^*), \tag{7.19} $$

and that the projections

$$\ker(i^*)/r^*H^2(K,\mathbb{Z}) \xrightarrow{\cong} \ker({}^{r^u}d_2^{1,1}) \xrightarrow{\cong} \ker({}^{r^u}d_2^{1,1})/(\ker({}^{\pi^u}d_2^{1,1}) \oplus \ker({}^{\hat{\pi}^u}d_2^{1,1})) \cong \mathbb{Z} \quad (7.20)$$

all are isomorphisms.

7.13 In the universal situation we have $\ker(i^*) = H_{\tau_2} = H^2(F^u \times_K \hat{F}^u, \mathbb{Z})$, and consequently

$$\mathtt{im}(\mu) \subset H_{\tau_2}/(p^*H^2(F^u,\mathbb{Z}) + \hat{p}^*H^2(\hat{F}^u,\mathbb{Z})). \quad (7.21)$$

By naturality this implies that (7.21) holds in general. This adds the missing detail in the construction of μ of (7.11).

7.14 We choose the representative

$$g := \sum_i (t^*a_i \otimes y_i - t^*\hat{a}_i \otimes \hat{y}_i) \in {}^{r^u}E_2^{1,1}$$

of the generator (see (7.20)) of $\ker({}^{r^u}d_2^{1,1})/(\ker({}^{\pi^u}d_2^{1,1}) \oplus \ker({}^{\hat{\pi}^u}d_2^{1,1}))$, and let $\tilde{g} \in \ker(i^*)$ be a lift. Let $[\tilde{g}]$ be the class represented by $\tilde{g}$ in the lower left corner of (7.16). Note that

$${}^{\pi^u}d_2^{1,1} \circ \mathtt{pr}_1(g) = \sum_i t^*a_i \cup t^*x_i = \sum_i t^*\hat{a}_i \cup t^*\hat{x}_i. \quad (7.22)$$

We must show that

$$\mu(\sum_i t^*a_i \cup t^*c_i) = \alpha([\tilde{g}]) \ .$$

Since the relevant group is cyclic, it suffices to show the equality of the leading parts in ${}^{r_u}E_\infty^{1,1} = \ker({}^{r_u}d_2^{1,1})$, i.e. that

$$\mu(\sum_i t^*a_i \cup t^*x_i)^{1,1} = \alpha([\tilde{g}])^{1,1} = \sum_i (t^*a_i \otimes y_i - t^*\hat{a}_i \otimes \hat{y}_i) \ . \quad (7.23)$$

7.15 We have $(\pi^u)^*t^*x_i = 0$ and $(\hat{\pi}^u)^*t^*\hat{x}_i = 0$. Thus we can choose a homotopy $[0,1] \times F^u \to K(\mathbb{Z},2)$ between the constant map and $(\pi^u)^*t^*x_i$, and a homotopy $[1,2] \times \hat{F}^u \to K(\mathbb{Z},2)$ between $(\hat{\pi}^u)^*t^*\hat{x}_i$ and the constant map. Since the transgression of the Chern class of an $U(1)$-bundle is represented by a generator of the first cohomology of the fiber (modulo the image of the first cohomology of the total space) we can choose these homotopies in such a way that their restrictions to $[0,1]\times F^u_b$ and $[1,2]\times \hat{F}^u_b$ (which are necessarily constant at both ends of the intervals) are the suspensions of the generators y_i or $\hat{y}_i$, respectively.

We now take the product of the above homotopies with the corresponding maps $(\pi^u)^*t^*a_i$ and $(\hat{\pi}^u)^*t^*\hat{a}_i$ respectively, and then the product over the index i in order to get homotopies

$$h_1\colon [0,1]\times F^u \to \prod_{i=1}^n (K(\mathbb{Z},2)\times K(\mathbb{Z},1))\ ,\quad \hat{h}_1\colon [1,2]\times \hat{F}^u \to \prod_{i=1}^n (K(\mathbb{Z},2)\times K(\mathbb{Z},1)).$$

We finally compose with the map $(K(\mathbb{Z},2)\times K(\mathbb{Z},1))^n \to K(\mathbb{Z},3)$ representing $\sum_i x_i\cup a_i$ to get the required homotopies $H_1\colon [0,1]\times F^u \to K(\mathbb{Z},3)$ and $\hat{H}_1\colon [1,2]\times \hat{F}^u \to K(\mathbb{Z},3)$ in the construction of $\mu(\sum t^*a_i\cup t^*x_i)$.

7.16 We consider the lifting problem

$$\begin{tikzcd} & K \arrow[d, "t"] \\ K(\mathbb{Z},1) \arrow[ur, "\alpha_1"] \arrow[r, "w"] & K(\mathbb{Z},2)^{2n}\times K(\mathbb{Z},1)^{2n} \end{tikzcd}\ ,$$

where w is the inclusion of the first $K(\mathbb{Z},1)$-factor —defined using the base points of the remaining factors. Since $w^*(\sum_i a_i\cup x_i - \hat{a}_i\cup\hat{x}_i)=0$ a lift α_1 exists.

Since $\alpha_1^*t^*x_i$ and $\alpha_1^*t^*\hat{x}_i$ are constant it follows that the bundles $\alpha_1^*F^u$ and $\alpha_1^*\hat{F}^u$ are trivialized. Since we can choose the map $\sum x_i\cup a_i\colon (K(\mathbb{Z},2)\times K(\mathbb{Z},1))^n \to K(\mathbb{Z},3)$ to factorize over the product of smash products $\prod(K(\mathbb{Z},2)\wedge K(\mathbb{Z},1))^n$ we see that the pull-back of H_1 along $[0,1]\times\alpha_1^*F^u\to[0,1]\times F^u$ has a factorization over the suspension $\Sigma(\alpha_1^*F^u_+)\to K(\mathbb{Z},3)$. It represents the suspension of the class $a\cup y_1\in H^2(\alpha_1^*F^u,\mathbb{Z})\cong H^2(K(\mathbb{Z},1)\times T^n,\mathbb{Z})$ since t^*a_i pulls back to $\delta_{i1}a_1$, and the null homotopy for $(\pi^u)^*t^*x_1$ gives the suspension of y_1 in the fiber by our choices above. In the same way the corresponding pull-back of $\hat{H}_1$ has a factorization $\Sigma(\alpha_1^*\hat{F}^u_+)\to K(\mathbb{Z},3)$ which in this case is actually constant and thus represents the suspension of $0\in H^2(\alpha_1^*\hat{F}^u,\mathbb{Z})$.

7.17 Let $A_1:\alpha_1^*(F^u\times_K\hat{F}^u)\to F^u\times_K\hat{F}^u$ be the induced map over α_1. We let A_1^* denote the induced map on the Leray-Serre spectral sequences. Then we have by the above discussion that $A_1^*\mu(\sum t^*a_i\cup x_i)^{1,1} = a\otimes y_1\in {}^{\alpha_1^*r_u}E^{1,1}_\infty$. On the other hand $A_1^*(\sum_i(t^*a_i\otimes y_i - t^*\hat{a}_i\otimes\hat{y}_i)) = a\otimes y_1$. This shows (7.23), since it is an equality in a cyclic group . □

This finishes the proof of Lemma 7.17 and therefore of Proposition 7.4. □

7.18 We consider the isomorphism class $[x_{n,univ}] \in \mathtt{Triple}(\mathbf{R}_n)$ of the universal T-duality triple $((\mathbf{F}_n, \mathcal{H}_n), (\hat{\mathbf{F}}_n, \hat{\mathcal{H}}_n), \mathbf{u}_n)$ (see Theorem 4.6). Furthermore, we consider the functor $B \mapsto P_n(B) := [B, \mathbf{R}_n]$ classified by $\mathbf{R}_n$. The element $[x_{n,univ}] \in \mathtt{Triple}_n(\mathbf{R}_n)$ induces a natural transformation of functors $\Psi_B : P_n(B) \to \mathtt{Triple}_n(B)$ which maps $[f] \in P_n(B)$ to $\Psi_B(f) := f^*[x_{n,univ}]$.

Theorem 7.24 *The natural transformation $\Psi : P_n \to \mathtt{Triple}_n$ is an isomorphism.*

In other words, $\mathbf{R}_n$ is the classifying space for T-duality triples. The following is then a consequence of 4.1.

Corollary 7.25 *For each space B there is a natural action of the T-duality group G_n on the set of triples $\mathtt{Triple}_n(B)$.*

7.19 *Proof.* [of Theorem 7.24] Given a space B we must show that Ψ_B is a bijection of sets. Therefore we show:

Lemma 7.26 *Ψ_B is surjective.*

Lemma 7.27 *Ψ_B is injective.*

7.20 The homotopy fiber of $(\mathbf{c}, \hat{\mathbf{c}}) : \mathbf{R}_n \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$ is $K(\mathbb{Z}, 3)$ (see 3.2). By obstruction theory the set of homotopy classes of lifts in the diagram

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & f \nearrow & (\mathbf{c}, \hat{\mathbf{c}}) \downarrow \\ B & \stackrel{(c,\hat{c})}{\to} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array}$$

is a torsor over $H^3(B,\mathbb{Z})$. So given two such lifts f_0, f_1 we have a well-defined difference element $\delta(f_1,f_0)\in H^3(B,\mathbb{Z})$.

Let $F := c^*U^n$ and $\hat{F} := \hat{c}^*U^n$ be the pull-backs of the universal T^n-bundle over $K(\mathbb{Z}^n,2)$. Note further that by construction $\mathbf{F}_n = \mathbf{c}^*U^n$ and $\hat{\mathbf{F}}_n = \hat{\mathbf{c}}^*U^n$. Since f lifts $(c,\hat{c})$ we have natural identifications $f^*\mathbf{F}_n \cong F$ and $f^*\hat{\mathbf{F}}_n \cong \hat{F}$. We can thus consider $f^*[x_{n,univ}] \in \mathtt{Triple}_n^{(F,\hat{F})}(B)$ in a natural way. Recall the action of $H^3(B,\mathbb{Z})$ on $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ introduced in 7.3.

Lemma 7.28 *We have $f_1^*[x_{n,univ}] = f_0^*[x_{n,univ}] + \delta(f_1,f_0)$.*

Proof. We can choose a model of $K(\mathbb{Z},3)$ which is an abelian group. Furthermore, we choose a model of the map $(\mathbf{c},\hat{\mathbf{c}}) : \mathbf{R}_n \to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$ which is a $K(\mathbb{Z},3)$-principal bundle. Let $a : \mathbf{R}_n \times K(\mathbb{Z},3) \to \mathbf{R}_n$ be the right-action. Let $\mathbf{z}\in H^3(K(\mathbb{Z},3),\mathbb{Z})$ be a generator, and let $\mathcal{W}$ be a twist in the corresponding isomorphism class. We claim that

$$a^*x_{n,univ} \cong \mathtt{pr}_1^*x_{n,univ} + \mathtt{pr}_2^*\mathcal{W}. \tag{7.29}$$

We have two maps $\mathtt{pr}, m : \mathbf{F}_n \times K(\mathbb{Z},3) \to \mathbf{F}_n$, namely the projection and the map induced by the right-action $a : \mathbf{R}_n \times K(\mathbb{Z},3)\to\mathbf{R}_n$. By the Künneth formula and Lemma 3.5 we have $H^3(\mathbf{F}_n\times K(\mathbb{Z},3),\mathbb{Z}) \cong \mathbb{Z}\mathtt{pr}^*\mathbf{h}_n \oplus \mathbb{Z}\mathtt{pr}_2^*\mathbf{z}$. Then $m^*\mathbf{h}_n = a\mathtt{pr}^*\mathbf{h}_n + b\mathtt{pr}_2^*\mathbf{z}$ with $a,b\in\mathbb{Z}$ to be determined. By restriction to $\mathbf{F}_n\times\{1\}\subset \mathbf{F}_n\times K(\mathbb{Z},3)$ we see that $a=1$. In order to calculate b we restrict the torus bundle to a fiber of $(\mathbf{c},\hat{\mathbf{c}})$. We identify this restriction with $T^n\times K(\mathbb{Z},3)$. By Lemma 3.6 we know that the restriction of $\mathbf{h}_n$ is $1\times\mathbf{z}$ (after choosing the appropriate sign of the generator $\mathbf{z}$). It follows that the restriction of $m^*\mathbf{h}_n$ is equal to $1\times\mathbf{z}\times 1 + 1\times 1\times b\mathbf{z}\in H^3(T^n\times K(\mathbb{Z},3)\times K(\mathbb{Z},3),\mathbb{Z})$. Let $\mu : K(\mathbb{Z},3)\times K(\mathbb{Z},3)\to K(\mathbb{Z},3)$ be the multiplication. Then we have $\mu^*\mathbf{z} = \mathbf{z}\times 1 + 1\times\mathbf{z}$. By a comparison of these two formulas we conclude that $b=1$.

We now know that we have an isomorphism $m^*\mathcal{H}_n \cong \mathtt{pr}^*\mathcal{H}_n \otimes \mathtt{pr}_2^*\mathcal{W}$. In a similar manner we have a map $\hat{m} : \hat{\mathbf{F}}_n\times K(\mathbb{Z},3)\to\hat{\mathbf{F}}_n$ and an isomorphism $\hat{m}^*\hat{\mathcal{H}}_n \cong \hat{\mathtt{pr}}^*\hat{\mathcal{H}}_n\otimes\hat{\mathtt{pr}}_2^*\mathcal{W}$. Note that $H^2(\mathbf{F}_n\times_{\mathbf{R}_n}\mathbf{F}_n\times K(\mathbb{Z},3)\times K(\mathbb{Z},3),\mathbb{Z})\cong\{0\}$. Therefore the identification of the isomorphism classes of the twists already implies (7.29).

We now consider maps $f_0, f_1 : B\to\mathbf{R}_n$. Then we can write $f_1 = a\circ(f_0,g)$, where $g : B \to$

$K(\mathbb{Z},3)$ represents $\delta(f_1,f_0)\in H^3(B,\mathbb{Z})$. It follows that $f_1^*x_{n,univ}\cong(f_0,g)^*a^*x_{n,univ}\cong(f_0,g)^*(\mathtt{pr}^*x_{n,univ}+\mathtt{pr}_2^*\mathcal{W})\cong f_0^*x_{n,univ}+g^*\mathcal{W}$. Therefore $f_1^*[x_{x,univ}]=f_0^*[x_{n,univ}]+\delta(f_1,f_0)$. $\square$

7.21 Proof. [of Lemma 7.26]. It suffices to show the Lemma under the assumption that B is connected. Let $((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ represent an isomorphism class $x\in\mathtt{Triple}_n(B)$. The bundles F and $\hat{F}$ give rise to classifying maps $c:B\to K(\mathbb{Z}^n,2)$ and $\hat{c}:B\to K(\mathbb{Z}^n,2)$. We thus have a map $(c,\hat{c}):B\to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$.

Let $c_i,\hat{c}_i$ be the components of $c,\hat{c}$. It follows from the structure of $\mathcal{H}$ that

$$0={}^\pi d_2^{2,1}(\sum_{i=1}^n y_i\otimes\hat{c}_i)=\sum_{i=1}^n c_i\cup\hat{c}_i\ .$$

This implies by the definition 3.1 of $\mathbf{R}_n$ the existence of a lift f in the diagram

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & f\nearrow & (\mathbf{c},\hat{\mathbf{c}})\downarrow \\ B & \stackrel{(c,\hat{c})}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)\end{array}\quad .$$

As already noticed in 7.20, the set of homotopy classes of such lifts is a torsor over $H^3(B,\mathbb{Z})$. Now $x':=f^*x_{n,univ}=:((F',\mathcal{H}'),(\hat{F}',\hat{\mathcal{H}}'),u')$. By construction we can assume that $F'=F$ and $\hat{F}'=\hat{F}$ for all such lifts. We therefore consider $[x],[x']\in\mathtt{Triple}_n^{(F,\hat{F})}(B)$. By Proposition 7.4 there exists a class $\alpha\in H^3(B,\mathbb{Z})$ such that $[x]=[x']+\alpha$. By Lemma 7.28, if we replace f by $f+\alpha$, then $[x]=[x']$. $\square$

7.22 Proof. [of Lemma 7.27] Let $f_0,f_1:B\to\mathbf{R}_n$ be given such that $f_0^*x_{n,univ}$ and $f_1^*x_{n,univ}$ are isomorphic. We must show that f_0 and f_1 are homotopic. First we choose an isomorphism $f_0^*x_{n,univ}\cong f_1^*x_{n,univ}$. Let $\Psi:f_0^*\mathbf{F}_n\to f_1^*\mathbf{F}_n$ and $\hat{\Psi}:f_0^*\hat{\mathbf{F}}_n\to f_1^*\hat{\mathbf{F}}_n$ denote the underlying isomorphisms of T^n-bundles. Their existence implies that the compositions $(\mathbf{c},\hat{\mathbf{c}})\circ f_0$ and $(\mathbf{c},\hat{\mathbf{c}})\circ f_1$ are homotopic. We choose such a homotopy H and consider the diagram

$$\begin{array}{ccccc}[0,1]\times f_0^*\mathbf{F}_n\times_B f_0^*\hat{\mathbf{F}}_n & \stackrel{\Phi}{\to} & H^*(U^n\times U^n) & \to & U^n\times U^n\\ \downarrow & & \downarrow & & \downarrow\\ [0,1]\times B & \stackrel{\mathtt{id}}{\to} & [0,1]\times B & \stackrel{H}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)\end{array}\quad . \tag{7.30}$$

The bundle isomorphism Φ is uniquely determined up to homotopy by the property that its restriction to $\{0\} \times B$ is the identity. Let $\Phi_1 : f_0^*\mathbf{F}_n \times_B f_0^*\hat{\mathbf{F}}_n \to f_1^*\mathbf{F}_n \times_B f_1^*\hat{\mathbf{F}}_n$ be the restriction of Φ to $\{1\} \times B$. Its homotopy class depends on the choice of H in the following way.

7.23 The homotopy H can be concatenated with a map $g : [0,1] \times B \to \Sigma(B_+) \to K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)$. Homotopy classes of such maps are classified by $H^1(B, \mathbb{Z}^n) \times H^1(B, \mathbb{Z}^n)$. Since T^n has the homotopy type of $K(\mathbb{Z}^n, 1)$ elements in this cohomology group can be represented by maps $B \to T^n \times T^n$, and these maps act by automorphisms on $f_0^*\mathbf{F}_n \times_B f_0^*\hat{\mathbf{F}}_n$. Let H' be obtained from H by concatenation with g, and let $\tilde{g} : B \to T^n \times T^n$ represent the corresponding homotopy class. As a consequence of the discussion in A.4 the homotopy class of the evaluation Φ'_1 (Φ' is defined by (7.30) for H' in the place of H) is obtained from Φ_1 by composition with g. We see that we can arrange H and the choice of Φ such that $\Phi_1 = (\Psi, \hat{\Psi})$. For the remainder of the proof we adopt this choice.

7.24 We choose a lift $\tilde{H}$ in the diagram

$$\begin{array}{ccc} B & \stackrel{f_0}{\to} & \mathbf{R}_n \\ b \mapsto (0,b) \downarrow & \tilde{H} \nearrow & (\mathbf{c}, \hat{\mathbf{c}}) \downarrow \\ I \times B & \stackrel{H}{\to} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} ,$$

and we let $\tilde{f}$ be the evaluation of $\tilde{H}$ at $\{1\} \times B$. Then $\tilde{f}$ and f_1 are two maps lifting $(\mathbf{c}, \hat{\mathbf{c}}) \circ f_1$. Furthermore, f_0 is homotopic to $\tilde{f}$. By Lemma 7.1 the pull-backs $f_0^* x_{n,univ}$ and $\tilde{f}^* x_{univ}$ are isomorphic. Note that the underlying T^n-bundles of $\tilde{f}^* x_{n,univ}$ are canonically isomorphic to $f_1^*\mathbf{F}_n$ and $f_1^*\hat{\mathbf{F}}_n$. Actually, by the choice of H above, the proof of 7.1 gives us an isomorphism $f_0^* x_{n,univ} \cong \tilde{f}^* x_{n,univ}$ such that the underlying bundle isomorphisms are Ψ and $\hat{\Psi}$. The composition

$$f_1^* x_{n,univ} \cong f_0^* x_{n,univ} \cong \tilde{f}^* x_{n,univ}$$

is therefore an isomorphism over $(f_1^*\mathbf{F}_n, f_1^*\hat{\mathbf{F}}_n)$. We conclude that $[f_1^* x_{n,univ}] = [\tilde{f}^* x_{n,univ}]$ holds in $\mathtt{Triple}_n^{(f_1^*\mathbf{F}_n, f_1^*\hat{\mathbf{F}}_n)}(B)$.

The maps f_1 and $\tilde{f}$ are lifts in the diagram

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & \nearrow & (\mathbf{c},\hat{\mathbf{c}}) \downarrow \\ B & \stackrel{(\mathbf{c},\hat{\mathbf{c}})\circ f_1}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2) \end{array} .$$

Their difference as homotopy classes of lifts is measured by $\delta(f_1,\tilde{f})\in H^3(B,\mathbb{Z})$. But by Proposition 7.4 and the equality $[f_1^* x_{n,univ}] = [\tilde{f}^* x_{n,univ}]$ we have $\delta(f_1,\tilde{f})=0$. Now f_0 is homotopic to $\tilde{f}$, and $\tilde{f}$ is homotopic to f_1 (even as a lift), so that f_0 is homotopic to f_1. $\square$

7.25 We fix classifying maps $(c,\hat{c}) : B \to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$. We consider a lift f in the diagram

$$\begin{array}{ccc} & & \mathbf{R}_n \\ & f \nearrow & (\mathbf{c},\hat{\mathbf{c}}) \downarrow \\ B & \stackrel{(c,\hat{c})}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2) \end{array} .$$

Let $x := f^* x_{n,univ}$ and write $x = ((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$. Let furthermore $\psi : F \to F$ and $\hat{\psi} : \hat{F} \to \hat{F}$ be automorphisms of T^n-bundles. We will use the notation introduced in 2.13

Proposition 7.31 *In* $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ *we have the following identity:*

$$[x^{(\psi,\hat{\psi})}] = [x] + \hat{c}\cup[\psi] + c\cup[\hat{\psi}] \ .$$

Proof. We have $\Omega K(\mathbb{Z}^n,2)\cong K(\mathbb{Z}^n,1)$. As in 7.23 we can concatenate the constant homotopy from $(c,\hat{c})$ to $(c,\hat{c})$ by a map corresponding to the class $([\psi],[\hat{\psi}])\in H^1(B,\mathbb{Z}^n)\times H^1(B,\mathbb{Z}^n)$. In this way we obtain a new homotopy $H : [0,1]\times B \to K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)$ from $(c,\hat{c})$ to $(c,\hat{c})$. We again consider the diagram

$$\begin{array}{ccccc} [0,1]\times F\times_B \hat{F} & \stackrel{(\phi,\hat{\phi})}{\to} & H^*(U^n\times U^n) & \to & U^n\times U^n \\ \downarrow & & \downarrow & & \downarrow \\ [0,1]\times B & \stackrel{\mathtt{id}}{\to} & [0,1]\times B & \stackrel{H}{\to} & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2) \end{array} .$$

As explained in 7.23 we can choose ϕ and $\hat{\phi}$ such that their restrictions ϕ_1 and $\hat{\phi}_1$ to $\{1\} \times B$ coincide with ψ and $\hat{\psi}$. We now choose a lift $\tilde{H}$ in

$$\begin{array}{ccc} B & \stackrel{f}{\to} & \mathbf{R}_n \\ i_0 \downarrow & \tilde{H} \nearrow & (\mathbf{c}, \hat{\mathbf{c}}) \downarrow \\ [0,1] \times B & \stackrel{H}{\to} & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2) \end{array} ,$$

where $i_0 : B \cong \{0\} \times B \to [0,1] \times B$ is the inclusion. Let $f_1 := \tilde{H}_{|\{1\} \times B} : B \to \mathbf{R}_n$. Then we have by construction $f_1^* x_{n,univ} \cong x^{(\psi,\hat{\psi})}$. By Lemma 7.28 we have $f_1^*[x_{n,univ}] = f^*[x_{n,univ}] + \delta(f_1, f_0)$ in $\mathtt{Triple}_n^{(F,\hat{F})}(B)$. Therefore it remains to show that $\delta(f_1, f_0) = \hat{c} \cup [\psi] + c \cup [\hat{\psi}]$. We want to apply the result of A.4. The homotopy H gives rise to a map $H' : B \to L(K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2))$ which restricts to $(c, \hat{c})$ at $0 \in S^1$. We must consider the composition

$$B \stackrel{H'}{\to} L(K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)) \stackrel{Lq}{\to} LK(\mathbb{Z}, 4) \stackrel{\sim}{\to} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 3) \stackrel{\mathtt{pr}_2}{\to} K(\mathbb{Z}^n, 3) \ .$$

By (A.3) we know that $\delta(f_0, f_1)$ is given by the cohomology class which classifies this map. In order to calculate this class we consider the following general result.

7.26 Fix integers $k, m \geq 1$. A homotopy class $f \in [B, LK(\mathbb{Z}, k)]$ is characterized by a pair of cohomology classes $(f_0, f_1) \in H^k(B, \mathbb{Z}) \times H^{k-1}(B, \mathbb{Z})$. Let $q : K(\mathbb{Z}, k) \times K(\mathbb{Z}, m) \to K(\mathbb{Z}, k) \wedge K(\mathbb{Z}, m) \to K(\mathbb{Z}, k+m)$ be the composition of the canonical projection and the cup product. It induces a map $Lq : LK(\mathbb{Z}, k) \times LK(\mathbb{Z}, m) \to LK(\mathbb{Z}, k+m)$. We let $q' : LK(\mathbb{Z}, k) \times LK(\mathbb{Z}, m) \to K(\mathbb{Z}, k+m-1)$ be the composition of Lq with the projection $LK(\mathbb{Z}, k+m) \cong K(\mathbb{Z}, k+m) \times \Omega K(\mathbb{Z}, k+m) \to \Omega K(\mathbb{Z}, k+m) \cong K(\mathbb{Z}, k+m-1)$.

Lemma 7.32 *We have $q'_*(f, g) = f_0 \cup g_1 + f_1 \cup g_0$.*

Proof. We consider (adjoint) representatives $f_1 : \Sigma B_+ \to K(\mathbb{Z}, k)$ and $g_1 : \Sigma B_+ \to K(\mathbb{Z}, m)$ such that f_1 is constant on $[0, 1/2] \times B$ and g_1 is constant on $[1/2, 1] \times B$. Then we see that $q'_*(f, g) = q_*(f_0, g_1) + q_*(f_1, g_0)$, where $+$ indicates concatenation of loops. This follows because of commutativity in the diagram

$$\begin{array}{ccccc} \Omega K(\mathbb{Z}, k-1) \wedge K(\mathbb{Z}, l) & \longrightarrow & \Omega(K(\mathbb{Z}, k-1) \wedge K(\mathbb{Z}, l)) & \xrightarrow{\Omega\cup} & \Omega K(\mathbb{Z}, k+l-1) \\ \downarrow \sim & & & & \downarrow \sim \\ K(\mathbb{Z}, k) \wedge K(\mathbb{Z}, l) & & \xrightarrow{\cup} & & K(\mathbb{Z}, k+l) \end{array}$$

where the first arrow in the upper row comes from the inclusion $K(\mathbb{Z},l)\to\Omega K(\mathbb{Z},l)$ as constant loops. The Lemma now follows by taking the adjoint. □

7.27 Let us finish the proof of Proposition 7.31. Recall that $\mathbf{R}_n$ is the homotopy fiber of the map $q: K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2)\to K(\mathbb{Z},4)$ classified by $\sum_{i=1}^n x_i\cup\hat{x}_i$. We consider the composition

$$L(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2))\stackrel{Lq}{\to}LK(\mathbb{Z},4)\cong K(\mathbb{Z},4)\times K(\mathbb{Z},3)\stackrel{\mathtt{pr}_2}{\to}K(\mathbb{Z},3)\ .$$

Furthermore note that

$$L(K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2))\cong K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,1)\times K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,1).$$

We let $x_i,a_i,\hat{x}_i,\hat{a}_i$ denote the corresponding canonical generators of the cohomology of the factors. Furthermore, we let $\mathbf{z}\in H^3(K(\mathbb{Z},3),\mathbb{Z})$ be the generator. We apply Lemma 7.32 to the summands of q and obtain

$$Lq^*\circ\mathtt{pr}_2^*(\mathbf{z})=\sum_{i=1}^n(x_i\cup\hat{a}_i+\hat{x}_i\cup a_i)\ .$$

We now see that $(H')^*x_i=c_i$, $(H')^*\hat{x}_i=\hat{c}_i$, and using the discussion in A.4, $(H')^*a_i=[\psi]_i$, $(H')^*\hat{a}_i=[\hat{\psi}]_i$. We get

$$(H')^*\sum_{i=1}^n(c_i\cup\hat{a}_i+\hat{c}_i\cup a_i)=c\cup[\hat{\psi}]+\hat{c}\cup[\psi]\ .$$

□

7.28 We fix a pair (F,h) over a space B. Then we consider n-dimensional T-duality triples $x=((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)$ which extend (F,h) (see Definition 2.21).

Definition 7.33 *An isomorphism of extensions*

$$x:=((F,\mathcal{H}),(\hat{F},\hat{\mathcal{H}}),u)\ ,\quad x':=((F,\mathcal{H}'),(\hat{F}',\hat{\mathcal{H}}'),u')$$

is an isomorphism of T-duality triples $x\cong x'$ (see 4.5) which induces the identity on F. The set of such isomorphism classes is denoted $\mathtt{Ext}(F,\mathcal{H})$.

7.29 *Proof.* [of Theorem 2.24] The first assertion of Theorem 2.24 is exactly Theorem 5.7. We continue with the existence statement of (2). Let $x := ((F,\mathcal{H}),(\hat{\mathcal{F}},\hat{\mathcal{H}}),u)$ represent a class in $\mathtt{Ext}(F,h)$. Let $B \in \mathtt{Mat}(n,n,\mathbb{Z})$ be antisymmetric and set $\hat{c}_i' := \hat{c}_i(x) + \sum_{j=1}^n B_{i,j}c_j(x)$. We compute $\sum_{i=1}^n c_i(x)\cup\hat{c}_i' = \sum_{i=1}^n c_i(x)\cup\hat{c}_i(x)+\sum_{j,i=1}^n B_{i,j}c_i(x)\cup c_j(x) = 0$. This implies the existence of a lift f' in

$$\begin{tikzcd} & \mathbf{R}_n \arrow[d, "(\mathbf{c},\hat{\mathbf{c}})"] \\ B \arrow[ur, "f'"] \arrow[r, "(c,\hat{c}')"] & K(\mathbb{Z}^n,2)\times K(\mathbb{Z}^n,2) \end{tikzcd} .$$

Then $\{x'\} := \{(f')^* x_{n,univ}\} \in \mathtt{Ext}(F,h)$ has the required properties. The remaining part of (2) was shown in Subsection 2.18.

We now show (3). We first consider the set of classes of triples $[x'] \in \mathtt{Triple}_n^{(F,\hat{F})}(B)$ which extend (F,h). It follows from Proposition 7.4 that $\ker(\pi^*)$ acts freely and transitively on this set. It follows that $\ker(\pi^*)$ acts transitively on any subset of $\mathtt{Ext}(F,\mathcal{H})$ with fixed $c(\hat{F})$. But note that the equivalence relation in $\mathtt{Triple}_n^{(F,\hat{F})}(B)$ is stronger than in $\mathtt{Ext}(F,h)$ since isomorphisms must induce the identity on $\hat{F}$. In $\mathtt{Ext}(F,h)$ we admit non-trivial bundle automorphism of $\hat{F}$. In view of Proposition 7.31 we see that the quotient $\ker(\pi^*)/\mathtt{im}(C)$ acts freely on the set of isomorphisms classes of extensions of (F,h) with prescribed isomorphism class of $\hat{F}$. $\square$

A Twists, spectral sequences and other conventions

A.1 Twists

We start with a description of twists. In the literature are various models for twists. Therefore we first describe the common core of these models and in particular the properties which we use in the present paper. Then we exhibit two of these models explicitly.

In each case one considers a pre-sheaf of monoidal groupoids $B \mapsto T(B)$ on the category of spaces. The objects of $T(B)$ are called twists. The unit of the monoidal structure is called the trivial twist and denoted by 0. If $f : A \to B$ is a map of spaces, then there is monoidal functor $f^* : T(B) \to T(A)$. Functoriality is implemented by natural transformations. We refer to [6, Section 3.1] for more details. The following three requirements provide the coupling to topology.

(1) We require that there is a natural monoidal transformation $T(B) \to H^3(B, \mathbb{Z})$, $\mathcal{H} \mapsto [\mathcal{H}]$, (the group $H^3(B, \mathbb{Z})$ is considered as a monoidal category which has only identity morphisms) which classifies the isomorphism classes of $T(B)$ for each B.

(2) If $\mathcal{H}, \mathcal{H}' \in T(B)$ are equivalent objects, then we require that $\mathtt{Hom}_{T(B)}(\mathcal{H}, \mathcal{H}')$ is a $H^2(B, \mathbb{Z})$-torsor such that the composition with fixed morphisms gives isomorphisms of torsors. Furthermore, we require that the torsor structure is compatible with the pull-back. Note that we have natural bijections $\mathtt{Hom}(\mathcal{H}, \mathcal{H}) \xrightarrow{\sim} H^2(B, \mathbb{Z})$ which map compositions to sums. In order to simplify the notation we will frequently identify automorphisms of twists and cohomology classes. The map $\mathtt{Hom}_{T(B)}(\mathcal{H}_0, \mathcal{H}'_0) \times \mathtt{Hom}_{T(B)}(\mathcal{H}_1, \mathcal{H}'_1) \to \mathtt{Hom}_{T(B)}(\mathcal{H}_0 \otimes \mathcal{H}_1, \mathcal{H}'_0 \otimes \mathcal{H}'_1)$, $(v, w) \mapsto v \otimes w$ is bilinear.

We now discuss two explicite realizations.

(1) Let K be the algebra of compact operators on a separable complex Hilbert space. The group of automorphisms of K is the projective unitary group PU. As a topological group we have $PU = U/U(1)$, where the topology of the unitary group of the Hilbert space is the strong topology. We define $T(B)$ to be the category of locally trivial bundles with fiber K such that the transition functions are continuous functions with values in PU. We further define $\mathtt{Hom}_{T(B)}(\mathcal{H}, \mathcal{H}')$ as the set of homotopy classes of algebra bundle isomorphisms. The monoidal structure is given by the fiberwise (completed) tensor product of bundles. In order to see that this preserves the category we use the fact that $K \otimes K \cong K$. Isomorphism classes of objects in $T(B)$ are classified by homotopy classes $[B, BPU]$. Since the classifying space BPU has the homotopy type $K(\mathbb{Z}, 3)$ we see that isomorphism classes in $T(B)$ are in one-to one correspondence with $H^3(B, \mathbb{Z})$. The group of automorphisms of a bundle $\mathcal{H} \in T(B)$ can be identified with the group of homotopy classes of maps $[B, PU]$ and is therefore in one-to-one correspondence to $H^2(B, \mathbb{Z})$.

This model of twists is very suitable for a quick definition of twisted K-theory. Assume that B is locally compact. Given a twist $\mathcal{H} \in T(B)$ we consider the C^*-algebra $C_0(B,\mathcal{H})$ of continuous sections of $\mathcal{H}$ vanishing at infinity. Then we can define $K(B,\mathcal{H}) := K(C_0(B,\mathcal{H}))$, where the right-hand side is K-theory of C^*-algebras.

(2) In our second model we fix an h-space model of $K(\mathbb{Z},3)$. Note that there exists in fact models which are topological abelian groups. A twist $\mathcal{H} \in T(B)$ is a map $\mathcal{H} : B \to K(\mathbb{Z},3)$. The monoidal structure is implemented by the h-space structure of $K(\mathbb{Z},3)$. The set $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ in this model is the set of homotopy classes of homotopies from $\mathcal{H}$ to $\mathcal{H}'$. It is obvious from the definition that isomorphism classes in $T(B)$ are classified by $H^3(B,\mathbb{Z})$. Furthermore, since $\Omega K(\mathbb{Z},3) \cong K(\mathbb{Z},2)$ we see that $\mathtt{Hom}_{T(B)}(\mathcal{H},\mathcal{H}')$ is empty or a torsor over $H^2(B,\mathbb{Z})$.

A.2 Spectral sequences

The cohomology of the total space F of a fiber bundle $\pi : F \to B$ has a decreasing filtration

$$0 \subset \mathcal{F}^n H^n(F,\mathbb{Z}) \subset \mathcal{F}^{n-1} H^n(F,\mathbb{Z}) \subset \cdots \subset \mathcal{F}^1 H^n(F,\mathbb{Z}) \subset \mathcal{F}^0 H^n(F,\mathbb{Z}) = H^n(F,\mathbb{Z})$$

such that $\mathcal{F}^n H^n(F,\mathbb{Z}) = \pi^*(H^n(B,\mathbb{Z}))$. By definition, $x \in \mathcal{F}^k H^n(F,\mathbb{Z})$ if for any $k-1$-dimensional CW-complex X and map $\phi : X \to B$ the condition $\Phi^*(x) = 0$ is satisfied. Here $\Phi : \phi^* F \to F$ is the induced map. The associated graded group is calculated by the Leray-Serre spectral sequence $({}^\pi E_r^{s,t}, {}^\pi d_r^{s,t})$. If $x \in \mathcal{F}^k H^n(F,Z)$, then we let $x^{k,n-k} \in {}^\pi E_\infty^{k,n-k}$ denote the leading part.

Let us assume that B is connected and choose a base point $b \in B$. Let $F_b := \pi^{-1}(b)$ be the fiber over b. We further assume that $\pi_1(B,b)$ acts trivially on $H^*(F_b,\mathbb{Z})$. In the present paper this assumption is always satisfied. Then we have isomorphisms ${}^\pi E_2^{p,q} \cong H^p(B,H^q(F_b,\mathbb{Z}))$. For various calculations we will employ naturality and multiplicativity of the spectral sequence.

Let us now fix some notation in the case that $\pi : F \to B$ is a T^n-principal bundle. We fix a generator of $H^1(U(1),\mathbb{Z})$. Since $T^n := U(1)^n$ this provides a natural set of generators of $H^1(T^n,\mathbb{Z})$. If we fix a base point in F_b, then using the right action we obtain a homeomorphism $F_b \cong T^n$. We let $y_1,\dots,y_n \in H^1(F_b,\mathbb{Z})$ denote the natural generators obtained in this way. They do not depend on the choice of the base point in F_b. If we consider a bundle $\hat{\pi} : \hat{F} \to B$, then we will write $\hat{y}_1,\dots,\hat{y}_n$ for the corresponding set of generators. Finally, the notation $y_1,\dots,y_n,\hat{y}_1,\dots,y_n$ for the generators of $H^1(F_b\times\hat{F}_b,\mathbb{Z})$ is self-explaining.

Note that $H^*(F_b,\mathbb{Z})$ is a free $\mathbb{Z}$-module. Therefore we have an isomorphism

$$^{\pi}E_2^{p,q} \cong H^p(B,H^q(F_b,\mathbb{Z})) \cong H^q(F_b,\mathbb{Z})\otimes H^p(B,\mathbb{Z}) \ .$$

Let $c_1,\dots,c_n \in H^2(B,\mathbb{Z})$ be the Chern classes of the T^n-bundle $\pi : F \to B$. Then we know that ${}^{\pi}d_2^{0,1}(y_i) = c_i$. By multiplicativity this leads to a complete calculation of ${}^{\pi}d_2$.

A.3 Let $T \xrightarrow{i} F \xrightarrow{\pi} B$ be a fibration (of pointed spaces) with base B and fiber T. Choose $\delta \in H^3(B,\mathbb{Z})$ such that $\pi^*\delta = 0$. We choose a twist $\mathcal{H}$ over F such that $[\mathcal{H}] = \delta$. Furthermore, we choose a trivialization $w\colon \pi^*\mathcal{H} \xrightarrow{\sim} 0$. Since $\pi\circ i$ is the constant map to the base point we have a natural isomorphism $0 \xrightarrow{can} \pi^*\mathcal{H}$. We obtain the automorphism $w_b : 0 \xrightarrow{can} \pi^*\mathcal{H} \xrightarrow{i^*w} 0$ of the trivial twist. We will consider $w_b \in H^2(T,\mathbb{Z})$.

Note that the set of trivializations w is a torsor over $H^2(F,\mathbb{Z})$. It follows that the class $[w_b] \in H^2(T,\mathbb{Z})/i^*H^2(F,\mathbb{Z})$ is independent of the choice of w. If we replace $\mathcal{H}$ by an isomorphic twist $\mathcal{H}'$ with isomorphism $u\colon \mathcal{H}\to\mathcal{H}'$, and we let w' be obtained from w by conjugation with π^*u, then we have $w_b = w'_b$. Therefore the class $[w_b] \in H^2(T,\mathbb{Z})/i^*H^2(F,\mathbb{Z})$ is well-defined independent of the choice of $\mathcal{H}$ in the class given by δ. Note that we can consider ${}^{\pi}E_3^{0,2}/\ker({}^{\pi}d_3^{0,2}) \subset H^2(T,\mathbb{Z})/i^*H^2(F,\mathbb{Z})$.

Lemma A.1 *We have* $[w_b] \in {}^{\pi}E_3^{0,2}/\ker({}^{\pi}d_3^{0,2})$ *and* ${}^{\pi}d_3^{0,2}([w_b])) = [\delta] \in H^3(B,\mathbb{Z})/\mathtt{im}({}^{\pi}d_2^{1,1})$.

Proof. We choose a model for $K(\mathbb{Z},3)$ and a (pointed) classifying map $f\colon B\to K(\mathbb{Z},3)$ such that $\delta = f^*\mathbf{z}$, where $\mathbf{z} \in H^3(K(\mathbb{Z},3),\mathbb{Z})$ is the canonical generator. We further choose a twist $\mathcal{K}$ on $K(\mathbb{Z},3)$ in the class $\mathbf{z}$. Then we can assume that $\mathcal{H} = f^*\mathcal{K}$.

We choose a (pointed) homotopy $H\colon [0,1]\times F\to K(\mathbb{Z},3)$ between $f\circ p$ and the constant map which exists because $\pi^*\delta = 0$. The homotopy class of such homotopies is well defined upto the action of $H^2(F,\mathbb{Z})\cong H^3(\Sigma F,\mathbb{Z})$, where ΣF is the reduced suspension of F. Elements of $H^3(\Sigma F,\mathbb{Z})$ are represented by maps $\Sigma F\to K(\mathbb{Z},3)$, and the action mentioned above is defined by concatenation with the homotopy given by the composition $[0,1]\times F\twoheadrightarrow \Sigma F\to K(\mathbb{Z},3)$.

Note that by the axioms of twists each such map H induces a well-defined isomorphism of twists

$$\pi^*\mathcal{H}\xrightarrow{\sim} H_0^*\mathcal{K}\xrightarrow{\sim} H_1^*\mathcal{K}\xrightarrow{\sim} 0, \tag{A.2}$$

where we use canonical identifications, and in particular that H_1 is the constant map. The axioms also imply that the action of $H^2(F,\mathbb{Z})$ on such isomorphisms corresponds exactly to the action of $H^2(F,\mathbb{Z})$ on the homotopies H as constructed above. Therefore, we can choose H such that the isomorphism (A.2) is exactly the isomorphism w chosen above.

If we restrict H to the fiber T, then we obtain a map $h\colon [0,1]\times T\to K(\mathbb{Z},3)$. Since $H_0 = f\circ p$ and $p\circ i$ is the constant map to the base point, it follows that h_0 is the constant map. Consequently, we have a factorization of h as $[0,1]\times T\twoheadrightarrow \Sigma T\xrightarrow{h'} K(\mathbb{Z},3)$. The axioms of twists give us an isomorphism $h_0^*\mathcal{K}\xrightarrow{\sim} h_1^*\mathcal{K}$ which is by naturality the pullback i^*w. We now see that $w_b\in H^2(T,\mathbb{Z})$ corresponds to $(h')^*\mathbf{z}\in H^3(\Sigma T,\mathbb{Z})$ under the suspension isomorphism.

A different interpretation uses adjunction to translate h' to a map $g\colon T\to\Omega K(\mathbb{Z},3) = K(\mathbb{Z},2)$. Then $w_b = g^*\mathbf{u}$, where $\mathbf{u}$ is the canonical generator of $H^2(K(\mathbb{Z},2),\mathbb{Z})$. We now consider the Leray-Serre spectral sequence of the fibration

$$\Omega K(\mathbb{Z},3)\to PK(\mathbb{Z},3)\xrightarrow{p} K(Z,3)\ ,$$

where $PK(\mathbb{Z},3)$ is the (contractible) space of pointed paths in $K(\mathbb{Z},3)$. Observe that ${}^pd_2^{0,2}(\mathbf{u}) = 0$ and ${}^pd_3^{0,2}(\mathbf{u}) = \mathbf{z}$. By naturality, $[w_b]\in {}^\pi E_3^{0,2}/\ker({}^\pi d_3^{0,2})$ and ${}^\pi d_3^{0,2}(w_b) = [\delta]\in H^3(B,\mathbb{Z})/\mathtt{im}({}^\pi d_2^{1,1})$. $\square$

A.4 Classification of lifts

We consider an integer $k \geq 1$ and an abelian group G. We form the loop space $LK(G,k) := \mathtt{Map}(S^1, K(G,k))$. Let $h : K(G,k) \times K(G,k) \to K(G,k)$ be the h-space structure. Then the map $\phi : K(G,k) \times \Omega K(G,k) \to LK(G,k)$ given by $\phi(x,l)(t) = h(x,l(t))$, $t \in S^1$, is a homotopy equivalence. We choose a homotopy inverse ψ of ϕ. Note that $\Omega K(G,k) \cong K(G,k-1)$.

We consider a space B and a map $c : B \to K(G,k)$. Furthermore we let $p : X \to B$ be the homotopy fiber of c. Then the homotopy fiber of p is a $K(G,k-1)$. By obstruction theory the set of lifts $\tilde{f}$ in the diagram

$$\begin{array}{ccc} & & X \\ & \tilde{f} \nearrow & p \downarrow \\ Y & \xrightarrow{f} & B \end{array}$$

is empty or a torsor over $H^{k-1}(Y,G)$. Given two lifts $\tilde{f}_0, \tilde{f}_1$ we have a difference element $\delta(\tilde{f}_0, \tilde{f}_1) \in H^{k-1}(Y,G)$.

In fact, we can choose a model for $K(G,k-1)$ which is an abelian group. Furthermore, we can choose a model for $p : X \to B$ which is a $K(G,k-1)$-principal bundle. Then there exists a unique map $g : Y \to K(G,k-1)$ such that $\tilde{f}_1 = \tilde{f}_0 g$ using the right-action of $K(G,k-1)$ on X. In this case the homotopy class of g is classified by $\delta(\tilde{f}_0, \tilde{f}_1)$.

We consider now a map $H : S^1 \times Y \to B$ such that $H_{|\{0\} \times Y} = f$ (we parameterize S^1 by $[0,1]$ with endpoints identified). Furthermore we consider a lift $\tilde{H}$ in the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & X \\ i_0 \downarrow & \tilde{H} \nearrow & p \downarrow \\ [0,1] \times Y & \xrightarrow{H \circ q} & B \end{array},$$

where $q : [0,1] \times Y \to S^1 \times Y$ is the quotient map, and $i_0(y) := (0,y)$. We define $\tilde{f}' := \tilde{H}_{|\{1\} \times Y} : Y \to X$.

Let $H' : Y \to LB$ be the adjoint of H. By composition with $Lc : LB \to LK(G,k)$, the homotopy inverse ψ, and the projection to the second component we obtain a map

$$\mathtt{pr}_2 \circ \psi \circ \circ Lc \circ H' : Y \to K(G, k-1) \ .$$

The homotopy class of this map is classified by a cohomology class $d(H) \in H^{k-1}(Y, G)$. Several times we need the following formula:

$$\delta(\tilde{f}', \tilde{f}) = d(H) \tag{A.3}$$

References

[1] P. Bouwknegt, J. Evslin and V. Mathai. Topology and H-flux of T-dual manifolds, *Phys. Rev. Lett.*, 92, no. 18, 181601 (2004). arXiv:hep-6h/0312284.

[2] P. Bouwknegt, J. Evslin and V. Mathai. T-duality: topology change from H-flux, *Comm. Math. Phys.*, 249 (2004), 383–415.

[3] P. Bouwknegt and K. Hannabuss and V. Mathai. T-Duality for principal torus bundles. *J. High Energy Phys.* (2004), no. 3, 018, 10 pp. (electronic). arXiv:hep-th/0312284.

[4] P. Bouwknegt and K. Hannabuss and V. Mathai. T-duality for principal torus bundles and dimensionally reduced Gysin sequences *Adv. Theor. Math. Phys.* 9 (2005), 749–773. arXiv:hep-th/0412268.

[5] P. Bouwknegt, K. Hannabuss, V. Mathai. Nonassociative tori and applications to T-duality. *Comm. Math. Phys.* 264 (2006), 41–69.

[6] U. Bunke and Th. Schick. On the topology of T-duality. *Rev. Math. Phys.* 17 (2005), 77–112. arXiv:math.GT/0405132.

[7] U. Bunke and Th. Schick. T-duality for non-free circle actions, arXiv:math.GT/0508550.

[8] U. Bunke, M. Spitzweck, Th. Schick and A. Thom . Pontrjagin duality for locally compact group stacks and T-duality. In preparation.

[9] T. Buscher. A symmetry of the string background field equations, *Phys. Lett. B* 194 (1987), 59–62.

[10] T. Buscher. Path integral derivation of quantum duality in nonlinear sigma models. *Phys. Lett. B* 201 (1988), 466–472.

[11] R. Donagi and T. Pantev. Torus fibrations, gerbes, and duality, arXiv:math.AG/0306213.

[12] N. Hitchin. Lectures on special Lagrangian submanifolds. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 151–182. Amer. Math. Soc., Providence, RI, 2001.

[13] D. Husemoller. *Fibre bundles*, Graduate Texts in Mathematics 20, Springer-Verlag, New York. 1994.

[14] V. Mathai and J. Rosenberg. T-Duality for torus bundles with H-fluxes via noncommutative topology. *Comm. Math. Phys.* 253 (2005), 705–721.

[15] V. Mathai and J. Rosenberg. On mysteriously missing T-duals, H-flux, and the T-duality group. arXiv:hep-th/0409073, to appear in *Proceedings of the XXXIII International Conference of Differential Geometric Methods in Mathematical Physics (August 2005)*, editors Mo-Lin Ge and Weiping Zhang, World Scientific (2006).

[16] V. Mathai and J. Rosenberg. T-duality for torus bundles with H-fluxes via noncommutative topology. II. The high-dimensional case and the T-duality group. *Adv. Theor. Math. Phys.*,10 (2006), 123–158.

[17] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its applications to Picard sheaves. *Nagoya Math J.* 81 (1981), 153–175.

[18] J. Polchinski. *String theory. Vol. I, II* , Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2005.

[19] A. Schneider. PhD thesis, Göttingen, to be finished 2006/07.

[20] A. Strominger, S.T. Yau and E. Zaslow. Mirror symmetry is T-duality, *Nuclear Phys. B* (479) (1996), 243–259.