On quantum ergodicity for vector bundles

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January 28, 2005

Abstract

In the present paper we develop a framework in which questions of quantum ergodicity for operators acting on sections of hermitian vector bundles over Riemannian manifolds can be studied. We are particularly interested in the case of locally symmetric spaces. For locally symmetric spaces, we extend the recent construction of Silberman and Venkatesh [7] of representation theoretic lifts to vector bundles.

Contents

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1 Introduction

1.1 We start with a brief review of the basic set-up for the study of quantum ergodicity of the Laplace operator acting on functions on a Riemannian manifold. Given a closed Riemannian manifold we can consider a sequence of normalized eigenfunctions of the Laplace operator associated to a sequence of eigenvalues tending to infinity. Taking the square of the absolute value of each of these functions we obtain a sequence of probability measures on the manifold. Note that the space of probability measures is weakly compact. So we can ask for a description of possible limit points of this sequence. In particular, in the framework of quantum unique ergodicity, we want to know under which circumstances there is a unique limit point, namely the measure determined by the Riemannian metric.

1.2 The natural way to study these limit measures is to lift them in a canonical way to a probability measure (called microlocal lift) on the unit sphere bundle of the manifold. One way to define a microlocal lift is as follows. Each eigenfunction defines a positive state on the algebra of zero-order pseudodifferential operators. If we choose a positivity-preserving operator convention (a right inverse of the symbol map), then this state induces a positive linear form on the algebra of symbols. Since the latter is the algebra of functions on the unit sphere bundle this linear form is just a measure on this bundle. Applying this to a sequence of eigenfunctions we get a bounded sequence of measures on the unit sphere bundle. Any limit point of this sequence is called a microlocal lift. It is now an interesting observation that *the set of microlocal lifts is independent of the choice of the operator convention*.

1.3 The unit sphere bundle carries a natural dynamical system, the geodesic flow. It could be considered as the classical counterpart of the quantum system described by the Laplace operator. The second basic observation is now that *all microlocal lifts are invariant with respect to the geodesic flow*. The combination of this observation with additional information about mixing properties of the geodesic flow is the starting point of a finer investigation of the shape of these microlocal lifts. In particular, under the assumption, that the geodesic flow is ergodic (with respect to Lebesgue class), it is natural to ask wether the microlocal lift is just the (normalized) Riemannian measure. This is the basic question of quantum ergodicity. We refer to the introduction of [7] for a detailed description of the current knowledge. Here we only mention the following. A manifold (or rather its Laplacian) for which the Riemannian measure is the only microlocal lift is called quantum uniquely ergodic (QUE). Rudnick and Sarnak [5] conjectured that negatively curved manifolds are always QUE. Recently, Lindenstrauss [3] has proved an arithmetic version of this conjecture for certain arithmetic hyperbolic surfaces.

1.4 The details of the construction of microlocal lifts and the verification of the two basic properties are not at all complicated. It is the purpose of Section 2 to give these arguments in a more general setting. In fact, if the Riemannian manifold comes equipped with a hermitian vector bundle with connection, then we can replace the Laplace operator on the manifold by the Laplace operator on this bundle. Then we are looking for microlocal lifts associated with sequences of eigensections of the operator. The new point is that the algebra of symbols is now the algebra of sections of the endomorphism bundle of the vector bundle lifted to the unit sphere bundle. In particular, this algebra can be non-commutative. This essentially leads to a change of terminology, the main instance of which is the replacement of probability measures by states.

The set of microlocal lifts is now a set of states on the algebra of symbols. We show in Proposition 2.1 that this set is naturally associated to the geometric data. The connection induces a natural lift of the geodesic flow to a flow of automorphisms of the algebra of symbols, and we verify in Proposition 2.3 that each microlocal lift is invariant.

Finer quantum ergodicity questions are left untouched in this paper and will be a topic of future research. Note that in the bundle case one cannot expect the microlocal lift to be unique (even for negatively curved manifolds). Let us consider e.g. the case of differential forms which can be decomposed into closed and coclosed ones. Associated with this decomposition is a natural splitting into two parts of the pull-back of bundle of differential forms to the unit sphere bundle. If we consider e.g sequence of closed eigenforms, then the associated microlocal lifts are annihilated by the projection onto the subbundle corresponding to coclosed eigenforms. The microlocal lifts associated to sequences of coclosed forms behave in the opposite way.

1.5 In Section 3 we start to develop a theory of representation theoretic lifts for the case of a compact locally symmetric space $\Gamma\backslash G/K$. Representation theoretic lifts serve as a substitute for the microlocal lifts discussed so far. They are designed to take into account the rich structure available in the locally symmetric situation. While defined without any reference to pseudodifferential operators it turns out (see the final Section 5) that they determine the microlocal lifts. Thus representation theoretic lifts should be considered as refined microlocal ones.

Our guide here is the recent paper by Silberman and Venkatesh [7], where the notion of a representation theoretic lift was introduced. The Laplace operator on functions on $\Gamma\backslash G/K$ commutes with a whole algebra of differential operators coming from the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra of the Lie algebra of G. Therefore the spectrum of the Laplace operator can be further decomposed with respect to this algebra. For studying the fine structure of the lifts associated with the locally symmetric situation it seems more appropriate to consider instead of a sequence of eigenfunctions of the Laplacian the corresponding sequence of embeddings of spherical unitary representations of G into $L^2(\Gamma\backslash G)$. Following earlier constructions for special cases due to Zelditch and Lindenstrauss, Silberman and Venkatesh associate to such a sequence of embeddings a representation theoretic lift.

Our main observation is that one can apply an analogous procedure if one wants to study sequences of eigensections of bundles of the form $\Gamma\backslash G\times_K V_\gamma$, where (γ, V_γ) is a unitary representation of K.

1.6 Let us remark at this point that locally symmetric spaces of higher rank do not have strictly negative curvature. In fact, they do not have the QUE-property defined in 1.3 as follows from the results of Section 5. In fact it turns out that the microlocal lifts associated to conveniently arranged sequences (see 3.4) of embeddings of principal series representations are supported on subsets of the unit sphere bundle of Lebesgue measure zero.

Thus the definition of QUE must be modified in the case of higher rank locally symmetric spaces. Replacing microlocal by representation theoretic lifts which live on $\Gamma\backslash G$ one can define an (arithmetic) QUE-property which has recently been verified in many cases (see the forthcoming second part of [7]).

1.7 We now describe our construction of the representation theoretic lifts which is presented in detail in Section 3. We follow quite closely the approach of [7]. Our contribution is essentially an adaptation of arguments and language to the non-commutative situation in the case of nontrivial K-types.

The main part of the spectrum of the Laplacian (and the other locally invariant differential operators) on $L^2(\Gamma\backslash G\times_K V_\gamma)$ is caused by embeddings of unitary principal series representations (associated with the minimal parabolic subgroup of G) into $L^2(\Gamma\backslash G)$. These pricipal series representations come in natural families. Let $G = KAN$ be an Iwasawa decomposition of G and $M := Z_K(A)$ be the centralizer of A in K. Then a family of unitary principal series representations (see 3.4) is determined by an element $\kappa \in \hat{M}$. The family parameter runs through $\mathfrak{a}^*$, the dual of the Lie algebra of A.

1.8 We fix now κ and therefore a family of principal series representations. We consider a sequence of embeddings of members of this family into $L^2(\Gamma\backslash G)$ with parameter tending to infinity approximately along a regular ray (see 3.5) in $\mathfrak{a}^*$. Such a sequence contributes to the spectrum in $L^2(\Gamma\backslash G \times_K V_\gamma)$ if and only if $[V_\kappa \otimes V_\gamma]^M \neq \{0\}$. In the function case ($\gamma = 1$) this condition is equivalent to $\kappa = 1$, i.e., the corresponding principal series representations are spherical. In contrast to the spherical case the dimension of $[V_\kappa \otimes V_\gamma]^M$ can be greater than one in general.

To any $T \in [V_\kappa \otimes V_\gamma]^M$ there is an associated vector ψ_T of "γ-spherical elements" in the corresponding principal series representation (see 3.4). Applying the embedding ξ of a principal series representation into $L^2(\Gamma\backslash G)$ to ψ_T we get an eigensection $\xi(\psi_T) \in L^2(\Gamma\backslash G \times_K V_\gamma)$ denoted by $\xi(\psi_T)$. This section defines a state $\sigma_{\xi(\psi_T)}$ on the algebra $C(\Gamma\backslash G \times_K \mathtt{End}(V_\gamma))$, which can be identified with the subalgebra of K-invariants in $C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$. Recall that we considered a sequence of embedded principal series representations, and therefore we get a sequence of such states. We are interested in "lifting" the limit states to a state of $C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$ which is not a priori K-invariant.

To this end we construct for each individual embedding ξ a functional $\sigma^{\xi}_{\psi_T,\delta_T}$ (see 3.17 for the definiton) on the smaller algebra $C^K(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$ (the C^K stands for K-finite functions). After choosing an appropriate subsequence of embeddings the corresponding sequence of functionals converges to a functional which extends to a state on the algebra $C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$ (see Proposition 3.10). The states which are obtained in this way are the representation theoretic lifts in question. Their restriction to the subalgebra of K-invariants in $C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$ coincides with the set of limit states associated to $\sigma_{\xi(\psi_T)}$ discussed above (see Proposition 3.16). On the one hand, this justifies the name "lift". On the other hand, the representation theoretic lifts contain additional microlocal information.

1.9 It is not apriori clear that a representation theoretic lift is a limit point of states associated to a sequence of functions in $L^2(\Gamma\backslash G) \otimes V_\gamma$. In Theorem 3.14 we show that this is indeed the case.

The space of embeddings of a fixed unitary representation of G into $L^2(\Gamma\backslash G)$ is acted on by Hecke operators. In particular, fixing a Hecke operator and a complex number, it makes sense to talk about eigenembeddings with this given eigenvalue.

If the representation theoretic lift is associated to a family of eigenembeddings (for fixed Hecke operator and eigenvalue), then we show further that the representation theoretic lift is a limit of states associated to a sequence of functions in $L^2(\Gamma\backslash G) \otimes V_\gamma$ which are also eigenfunctions of the Hecke operator with the given eigenvalue. Such a property played a fundamental role in the study of the quantum ergodic properties in [2]. In particular, it implies restrictions on the support of the representation theoretic lifts.

1.10 In Section 4 we investigate various invariance properties of the representation theoretic lifts. Note that the group A acts on $\Gamma\backslash G$ by right multiplication. This induces an action of A by automorphisms on $C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$. The main result of the section (Theorem 4.4) now states that *all representation theoretic lifts are invariant with respect to the action of A.* This is the locally symmetric counterpart of the invariance of the microlocal lifts with respect to the geodesic flow.

We show further (Proposition 4.2) that the microlocal lifts are M-invariant and, when associated with $T \in [V_\kappa \otimes V_\gamma]^M$, are essentially states on the smaller algebra $C(\Gamma\backslash G \times_M \mathtt{End}(V_T))$, where the irreducible M-representation space $V_T \subset V_\gamma$ is given by $\{< v, T > \mid v \in V_\kappa\}$. In particular, a pair of sequences of eigensections which corresponds to a pair of linearly independent T's has a disjoint pair of sets of representation theoretic lifts. In view of the comparison result (Corollary 5.2) between representation theoretic and microlocal lifts obtained in the final Section 5 this is another manifestation of the non-uniqueness of the microlocal lifts mentioned at the end of 1.4.

2 Microlocal lifts

2.1 Let M be a closed smooth manifold with Riemannian metric g. Let $E \to M$ be a complex vector bundle of dimension n with hermitian metric h and metric connection ∇. By $\Delta = \nabla^*\nabla$ we denote the Laplace operator. With the smooth sections $C^\infty(M,E)$ as domain it can be considered as an unbounded essentially selfadjoint operator on the Hilbert space $L^2(M,E)$ of square integrable sections of E.

2.2 Let $\pi : SM \to M$ denote the unit sphere bundle of the cotangent bundle $T^*M \to M$. Let $\Psi DO^i(M,E)$ denote the algebra of classicial i'th-order pseudodifferential operators on M. Then we have an exact sequence of algebras

$$0 \to \Psi DO^{-1}(M,E) \to \Psi DO^0(M,E) \xrightarrow{s} C^\infty(SM, p^*\mathtt{End}(E)) \to 0 \ ,$$

where s is the principal symbol map.

A linear continuous right-inverse $\mathtt{Op} : C^\infty(SM, p^*\mathtt{End}(E)) \to \Psi DO^0(M,E)$ of s is called a quantization or operator convention. The algebra pseudo-differential operators is here topologized as algebra of continuous operators on $C^\infty(M,E)$. In general, a quantization does not extend continuously to a map $C(SM, p^*\mathtt{End}(E)) \to B(L^2(M,E))$. But using a construction of Friedrichs (see [8], p. 142) we can choose the quantization $\mathtt{Op}$ such that it preserves positivity, i.e. if $a \in C^\infty(SM, p^*\mathtt{End}(E))$ is a non-negative element in the C^*-algebra $C(SM, p^*\mathtt{End}(E))$, then $\mathtt{Op}(a) \geq 0$ in the C^*-Algebra $B(L^2(M,E))$.

2.3 Consider a C^*-algebra A and a dense subalgebra $A_\infty \subset A$. A linear map $\sigma : A_\infty \to \mathbb{C}$ is called positive if $a \geq 0$ implies that $\sigma(a) \geq 0$.

If $\sigma : A_\infty \to \mathbb{C}$ is positive and $\sigma(1) < \infty$, then σ extends uniquely to a continuous linear positive map $\sigma : A \to \mathbb{C}$. A state on A is a normalized (i.e. $\sigma(1) = 1$) linear positive map $\sigma : A \to \mathbb{C}$.

2.4 Let $\psi \in C^\infty(M,E)$ be a unit vector in $L^2(M,E)$. We then consider the linear map

$$\sigma_\psi : C^\infty(SM, p^*\mathtt{End}(E)) \to \mathbb{C}$$

given by

$$\sigma_\psi(a) = <\psi, \mathtt{Op}(a)\psi> \ .$$

Since the quantization preserves positivity, σ_ψ is a positive. Since $\sigma_\psi(1) < \infty$ it follows that σ_ψ extends to a continuous positive linear functional

$$\sigma_\psi : C(SM, p^*\mathtt{End}(E)) \to \mathbb{C} \ .$$

In fact, we have the uniform estimate

$$\|\sigma_\psi\| \leq \|\mathtt{Op}(1)\| \ .$$

2.5 We now consider the countable set of functionals on $C(SM, p^*\mathtt{End}(E))$ of the form σ_ψ, where ψ is a normalized eigenvector of Δ. This set is a bounded set in the Banach dual of $C(SM, p^*\mathtt{End}(E))$ and therefore weak-$*$-precompact. By $V \subset C(SM, p^*\mathtt{End}(E))^*$ we denote then non-empty set of all its accumulation points.

Proposition 2.1 *The set V is independent of the choice of the positive quantization map and consists of states.*

Proof. Let V' denote the set defined with another choice $\mathtt{Op}'$ of the quantization. For the first assertion it suffices to show that $V \subset V'$. We consider $\sigma \in V$. Then there exists a sequence of normalized eigenvectors ψ_n of Δ to eigenvalues $\lambda_n \to \infty$ such that for all $f \in C^\infty(SM, p^*\mathtt{End}(E))$ we have $\sigma(f) = \lim < \psi_n, \mathtt{Op}(f)\psi_n >$. Note that $\mathtt{Op}(f) - \mathtt{Op}'(f) \in \Psi DO^{-1}(M,E)$. But for $A \in \Psi DO^{-1}(M,E)$ we have $\lim < \psi_n, A\psi_n >= 0$ since $\lambda_n \to \infty$. This shows that

$$\sigma(f) = \lim < \psi_n, \mathtt{Op}'(f)\psi_n > \ .$$

We conclude that $\sigma \in V'$.

We now show that V consists of states. It is clear that the elements of V are positive. We must verify normalization. Note that $\mathtt{Op}(1) = 1 + A$, where $A \in \Psi DO^{-1}(M,E)$. We conclude that $\sigma(1) = 1$. □

2.6 Using functional calculus we can define the strongly continuous group of unitary operators $\exp(it\sqrt{\Delta})$. One can show that these are Fourier-integral operators. Note that the conjugation by a Fourier-integral operator preserves pseudodifferential operators. Let $f \in C^\infty(SM, p^*\mathtt{End}(E))$. Then in principle one can calculate the symbol of

$$\exp(it\sqrt{\Delta})\mathtt{Op}(f)\exp(-it\sqrt{\Delta})$$

using the calculus of Fourier-integral operators. Here we prefer a simpler way by computing the infinitesimal action which amounts to a calculation of the symbol $s([i\sqrt{\Delta}, \mathtt{Op}(f)]) \in C^\infty(SM, p^*\mathtt{End}(E))$. Let $X \in C^\infty(SM, TSM)$ denote the generator of the geodesic flow. Note that we have an induced connection on $p^*\mathtt{End}(E)$ which we will also denote by ∇.

Lemma 2.2 *We have* $s([i\sqrt{\Delta}, \mathtt{Op}(f)]) = \nabla_X f$.

Proof. This is a local computation and independent of the choice of the quantization.

We consider a point in M and choose geodesic normal coordinates x. Let (x, ξ) denote the corresponding coordinates of T^*M. We want to compute $s([i\sqrt{\Delta}, \mathtt{Op}(f)])$ in the point $(0, \xi)$ with $\|\xi\| = 1$. We further trivialize E using radial parallel transport. We now use the standard quantization. In these coordinates the full symbol of $\sqrt{\Delta}$ is given by $\|\xi\| + O(x^2)$. The principal symbol of a commutator of pseudodifferential operators is given by the Poisson bracket of their symbols. Therefore we get (on the sphere $\{\|\xi\| = 1\}$)

$$s([i\sqrt{\Delta}, \mathtt{Op}(f)])(0,\xi) = \xi^i \partial_{x_i} f(0,\xi) \ .$$

This implies the assertion in view of the choice of the trivializations, since $\xi^i \partial_{x_i}$ is the value of the generator of the geodesic flow at $(0, \xi)$. □

2.7 Using the connection ∇ on $p^*\mathtt{End}(E)$ we can lift the geodesic flow Φ_t on SM to a flow $\tilde{\Phi}_t$ on $p^*\mathtt{End}(E)$. We denote the action of this flow on sections by the same symbol. We have $\frac{d}{dt}_{|t=0} \tilde{\Phi}_t(f) = \nabla_X f$. Thus the algebra $C(SM, p^*\mathtt{End}(E))$ comes with a flow of automorphisms $\tilde{\Phi}_t$. By Lemma 2.2 we have

$$s(\exp(it\sqrt{\Delta})\mathtt{Op}(f)\exp(-it\sqrt{\Delta})) = \tilde{\Phi}_t(f) \ .$$

Proposition 2.3 *Every limit state* $\sigma \in V$ *is invariant under this flow.*

Proof. Let ψ_n be a sequence of normalized eigenvectors of Δ to eigenvalues λ_n such that for all $f \in C^\infty(SM, p^*\mathtt{End}(E))$ we have $\sigma(f) = \lim < \psi_n, \mathtt{Op}(f)\psi_n >$. We compute

$$\begin{aligned}
\sigma(f) &= \lim < \psi_n, \mathtt{Op}(f)\psi_n > \\
&= \lim < \exp(-it\sqrt{\Delta})\psi_n, , \mathtt{Op}(f)\exp(-it\sqrt{\Delta})\psi_n > \\
&= \lim < \psi_n, \exp(it\sqrt{\Delta})\mathtt{Op}(f)\exp(-it\sqrt{\Delta})\psi_n > \\
&= \lim < \psi_n, \mathtt{Op}(s(\exp(it\sqrt{\Delta})\mathtt{Op}(f)\exp(-it\sqrt{\Delta})))\psi_n > \\
&= \sigma(\tilde{\Phi}_t(f)) \ .
\end{aligned}$$

□

2.8 A state on the algebra of functions $C(SM)$ is the same thing as a probability measure on SM. A state σ on the algebra $C(SM, p^*\mathtt{End}(E))$ determines and is determined by a pair (μ, M), where μ is a probability measure on SM, $M \in L^\infty(SM, p^*\mathtt{End}(E), \mu)$ gives a measurable family of states on the local algebras, and $\sigma(f) = \mu(\mathtt{tr}Mf)$. Here $\mathtt{tr} : p^*\mathtt{End}(E) \to \mathbb{C}$ denotes the local trace and $\mathtt{tr}Mf \in L^\infty(SM, \mu)$.

If σ is invariant under the flow $\tilde{\Phi}_t$, then μ is invariant under the geodesic flow, and M is invariant under its lift $\tilde{\Phi}_t$.

2.9 The picture which we have described so far is a simple generalization of a well-known construction (see [6], [1], [9]) from the case of the trivial bundle $E = M \times \mathbb{C}$ to arbitrary bundles $E \to M$. It is by now an interesting piece of mathematics to obtain more information about the size of the set of limiting states V and the properties of its elements under ergodicity assumptions on the geodesic flow Φ_t.

3 Representation theoretic lifts

3.1 Let G be a semisimple Lie group, $K \subset G$ be a maximal compact subgroup of G and Γ be a cocompact torsion free discrete subgroup. Then we can consider the locally symmetric space $M = \Gamma\backslash G/K$ with a Riemannian metric given by the Killing form of G. A unitary representation

(γ, V_γ) of K gives rise to a vector bundle $V(\gamma) := \Gamma\backslash G \times_K V_\gamma$ over M which comes with a natural connection. In this situation we can consider the limit states as discussed in Subsection 2. But because of the locally-symmetric structure we can perform a refined construction which we will describe below.

3.2 We consider the C^*-algebra of functions $A_\gamma := C(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$. Let (π, H) be an irreducible unitary representation of G. By $H_{\pm\infty}$ we denote the distribution or smooth vectors of H. Let $\xi \in [H^*_{-\infty}]^\Gamma$ be an invariant distribution vector. Equivalently, we can consider ξ as an embedding $H_\infty \to C^\infty(\Gamma\backslash G)$. We shall assume that ξ is normalized such that this embedding extends to a unitary embedding $H \to L^2(\Gamma\backslash G)$. Let $\phi, \psi \in H_\infty \otimes V_\gamma$. We have $\xi(\phi) \in C^\infty(\Gamma\backslash G) \otimes V_\gamma$. Then we can define a functional $\sigma_{\phi,\psi}$ on A_γ by

$$\sigma^\xi_{\phi,\psi}(f) = \int_{\Gamma\backslash G} < \xi(\phi)(\Gamma g), f(\Gamma g)\xi(\psi)(\Gamma g) > \ . \qquad (3.1)$$

If $\|\phi\| = 1$, then $\sigma^\xi_{\phi,\phi}$ is a state.

3.3 Let $\phi, \psi \in H_\infty$ and $f \in A^\infty_\gamma := C^\infty(\Gamma\backslash G) \otimes \mathtt{End}(V_\gamma)$. Furthermore, let $\mathfrak{g}$ denote the Lie algebra of G and $X \in \mathfrak{g}$. We set $Xf(g) = f(gX) := \frac{d}{dt}_{|t=0} f(g\ \exp(tX))$.

Lemma 3.2 *We have*

$$\sigma^\xi_{\pi(X)\phi,\psi}(f) + \sigma^\xi_{\phi,\pi(X)\psi} + \sigma^\xi_{\phi,\psi}(Xf) = 0\ .$$

Proof. This follows from the fact that the G-action on $\Gamma\backslash G$ preserves the measure. □

3.4 Let $G = KAN$, $g = k(g)a(g)n(g)$, be an Iwasawa decomposition, and let $M := Z_K(A) \subset K$ be the centralizer of A in K. Let (κ, V_κ) be an irreducible unitary representation of M.

Let $\mathfrak{a}$ denote the Lie algebra of A. The choice of κ gives rise to a family of unitary principal series representations $(\pi^{\kappa,i\lambda}, H^{\kappa,i\lambda})$, $\lambda \in \mathfrak{a}^*$, of G. In the compact picture we set $H^{\kappa,i\lambda} := L^2(K \times_M V_\kappa)$. Then $(\pi^{\kappa,i\lambda}(g)\phi)(k) = \phi(k(g^{-1}k))a(g^{-1}k)^{i\lambda-\rho}$. Here for $a \in A$ and $\lambda \in \mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$ the symbol a^λ is a short-hand for $\exp(\lambda(\log(a)))$. Moreover, $\rho \in \mathfrak{a}^*$ is given by $\rho(H) = \frac{1}{2}\mathtt{Tr}\ \mathtt{Ad}(H)_{|\mathfrak{n}}$, $H \in \mathfrak{a}$, where $\mathfrak{n}$ denotes the Lie algebra of N.

By Frobenius reciprocity we have $[C^\infty(K\times_M V_\kappa)\otimes V_\gamma]^K\cong[V_\kappa\otimes V_\gamma]^M$. Explicitly the isomorphism is given by evaluation at $1\in K$. For $T\in[V_\kappa\otimes V_\gamma]^M$ we let $\psi_T\in[C^\infty(K\times_M V_\kappa)\otimes V_\gamma]^K$ denote the corresponding element.

3.5 The Weyl group $W(G,A):=N_K(A)/M$ acts by reflections on $\mathfrak{a}^*$. A point $\lambda\in\mathfrak{a}^*$ is called singular, if it is fixed by some element of the Weyl group. Otherwise it is called regular. Let $S(\mathfrak{a}^*):=(\mathfrak{a}^*\setminus\{0\})/\mathbb{R}^+$ be the space of rays in $\mathfrak{a}^*$. We have a corresponding decomposition of $S(\mathfrak{a}^*)$ into singular and regular rays. Using the metric $\|.\|$ induced by the Killing form on $\mathfrak{g}$ we will identify $S(\mathfrak{a}^*)$ with the unit sphere in $\mathfrak{a}^*$.

3.6 For $0\neq\lambda\in\mathfrak{a}^*$ let $[\lambda]\in S(\mathfrak{a}^*)$ denote the corresponding ray.

Definition 3.3 *We define the closed subset $L(\kappa)\subset S(\mathfrak{a}^*)$ as the set points $l\in S(\mathfrak{a}^*)$ such that there exists a sequence $\xi_n : H^{\kappa,i\lambda_n}\to L^2(\Gamma\backslash G)$ of unitary embeddings such that $\lambda_n\to\infty$ and $[\lambda_n]\to l$.*

3.7 Let $C^K(K\times_M V_\kappa)\subset C^\infty(K\times_M V_\kappa)$ denote the subspace of K-finite vectors. Let us fix a regular point $l\in L(\kappa)$.

Definition 3.4 *We call a sequence $\xi_n : H^{\kappa,i\lambda_n}\to L^2(\Gamma\backslash G)$ of unitary embeddings l-conveniently arranged if $\lambda_n\to\infty$, $[\lambda_n]\to l$, all λ_n are regular, and for all $\phi,\psi\in C^K(K\times_M V_\kappa)\otimes V_\gamma$ the sequence of functionals $\sigma^{\xi_n}_{\phi,\psi}$ converges weakly.*

3.8

Lemma 3.5 *For each regular $l\in L(\kappa)$ there exists a l-conveniently arranged sequence.*

Proof. Consider a sequence $\xi_n : H^{\kappa,i\lambda_n}\to L^2(\Gamma\backslash G)$ of unitary embeddings such that $\lambda_n\to\infty$ and $[\lambda_n]\to l$. By taking a subsequence we can assume that all λ_n are regular.

For fixed $\phi,\psi\in C^K(K\times_M V_\kappa)\otimes V_\gamma$ the bounded set $\{\sigma^{\xi_n}_{\phi,\psi}|n\in\mathbb{N}\}$ of functionals on A_γ is weak-$*$-precompact. Therefore, by taking a subsequence, we can assume that $\sigma^{\xi_n}_{\phi,\psi}$ weakly converges.

Finally note that $C^K(K\times_M V_\kappa)\otimes V_\gamma$ is a vector space with a countable base. Therefore by a diagonal sequence argument we can again choose a subsequence such that $\sigma^{\xi_n}_{\phi,\psi}$ converges for all $\phi,\psi\in C^K(K\times_M V_\kappa)\otimes V_\gamma$. □

3.9 Note that the C^*-algebra $C(K/M)$ acts naturally on the Hilbert space $L^2(K\times_M V_\kappa)\otimes V_\gamma$ by multiplication on the first factor. The subalgebra $C^K(K/M)$ of K-finite functions acts on $C^K(K\times_M V_\kappa)\otimes V_\gamma$.

We consider a regular $l\in L(\kappa)$ and let ξ_n be a l-conveniently arranged sequence as in Definition 3.4. By $\sigma_{\phi,\psi}$ we denote the weak limit of the sequence of functionals $\sigma^{\xi_n}_{\phi,\psi}$ for fixed $\phi,\psi\in C^K(K\times_M V_\kappa)\otimes V_\gamma$.

Note that if $\|\phi\|=1$, then $\sigma_{\phi,\phi}$ is a state.

3.10 Let $h\in C^K_{\mathbb{R}}(K/M)$ be a real-valued K-finite function.

Lemma 3.6 *We have*

$$\sigma_{h\phi,\psi}=\sigma_{\phi,h\psi}\ .$$

Proof. Let $P=MAN\subset G$ be the minimal parabolic subgroup of G. For $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ we consider the representation $\kappa_\lambda(man):=\kappa(m)a^{\rho-\lambda}$ of P on V_κ. We identify $C^\infty(K\times_M V_\kappa)\otimes V_\gamma$ with $C^\infty(G\times_P V_{\kappa_\lambda})\otimes V_\gamma$ by restriction from G to K. This restriction intertwines the G-action π^L on $C^\infty(G\times_P V_{\kappa_\lambda})\otimes V_\gamma$ by left translations with the action $\pi^{\kappa,\lambda}$ (see 3.4) on $C^\infty(K\times_M V_\kappa)\otimes V_\gamma$.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ be the Iwasawa decomposition, and let $X=X_\mathfrak{k}+X_\mathfrak{a}+X_\mathfrak{n}$ denote the corresponding decomposition of $X\in\mathfrak{g}$. Furthermore, for $k\in K$ let $X(k):=\mathtt{Ad}(k^{-1})X$ and Iwasawa decompose $X(k)=X_\mathfrak{k}(k)+X_\mathfrak{a}(k)+X_\mathfrak{n}(k)$. If $\phi\in C^\infty(G\times_P V_{\kappa_\lambda})\otimes V_\gamma$, then we have

$$\begin{aligned}(\pi^L(X)\phi)(k) &= \phi(Xk)\\ &= \phi(kX(k))\\ &= \phi(kX_\mathfrak{k}(k))+\phi(kX_\mathfrak{a}(k))+\phi(kX_\mathfrak{n}(k))\\ &= \phi(kX_\mathfrak{k}(k))-(\rho-\lambda)(X_\mathfrak{a}(k))\phi(k)\end{aligned}$$

Let now $f \in A^\infty_\gamma$ and $\phi, \psi \in C^K(K \times_M V_\kappa) \otimes V_\gamma$. Note that the action of $\mathfrak{g}$ preserves K-finite vectors. Since $K \ni k \mapsto (\rho - \lambda)(X_{\mathfrak{a}}(k)) =: p_{X,\rho-\lambda}(k) \in \mathbb{C}$ is a K-finite function on K/M, we see that $k \mapsto \phi_X(k) := \phi(kX_{\mathfrak{k}}(k))$ is K-finite, too.

By Lemma 3.2 we have

$$0 = \sigma^{\xi_n}_{\pi^L(X)\phi,\psi}(f) + \sigma^{\xi_n}_{\phi,\pi^L(X)\psi}(f) + \sigma^{\xi_n}_{\phi,\psi}(Xf) = 0 \; .$$

This implies

$$\begin{aligned} \sigma^{\xi_n}_{p_{X,i\frac{\lambda_n}{\|\lambda_n\|}}\phi,\psi}(f) + \sigma^{\xi_n}_{\phi,p_{X,i\frac{\lambda_n}{\|\lambda_n\|}}\psi}(f) &= \frac{1}{\|\lambda_n\|}\big(\sigma^{\xi_n}_{p_{X,\rho}\phi,\psi}(f) + \sigma^{\xi_n}_{\phi,p_{X,\rho}\psi}(f) \\ &\quad -\sigma^{\xi_n}_{\phi_X,\psi}(f) - \sigma^{\xi_n}_{\phi,\psi_X}(f) - \sigma^{\xi_n}_{\phi,\psi}(Xf)\big) \; . \end{aligned}$$

The right-hand side of this equation converges to zero as n tends to infinity. We set

$$p_{X,l}(k) := -i \lim_{n\to\infty} p_{X,i\frac{\lambda_n}{\|\lambda_n\|}}(k) \; .$$

The sequences of functions $p_{X,i\frac{\lambda_n}{\|\lambda_n\|}}\phi$ and $p_{X,i\frac{\lambda_n}{\|\lambda_n\|}}\psi$ span finite-dimensional spaces. Since $\sigma_{\phi,\psi}$ is conjugated linear in ϕ we conclude that

$$\sigma_{p_{X,l}\phi,\psi}(f) - \sigma_{\phi,p_{X,l}\psi}(f) = 0 \; .$$

Let $\mathcal{F} \subset C^K_{\mathbb{R}}(K/M)$ denote the algebra of functions generated by the constant functions and the functions $p_{X,l}$, $X \in \mathfrak{g}$. Then we have shown that for all $h \in \mathcal{F}$ we have

$$\sigma_{h\phi,\psi}(f) = \sigma_{\phi,h\psi}(f) \; .$$

It remains to show that $\mathcal{F} = C^K_{\mathbb{R}}(K/M)$. It suffices to show that $\bar{\mathcal{F}} = C(K/M)$, where $\bar{\mathcal{F}}$ is the closure of $\mathcal{F}$ in $C_{\mathbb{R}}(K/M)$. The algebra $\mathcal{F}$ contains the identity and separates points. This can be seen as follows. Using the Iwasawa decomposition we extend l to $\tilde{l} \in \mathfrak{g}^*$. Then we can write $p_{X,l}(k) = \tilde{l}(\mathtt{Ad}(k^{-1})X) = \mathtt{Ad}(k)(\tilde{l})(X)$. If $p_{X,l}(k_1) = p_{X,l}(k_2)$ for all $X \in \mathfrak{g}$, then we have $\mathtt{Ad}(k_1)(\tilde{l}) = \mathtt{Ad}(k_2)(\tilde{l})$. Since l is regular, this implies that $k_1^{-1}k_2 \in M$, hence $k_1M = k_2M$. We conclude by the Stone-Weierstrass theorem that $\bar{\mathcal{F}} = C_{\mathbb{R}}(K/M)$. □

Corollary 3.7 *For $\phi, \psi \in C^K(K \times_M V_\kappa) \otimes V_\gamma$ and $h \in C^K(K/M)$ we have*

$$\sigma_{h\phi,\psi} = \sigma_{\phi,\bar{h}\psi} \; .$$

3.11 Let $A_{\gamma,K} \subset A_\gamma$ denote the subalgebra of K-finite elements. Let $\phi \in C^K(K \times_M V_\kappa) \otimes V_\gamma$ and $\psi \in C^{-\infty}(K \times_M V_\kappa) \otimes V_\gamma$. Then we have $\psi = \sum_{\mu \in \hat{K}} \psi_\mu$ in the sense of distributions, where $\psi_\mu \in C^K(K \times_M V_\kappa) \otimes V_\gamma$ is the component of ψ in the μ-isotypic subspace of $C^{-\infty}(K \times_M V_\kappa)$. For $F \subset \hat{K}$ we set $\psi_F := \sum_{\mu \in F} \psi_\mu$. It is easy to see that for $f \in A_{\gamma,K}$ the sum $\sum_{\mu \in \hat{K}} \sigma_{\phi,\psi_\mu}(f)$ is finite. In fact, there exists a finite set $F = F(\phi, f)$, independent of ψ (but which depends on ϕ and f), of K-types which can contribute to this sum.

Definition 3.8 *We define the functional* $\sigma_{\phi,\psi} : A_{\gamma,K} \to \mathbb{C}$ *by*

$$\sigma_{\phi,\psi}(f) := \sum_{\mu \in \hat{K}} \sigma_{\phi,\psi_\mu}(f) \ .$$

3.12 Now we fix $T \in [V_\kappa \otimes V_\gamma]^M$ such that $\|T\| = 1$ and let $\psi_T \in C^K(K \times_M V_\kappa) \otimes V_\gamma$ as in 3.4. Furthermore, we let $\delta_T \in C^{-\infty}(K \times_M V_\kappa) \otimes V_\gamma$ be the distribution $\phi \mapsto < T, \phi(1) >$.

Definition 3.9 *We define the functional* σ_T *on* $A_{\gamma,K}$ *by*

$$\sigma_T := \sigma_{\psi_T,\delta_T}$$

.

Proposition 3.10 *The functional* σ_T *extends continuously to a state* σ_T *on* A_γ*.*

Proof. We choose a sequence $f_j \in C^K(K/M)$ such that $|f_j|^2$ is a δ-sequence located at $1M$. In particular we require that $\|f_j\|_{L^2(K/M)} = 1$ for all j. Note that $\lim_j |f_j|^2 \psi_T = \delta_T$ and

$$\|f_j \psi_T\|_{L^2(K \times_M V_\kappa) \otimes V_\gamma} = 1 \ .$$

Let $f \in A_{\gamma,K}$. We consider the finite subset $F = F(\psi_T, f) \subset \hat{K}$. Then we can write

$$\begin{aligned} \sigma_T(f) &= \sigma_{\psi_T,\delta_{T,F}}(f) \\ &= \lim_j \sigma_{\psi_T,[|f_j|^2\psi_T]_F}(f) \\ &= \lim_j \sigma_{\psi_T,|f_j|^2\psi_T}(f) \\ &= \lim_j \sigma_{f_j\psi_T,f_j\psi_T}(f) \end{aligned}$$

using 3.7 in the last step. Since $\sigma_{f_j\psi_T,f_j\psi_T}$ is a state the assertion follows.

3.13 Recall that σ_T may depend on the choice of the sequence ξ_n. Let us fix $\kappa \in \hat{M}$, $\gamma \in \hat{K}$, and $T \in [V_\kappa \otimes V_\gamma]^M$ with $\|T\| = 1$.

Definition 3.11 *For each regular $l \in L(\kappa)$ we define the set $V(l,\kappa,\gamma,T)$ of states on A_γ of the form σ_T for the various l-conveniently arranged sequences ξ_n. These states are called representation theoretic lifts.*

3.14 Let $\sigma_T \in V(l,\kappa,\gamma,T)$ be associated to the l-conveniently arranged sequence ξ_n. We again consider the δ-sequence $f_j \in C^K(K/M)$ as in the proof of Proposition 3.10.

Proposition 3.12 *There exists a sequence of integers $n_j \to \infty$ such that $\sigma^{\xi_{n_j}}_{f_j\psi_T, f_j\psi_T}$ weakly converges to σ_T as $j \to \infty$.*

Proof. Let $f \in A_{\gamma,K}$. We have $\sigma_T(f) = \lim_n \sigma^{\xi_n}_{\psi_T,\delta_{T,F}}(f)$, where the finite subset $F := F(\psi_T, f) \subset \hat{K}$ only depends on T and f. We estimate

$$\begin{aligned} |\sigma^{\xi_n}_{\psi_T,\delta_{T,F}}(f) - \sigma^{\xi_n}_{f_j\psi_T,f_j\psi_T}(f)| &\leq |\sigma^{\xi_n}_{\psi_T,\delta_{T,F}}(f) - \sigma^{\xi_n}_{\psi_T,[|f_j|^2\psi_T]_F}(f)| \\ &\quad + |\sigma^{\xi_n}_{\psi_T,|f_j|^2\psi_T}(f) - \sigma^{\xi_n}_{f_j\psi_T,f_j\psi_T}(f)| \ . \end{aligned}$$

We choose $n_j > j$ sufficiently large such that (by Corollary 3.7)

$$|\sigma^{\xi_{n_j}}_{\psi_T,|f_j|^2\psi_T}(f) - \sigma^{\xi_{n_j}}_{f_j\psi_T,f_j\psi_T}(f)| \leq j^{-1} \ .$$

Since $\lim_j [|f_j|^2\psi_T]_F = \delta_{T,F}$ inside a finite dimensional vector space we conclude that

$$\lim |\sigma^{\xi_{n_j}}_{\psi_T,\delta_{T,F}}(f) - \sigma^{\xi_{n_j}}_{f_j\psi_T,f_j\psi_T}(f)| = 0 \ .$$

□

3.15 If Γ is arithmetic (this is automatic if G has higher real rank and $\Gamma \subset G$ is irreducible), then we can consider Hecke operators. Let $H^\Gamma \subset H$ denote the subset of Γ-invariant vectors in a representation (π, H) of G. If $h \in G$ is in the commensurator of Γ, i.e. $\Gamma^h := h\Gamma h^{-1}$ and Γ are commensurable, then we define the following operator $T_h : H^\Gamma \to H^\Gamma$.

Definition 3.13 $T_h(\phi) = \sum_{[\gamma]\in\Gamma/(\Gamma\cap\Gamma^h)} \pi(\gamma)\pi(h)\phi$.

3.16 We can apply the Hecke operator T_h to $[H^*_{-\infty}]^\Gamma$ and to $C^\infty(\Gamma\backslash G)\otimes V_\gamma$. Let $\eta\in\mathbb{C}$ and assume that ξ_n is a l-conveniently arranged sequence of unitary embeddings $H^{\kappa,i\lambda_n}\to L^2(\Gamma\backslash G)$ for some regular $l\in L(\kappa)$ such that $T_h\xi_n=\eta\xi_n$ for all n. Let σ_T denote a limit state as in 3.12. For $u\in L^2(\Gamma\backslash G)\otimes V_\gamma$ let σ_u be the functional on A_γ given by

$$\sigma_u(f)=\int_{\Gamma\backslash G}< u(\Gamma g), f(\Gamma g)u(\Gamma g)> \ .$$

Theorem 3.14 *There exists a sequence $u_j\in C^\infty(\Gamma\backslash G)\otimes V_\gamma$ of eigenvectors of T_h to the eigenvalue η such that $\|u_j\|_{L^2(\Gamma\backslash G)\otimes V_\gamma}=1$ and σ_T is the weak limit of the sequence of states σ_{u_j}.*

Proof. We choose the sequence n_j as in Proposition 3.12 and set

$$u_j:=\xi_{n_j}(f_j\psi_T)\ .$$

Since $\|f_j\psi_T\|_{L^2(K\times_M V_\kappa)\otimes V_\gamma}=1$ and ξ_{n_j} is an unitary embedding, the section u_j is a unit vector. It is a Hecke-eigenvector since ξ_{n_j} is so. □

3.17 Let $A^K_\gamma\subset A_\gamma$ be the subalgebra of K-invariants, i.e. the algebra of sections of the bundle of endomorphisms of the vector bundle $\Gamma\backslash G\times_K V_\gamma\to\Gamma\backslash G/K$. For each unit vector $u\in L^2(\Gamma\backslash G\times_K V_\gamma)$ we consider the state σ_u as defined in 3.16.

Let $\xi: H^{\kappa,i\lambda}\to L^2(\Gamma\backslash G)$ be a unitary embedding. Then we have a unit vector $\xi(\psi_T)\in L^2(\Gamma\backslash G\times_K V_\gamma)=[L^2(\Gamma\backslash G)\otimes V_\gamma]^K$. Let $f\in A^K_\gamma$. As in 3.11 the functional $\phi\mapsto\sigma^\xi_{\psi_T,\phi}(f)$ extends to distributions ϕ.

Lemma 3.15 *We have $\sigma^\xi_{\psi_T,\delta_T}(f)=\sigma_{\xi(\psi_T)}(f)$.*

Proof. Let $F=F(\psi_T,f)\subset\hat{K}$ be the finite subset of K-types as in 3.11. We may assume that F contains the trivial K-type. Note that

$$\int_K\gamma(k)^{-1}\pi^{\kappa,i\lambda}(k^{-1})\delta_{T,F}=\int_K\gamma(k)^{-1}\pi^{\kappa,i\lambda}(k^{-1})\delta_T=\psi_T\ .$$

In fact, the integral defines a K-invariant vector in $H^{\kappa,i\lambda}\otimes V_\gamma$ which by Frobenius reciprocity is determined by its value at $1\in K$. In view of the definition of δ_T this evaluation is $T\in[V_\kappa\otimes V_\gamma]^M$. We compute

$$\begin{aligned}
\sigma^{\xi}_{\psi_T,\delta_T}(f) &= \sigma^{\xi}_{\psi_T,\delta_{T,F}}(f)\\
&= \int_{\Gamma\backslash G} <\xi(\psi_T)(\Gamma g), f(\Gamma g)\xi(\delta_{T,F})(\Gamma g)>\\
&= \int_{\Gamma\backslash G}\int_K <\xi(\psi_T)(\Gamma g), \gamma(k) f(\Gamma gk)\gamma(k)^{-1}\xi(\delta_{T,F})(\Gamma g)>\\
&= \int_{\Gamma\backslash G}\int_K <\gamma(k^{-1})\xi(\psi_T)(\Gamma gk^{-1}), f(\Gamma g)\gamma(k)^{-1}\xi(\delta_{T,F})(\Gamma gk^{-1})>\\
&= \int_{\Gamma\backslash G}\int_K <\xi(\psi_T)(\Gamma g), f(\Gamma g)\gamma(k)^{-1}\xi(\pi^{\kappa,i\lambda}(k^{-1})\delta_{T,F})(\Gamma g)>\\
&= \int_{\Gamma\backslash G}\int_K <\xi(\psi_T)(\Gamma g), f(\Gamma g)\xi(\psi_T)(\Gamma g)>\\
&= \sigma^{\xi}_{\psi_T,\psi_T}(f)\\
&= \sigma_{\xi(\psi_T)}(f)\ .
\end{aligned}$$

□

3.18 Let $l\in L(\kappa)$ be regular and $\sigma_T\in V(l,\kappa,\gamma,T)$ associated to the l-conveniently arranged sequence of unitary embeddings ξ_n.

Proposition 3.16 *The sequence of states $\sigma_{\xi_n(\psi_T)}$ on A^K_γ has a weak limit which is given by the restriction of σ_T from A_γ to A^K_γ.*

Proof. This is a consequence of Lemma 3.15. □

4 Invariance of representation theoretic lifts

4.1 Throughout this section we fix a unitary K-representation (γ,V_γ) and an element $\kappa\in\hat{M}$. We further fix a regular $l\in L(\kappa)$, and $T\in[V_\kappa\otimes V_\gamma]^M$ such that $\|T\|=1$.

We want to study the invariance properties of the states $\sigma \in V(l,\kappa,\gamma,T)$ constructed in Section 3.

4.2 We first consider the action of the subgroup M. Note that K acts on the C^*-algebra $A_\gamma := C(\Gamma\backslash G)\otimes \mathtt{End}(V_\gamma)$ by

$$k(f\otimes A)(g) := (f\otimes \gamma(k)A\gamma(k^{-1}))(gk)\ , \quad f\in C(\Gamma\backslash G), A\in \mathtt{End}(V_\gamma), k\in K\ . \tag{4.1}$$

By duality, (4.1) induces a K-action on the set of states of A_γ.

4.3 We consider T as an element of $\mathtt{Hom}_M(V_{\tilde\kappa},V_\gamma)$, where $\tilde\kappa$ is the dual representation of κ. Then $\dim(V_\kappa)TT^*\in \mathtt{End}(V_\gamma)$ is an M-equivariant projection. Let $P_T := 1\otimes \dim(V_\kappa)TT^*\in A_\gamma$ be the corresponding projection in A_γ.

Proposition 4.2 *Let* $\sigma\in V(l,\kappa,\gamma,T)$.

(1) σ *is M-invariant w.r.t. the action induced by (4.1).*

(2) For each $f\in A_\gamma$ *we have* $\sigma(P_T f P_T)=\sigma(f)$.

Proof. Let $\lambda\in\mathfrak{a}^*$, and let $\xi: H^{\kappa,i\lambda}\to L^2(\Gamma\backslash G)$ be a unitary embedding. Using that

$$\psi_T,\delta_T\in[H^{\kappa,i\lambda}_{-\infty}\otimes V_\gamma]^M \ \text{ and } \ (\mathtt{id}\otimes\dim(V_\kappa)TT^*)\delta_T=\delta_T$$

we see that $m\sigma^\xi_{\psi_T,\delta_T}=\sigma^\xi_{\psi_T,\delta_T}$ for all $m\in M$, and that $\sigma^\xi_{\psi_T,\delta_T}(P_T f)=\sigma^\xi_{\psi_T,\delta_T}(f)$ for all $f\in A^\infty_{\gamma,K}$. Taking the limit over an l-conveniently arranged sequence ξ_n we obtain the first assertion and that

$$\sigma(P_T f)=\sigma(f)\ . \tag{4.3}$$

A state σ on a C^*-algebra is a real functional, i.e., it satisfies $\sigma(f^*)=\overline{\sigma(f)}$ for all f. Using (4.3) we obtain

$$\sigma(P_T f P_T)=\sigma(fP_T)=\overline{\sigma(P_T f^*)}=\overline{\sigma(f^*)}=\sigma(f)\ .$$

Since $A^\infty_{\gamma,K}\subset A_\gamma$ is dense this finishes the proof of the proposition. □

4.4 The proposition tells us that $\sigma \in V(l,\kappa,\gamma,T)$ is given as a pull back of a state σ^0 on the smaller algebra $P_T A_\gamma^M P_T$ which is in fact isomorphic to $C(\Gamma\backslash G \times_M \mathtt{End}(V_{\tilde\kappa}))$.

It seems to be likely that σ^0 is actually a trace on $C(\Gamma\backslash G \times_M \mathtt{End}(V_{\tilde\kappa}))$. Up to now we do not know how to prove this property. It would imply that σ is determined by a probability measure on $\Gamma\backslash G/M$ alone. By Theorem 4.4 below this measure would be right A-invariant.

4.5 The right-regular representation of G on $C(\Gamma\backslash G)$ induces an action of G by automorphisms on the C^*-algebra $A_\gamma = C(\Gamma\backslash G)\otimes \mathtt{End}(V_\gamma)$. Dually, we obtain a G-action on the states of A_γ. The main goal of this section is to prove the following higher rank analog of Proposition 2.3. Recall the Iwasawa decomposition $G = KAN$ (see 3.4).

Theorem 4.4 *The states $\sigma \in V(l,\kappa,\gamma,T)$ are A-invariant.*

Let $\xi_n : H^{\kappa,i\lambda_n} \to L^2(\Gamma\backslash G)$ be a l-conveniently arranged sequence of unitary embeddings giving rise to σ. Following the approach of [7], Section 4, we will exhibit a certain family of differential operators $D(\lambda)$, depending polynomially on $\lambda \in \mathfrak{a}^*$, such that $D(\lambda_n)\sigma^{\xi_n}_{\psi_T,\delta_T} = 0$. In the limit $n \to \infty$ this will imply A-invariance of σ.

4.6 For any real or complex Lie algebra $\mathfrak{l}$ let $\mathcal{U}(\mathfrak{l})$ be its universal enveloping algebra over $\mathbb{C}$. The algebra $\mathcal{D}_\gamma := \mathcal{U}(\mathfrak{g})\otimes \mathtt{End}(V_\gamma)\otimes \mathtt{End}(V_\gamma)^{\mathtt{opp}}$ acts by differential operators on $A_\gamma^\infty = C^\infty(\Gamma\backslash G)\otimes \mathtt{End}(V_\gamma)$:

$$(X\otimes A\otimes B)(f\otimes C) := Xf\otimes ABC\ , \quad X \in \mathcal{U}(\mathfrak{g}),\ A,B,C \in \mathtt{End}(V_\gamma),\ f \in C^\infty(\Gamma\backslash G)\ .$$

The subspace of K-finite elements $A^\infty_{\gamma,K} := A^\infty_\gamma \cap A_{\gamma,K}$ is invariant w.r.t. this action. Therefore we have an action of $\mathcal{D}_\gamma$ on the space of functionals on $A^\infty_{\gamma,K}$ given by

$$D\sigma(f) := \sigma(D^t f)\ , \tag{4.5}$$

where $D \mapsto D^t$ is the anti-automorphism of $\mathcal{D}_\gamma$ induced by

$$(X\otimes 1)^t = -X\otimes 1\otimes 1,\ X \in \mathfrak{g}, \quad (1\otimes A\otimes B)^t = 1\otimes B\otimes A,\ A,B \in \mathtt{End}(V_\gamma)\ .$$

We are mainly concerned with the subalgebra

$$\mathcal{L}_\gamma := \mathcal{U}(\mathfrak{n}\oplus\mathfrak{a})\otimes \mathtt{End}(V_\gamma)^{\mathtt{opp}} \subset \mathcal{U}(\mathfrak{g})\otimes \mathtt{End}(V_\gamma)\otimes \mathtt{End}(V_\gamma)^{\mathtt{opp}} = \mathcal{D}_\gamma\ .$$

There is a linear map $q_\gamma : \mathcal{U}(\mathfrak{g}) \to \mathcal{L}_\gamma$ which sends $X \otimes Y \in \mathcal{U}(\mathfrak{n} \oplus \mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k}) \cong \mathcal{U}(\mathfrak{g})$ to $X \otimes \gamma(Y) \in \mathcal{U}(\mathfrak{n} \oplus \mathfrak{a}) \otimes \mathrm{End}(V_\gamma)^{\mathrm{opp}} = \mathcal{L}_\gamma$.

If $\mathfrak{l}$ is a Lie algebra and $\varphi : \mathfrak{l} \to \mathbb{C}$ is a Lie algebra homomorphism, then there is a corresponding translation automorphism $\tau_\varphi : \mathcal{U}(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l})$ characterized by $\tau_\varphi(X) = X + \varphi(X) \cdot 1$, $X \in \mathfrak{l}$.

4.7 Let $\mathfrak{m}$ be the Lie algebra of M. We choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{m}$. Then $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $W_\mathbb{C}$ be the Weyl group of $\mathfrak{h}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Let $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$. We view P as a complex-valued polynomial on $\mathfrak{h}_\mathbb{C}^* = \mathfrak{t}_\mathbb{C}^* \oplus \mathfrak{a}_\mathbb{C}^*$. Its differential P' is a polynomial on $\mathfrak{h}_\mathbb{C}^*$ with values in $\mathfrak{h}_\mathbb{C} \cong (\mathfrak{h}_\mathbb{C}^*)^*$. Let $\mu_\kappa \in i\mathfrak{t}^* \subset \mathfrak{h}_\mathbb{C}^*$ be an extremal weight of κ, i.e., it is the highest weight of κ w.r.t. some positive root system of $\mathfrak{t}$ in $\mathfrak{m}_\mathbb{C}$.

Proposition 4.6 *Fix $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$ of degree $\leq d \in \mathbb{N}_0$. Then there exists a $\mathcal{L}_\gamma$-valued polynomial J_P on $\mathfrak{a}_\mathbb{C}^*$ of degree at most $d-2$ such that for all unitary G-maps $\xi : H^{\kappa,i\lambda} \to L^2(\Gamma\backslash G)$*

$$\left(q_\gamma \circ \tau_{\mu_\kappa}(P'(i\lambda)) + J_P(i\lambda)\right) \sigma^\xi_{\psi_T,\delta_T} = 0 \ .$$

Here $P'(i\lambda) \in \mathfrak{h}_\mathbb{C}$ is viewed as an element of $\mathcal{U}(\mathfrak{h})$. Then $\tau_{\mu_\kappa}(P'(i\lambda)) \in \mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})$, and q_γ can be applied.

The Weyl group W_0 of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{m}_\mathbb{C}$ considered as subgroup of $W_\mathbb{C}$ fixes the element $i\lambda \in \mathfrak{h}_\mathbb{C}^*$. It follows that $P'(i\lambda) \in \mathfrak{h}_\mathbb{C}^{W_0}$. Since all extremal weights of κ are conjugated by W_0, the element $\tau_{\mu_\kappa}(P'(i\lambda)) \in \mathcal{U}(\mathfrak{h})$ does not depend on the choice of μ_κ.

4.8 The proof of Proposition 4.6 starts in the next paragraph 4.9 and will then occupy the remainder of this section. Here we argue as in [7], Corollary 4.6, Lemma 4.7 and Corollary 4.8 in order to conclude that Proposition 4.6 implies Theorem 4.4.

We first assume $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$ to be homogeneous of degree d. Fix $f \in A^\infty_{\gamma,K}$. Then by Proposition 4.6 the equation

$$\sigma^{\xi_n}_{\psi_T,\delta_T}\left(\left(q_\gamma \circ \tau_{\mu_\kappa}\left(P'\left(\frac{i\lambda_n}{\|\lambda_n\|}\right)\right) + \frac{J_P(i\lambda_n)}{\|\lambda_n\|^{d-1}}\right)^t f\right) = 0$$

holds for all n. There is a finite dimensional subspace $V \subset A^\infty_{\gamma,K}$ such that

$$f_n := \left(q_\gamma \circ \tau_{\mu_\kappa}\left(P'\left(\frac{i\lambda_n}{\|\lambda_n\|}\right)\right) + \frac{J_P(i\lambda_n)}{\|\lambda_n\|^{d-1}}\right)^t f \in V$$

for all n. Moreover, f_n converges in V to $\left(q_\gamma \circ \tau_{\mu_\kappa}(P'(il))\right)^t (f)$. We obtain

$$\left(q_\gamma \circ \tau_{\mu_\kappa}(P'(il))\sigma_T\right)(f) = 0 \ .$$

Since this is valid for all f, and each $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$ can be decomposed into homogeneous components, we conclude that σ is annihilated by all the operators $q_\gamma \circ \tau_{\mu_\kappa}(P'(il))$, $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$.

For each $H \in \mathfrak{a}$ there exists an element $P_H \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$ such that $P'_H(il) = H$ (see [7], Lemma 4.7). Here the regularity of l is crucial. Then $q_\gamma \circ \tau_{\mu_\kappa}(P'_H(il)) = H \otimes 1$. It follows that σ is $\mathfrak{a}$-invariant, and hence A-invariant. This proves Theorem 4.4 assuming Proposition 4.6. $\square$

4.9 Let $\tilde{\kappa}$, $\tilde{\gamma}$ be the representations dual to κ, γ. Let $\nu \in \mathfrak{a}^*_\mathbb{C}$. The algebra $\mathcal{U}(\mathfrak{g})$ acts on the tensor product representation $H^{\tilde{\kappa},-\nu}_\infty \otimes H^{\kappa,\nu}_{-\infty}$. We let $A \otimes B \in \mathtt{End}(V_\gamma) \otimes \mathtt{End}(V_\gamma)^{\mathrm{opp}}$ act on $V_{\tilde{\gamma}} \otimes V_\gamma$ by $B^\top \otimes A$, where $B^\top \in \mathtt{End}(V_{\tilde{\gamma}})$ is the dual operator of $B \in \mathtt{End}(V_\gamma)$. We obtain an action of $\mathcal{D}_\gamma$ on $(H^{\tilde{\kappa},-\nu}_\infty \otimes V_{\tilde{\gamma}}) \otimes (H^{\kappa,\nu}_{-\infty} \otimes V_\gamma)$. If $\nu \in i\mathfrak{a}^*$, then we have a canonical antilinear identification $R : H^{\kappa,\nu} \otimes V_\gamma \to H^{\tilde{\kappa},-\nu} \otimes V_{\tilde{\gamma}}$. It is a direct consequence of Lemma 3.2 and (4.5) that,

$$\text{if } \ D(R\psi \otimes \phi) = \sum_i \psi_i \otimes \phi_i \,, \qquad \text{then } \ D\sigma^\xi_{\psi,\phi} = \sum_i \sigma^\xi_{R^{-1}\psi_i,\phi_i} \,. \tag{4.7}$$

Here $\xi : H^{\kappa,\nu} \to L^2(\Gamma\backslash G)$, $\psi \in H^{\kappa,\nu}_\infty \otimes V_\gamma$, $\phi \in H^{\kappa,\nu}_{-\infty} \otimes V_\gamma$, $D \in \mathcal{D}_\gamma$.

4.10 The composition of $\nu \in \mathfrak{a}^*_\mathbb{C}$ with the projection of $\mathfrak{n} \oplus \mathfrak{a} \to \mathfrak{a}$ defines a Lie algebra homomorphism $\nu : \mathfrak{n} \oplus \mathfrak{a} \to \mathbb{C}$. Let τ_ν be the corresponding translation automorphism of $\mathcal{U}(\mathfrak{n} \oplus \mathfrak{a})$ (see 4.6). Then $\tau_\nu \otimes \mathtt{id}_{\mathtt{End}(V_\gamma)^{\mathrm{opp}}}$ is an automorphism of $\mathcal{L}_\gamma$ which will be denoted by τ_ν as well.

Lemma 4.8 *For* $\nu \in \mathfrak{a}^*_\mathbb{C}$, $\psi \in H^{\tilde{\kappa},-\nu}_\infty$, $T \in [V_\kappa \otimes V_\gamma]^M$, *and* $D \in \mathcal{L}_\gamma$ *we have*

$$\tau_{\nu+\rho}(D)(\psi \otimes \delta_T) = (D\psi) \otimes \delta_T \ .$$

Proof. It suffices to check the assertion for the generators $X \in \mathfrak{a}$, $Y \in \mathfrak{n}$, and $B \in \mathtt{End}(V_\gamma)^{\mathrm{opp}}$ of $\mathcal{L}_\gamma$. For $D = B$ the assertion holds by definition while for $D = Y$ it follows from $Y\delta_T = 0$. Now let $X \in \mathfrak{a}$ and $\phi \in H^{\tilde{\kappa},-\nu}_\infty$. Then

$$\langle X\delta_T, \phi\rangle = -\langle \delta_T, X\phi\rangle = -\langle T, X\phi(1)\rangle = -\langle T, (\nu+\rho)(X)\phi(1)\rangle = -(\nu+\rho)(X)\langle \delta_T, \phi\rangle \ .$$

Hence $X\delta_T = -(\nu+\rho)(X)\delta_T$ and

$$\tau_{\nu+\rho}(X)(\psi\otimes\delta_T) = (X\psi)\otimes\delta_T + \psi\otimes(X\delta_T) + (\nu+\rho)(X)(\psi\otimes\delta_T) = (X\psi)\otimes\delta_T \ .$$

□

4.11 Let $\mathfrak{u}$ ($\bar{\mathfrak{u}}$, resp.) be the sum of positive (negative) root spaces in $\mathfrak{g}_\mathbb{C}$ w.r.t. a chosen positive Weyl chamber in $\mathfrak{h}_\mathbb{R} := i\mathfrak{t}\oplus\mathfrak{a}$. We arrange this choice such that $\mathfrak{n}_\mathbb{C}\subset\mathfrak{u}$. Then we have a decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{u}\oplus\mathfrak{h}_\mathbb{C}\oplus\bar{\mathfrak{u}} \ .$$

It induces decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h})\oplus(\mathfrak{u}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\bar{\mathfrak{u}}) \ .$$

Let p be the projection onto the first summand. Note that p is equivariant w.r.t. the adjoint action of $\mathfrak{h}_\mathbb{C}$ on $\mathcal{U}(\mathfrak{g})$. By $\mathcal{U}^{\le d}(\mathfrak{l})$ we denote the subspace of elements of $\mathcal{U}(\mathfrak{l})$ of degree at most d. The following lemma is essentially Lemma 4.3 of [7].

Lemma 4.9 *If $Z\in\mathcal{U}^{\le d}(\mathfrak{g})^{\mathfrak{h}_\mathbb{C}}$, then $Z-p(Z)\in\mathcal{U}(\mathfrak{n})\mathcal{U}^{\le d-2}(\mathfrak{a})\mathcal{U}(\mathfrak{k})$.*

Proof. Observe that

$$(\mathfrak{u}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\bar{\mathfrak{u}})^{\mathfrak{h}_\mathbb{C}} \subset \mathfrak{u}\mathcal{U}(\mathfrak{g})\bar{\mathfrak{u}} \ .$$

If $Z\in\mathcal{U}^{\le d}(\mathfrak{g})^{\mathfrak{h}_\mathbb{C}}$, then by $\mathfrak{h}_\mathbb{C}$-equivariance of p

$$Z-p(Z)\in\mathfrak{u}\mathcal{U}^{\le d-2}(\mathfrak{g})\bar{\mathfrak{u}}\subset\mathcal{U}(\mathfrak{u})\mathcal{U}^{\le d-2}(\mathfrak{h})\mathcal{U}(\bar{\mathfrak{u}}) \ . \tag{4.10}$$

Using that $\mathfrak{h}_\mathbb{C}\subset\mathfrak{a}_\mathbb{C}\oplus\mathfrak{m}_\mathbb{C}$, $\mathfrak{u}\subset\mathfrak{n}_\mathbb{C}\oplus\mathfrak{m}_\mathbb{C}$, $\mathfrak{m}_\mathbb{C}\subset\mathfrak{k}_\mathbb{C}$, and $\bar{\mathfrak{u}}\subset\mathfrak{n}_\mathbb{C}\oplus\mathfrak{k}_\mathbb{C}$ one shows inductively that the right hand side of (4.10) is contained in $\mathcal{U}(\mathfrak{n})\mathcal{U}^{\le d-2}(\mathfrak{a})\mathcal{U}(\mathfrak{k})$. This proves the lemma. □

4.12 Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. If $\nu\in\mathfrak{a}^*_\mathbb{C}$, then $\mathcal{Z}(\mathfrak{g})$ acts on $H^{\tilde{\kappa},-\nu}_\infty$ by a certain character denoted by $\chi_{\kappa,\nu}$. Recall the definition of $q_\gamma : \mathcal{U}(\mathfrak{g})\to\mathcal{L}_\gamma$ from 4.6. For $Z\in\mathcal{Z}(\mathfrak{g})$ we consider the elements $p_\gamma(Z) := q_\gamma(p(Z))\in\mathcal{L}_\gamma$ and $b_\gamma(Z) := q_\gamma(Z-p(Z))\in\mathcal{L}_\gamma$.

Lemma 4.11 *If $\psi \in [H_\infty^{\tilde{\kappa},-\nu} \otimes V_{\tilde{\gamma}}]^K$, then we have for all $Z \in \mathcal{Z}(\mathfrak{g})$*

$$\big(p_\gamma(Z) - \chi_{\kappa,\nu}(Z) + b_\gamma(Z)\big)\,\psi = 0\ .$$

Proof. Let $W \in \mathcal{U}(\mathfrak{g})$, $Y \in \mathfrak{k}$ and $\psi \in [H_\infty^{\tilde{\kappa},-\nu} \otimes V_{\tilde{\gamma}}]^K$. Then we have $q_\gamma(WY) = q_\gamma(Y)q_\gamma(W)$ and

$$(WY \otimes 1)\psi = (W \otimes \tilde{\gamma}(-Y))\psi = (W \otimes \gamma(Y)^\top)\psi = \big(q_\gamma(Y)(W \otimes 1)\big)\,\psi\ .$$

It follows by induction that $(X \otimes 1)\psi = q_\gamma(X)\psi$ for any $X \in \mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{n} \oplus \mathfrak{a})\,\mathcal{U}(\mathfrak{k})$. Applying this to $X = Z \in \mathcal{Z}(\mathfrak{g})$ we obtain

$$0 = ((Z \otimes 1) - \chi_{\kappa,\nu}(Z))\,\psi = \big(q_\gamma(Z) - \chi_{\kappa,\nu}(Z)\big)\,\psi = \big(p_\gamma(Z) + b_\gamma(Z) - \chi_{\kappa,\nu}(Z)\big)\,\psi\ .$$

□

Combining Lemma 4.11 with Lemma 4.8 we obtain

Corollary 4.12 *If $\psi \in [H_\infty^{\tilde{\kappa},-\nu} \otimes V_{\tilde{\gamma}}]^K$ and $T \in [V_\kappa \otimes V_\gamma]^M$, then we have for all $Z \in \mathcal{Z}(\mathfrak{g})$*

$$\big(\tau_{\nu+\rho}(p_\gamma(Z)) - \chi_{\kappa,\nu}(Z) + \tau_{\nu+\rho}(b_\gamma(Z))\big)\,(\psi \otimes \delta_T) = 0\ .$$

4.13 Let $\rho_\mathfrak{h} \in \mathfrak{h}_\mathbb{C}^*$ be given by $\rho_\mathfrak{h}(H) = \frac{1}{2}\mathtt{Tr}\,\mathtt{Ad}(H)_{|\mathfrak{u}}$. Then $\rho_\mathfrak{t} := \rho_\mathfrak{h} - \rho \in i\mathfrak{t}^* \subset \mathfrak{h}_\mathbb{C}^*$. The composition $\tau_{\rho_\mathfrak{h}} \circ p$ maps the algebra $\mathcal{Z}(\mathfrak{g})$ isomorphically onto $\mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$. In fact, this map is the celebrated Harish-Chandra isomorphism. If $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$, then we denote its preimage under the Harish-Chandra isomorphism by Z_P. The roots of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{m}_\mathbb{C} \cap \mathfrak{u}$ form a positive root system of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{m}_\mathbb{C}$. Let $\mu_\kappa \in i\mathfrak{t}^*$ be the highest weight of κ with respect to this system of positive roots.

Lemma 4.13 *Fix $P \in \mathcal{U}(\mathfrak{h})^{W_\mathbb{C}}$ of degree $\le d$. Then the $\mathcal{L}_\gamma$-valued polynomial J_P on $\mathfrak{a}_\mathbb{C}^*$ defined by*

$$J_P(\nu) := \tau_{\nu+\rho}(p_\gamma(Z_P)) - \chi_{\kappa,\nu}(Z_P) + \tau_{\nu+\rho}(b_\gamma(Z_P)) - q_\gamma \circ \tau_{\mu_\kappa}(P'(\nu))$$

has degree at most $d-2$. Here the expression $q_\gamma \circ \tau_{\mu_\kappa}(P'(\nu))$ is interpreted as in Proposition 4.6.

Proof. If P has degree d, then $Z_P \in \mathcal{U}(\mathfrak{g})^{\le d}$. Now Lemma 4.9 implies that $\nu \mapsto \tau_{\nu+\rho}(b_\gamma(Z_P))$ has degree at most $d-2$.

We now analyze the first two terms appearing in the definition of J_P. It is well-known that $\chi_{\kappa,\nu}(Z_P) = P(\nu - \rho_\mathfrak{t} - \mu_\kappa)$. The reader may verify this by computing the value $\langle w, Z_P\psi(1)\rangle$, where $w \in V_\kappa$ is the highest weight vector and $\psi \in H^{\tilde{\kappa},-\nu}$ (compare the proof of Lemma 4.8). Let $S \in \mathcal{U}(\mathfrak{h})$. Choosing a basis of V_γ consisting of weight vectors w.r.t. the action of $\mathfrak{t}$ we may view $q_\gamma(S) \in \mathcal{U}(\mathfrak{a}) \otimes \mathtt{End}(V_\gamma)^{\mathrm{opp}}$ as a diagonal matrix whose entries are polynomials on $\mathfrak{a}^*_\mathbb{C}$. The matrix entry corresponding to a weight $\mu_i \in i\mathfrak{t}^*$ is then given by the polynomial $\mathfrak{a}^*_\mathbb{C} \ni x \mapsto S(x+\mu_i)$. Therefore the matrix entries of $p_\gamma(Z_P) = q_\gamma(p(Z_P))$ are $\mathfrak{a}^*_\mathbb{C} \ni x \mapsto P(x - \rho_\mathfrak{h} + \mu_i)$, and the ones of $\tau_{\nu+\rho}(p_\gamma(Z_P)) - \chi_{\kappa,\nu}(Z_P)$ are given by $x \mapsto P(x+\nu-\rho_\mathfrak{t}+\mu_i) - P(\nu-\rho_\mathfrak{t}-\mu_\kappa)$. The Taylor expansion of P at $\nu - \rho_\mathfrak{t} - \mu_\kappa$ yields

$$P(x+\nu-\rho_\mathfrak{t}+\mu_i) - P(\nu-\rho_\mathfrak{t}-\mu_\kappa) = P'(\nu-\rho_\mathfrak{t}-\mu_\kappa)(x+\mu_i+\mu_\kappa) + Q_\nu(x+\mu_i+\mu_\kappa) \ ,$$

where $P'(\nu-\rho_\mathfrak{t}-\mu_\kappa)$ is viewed as a linear form on $\mathfrak{h}^*_\mathbb{C}$ and the polynomial Q_ν is formed by partial derivatives of P at $\nu-\rho_\mathfrak{t}-\mu_\kappa$ of degree at least 2. Therefore the degree of Q_ν w.r.t. ν is bounded by $d-2$. Using that the degree of $\nu \mapsto P'(\nu-\rho_\mathfrak{t}-\mu_\kappa) - P'(\nu)$ is also bounded by $d-2$ we conclude that the same is true for

$$\nu \mapsto P(x+\nu-\rho_\mathfrak{t}+\mu_i) - P(\nu-\rho_\mathfrak{t}-\mu_\kappa) - P'(\nu)(x+\mu_i+\mu_\kappa) \ .$$

It follows that the degree of the $\mathcal{L}_\gamma$-valued polynomial

$$\nu \mapsto \tau_{\nu+\rho}(p_\gamma(Z_P)) - \chi_{\kappa,\nu}(Z_P) - q_\gamma \circ \tau_{\mu_\kappa}(P'(\nu))$$

is at most $d-2$. Since we have already estimated the degree of $\nu \mapsto \tau_{\nu+\rho}(b_\gamma(Z_P))$ the proof of the lemma is now complete. □

4.14 It is now easy to finish the proof of Proposition 4.6. Let J_P be as in Lemma 4.13. Put $\lambda \in \mathfrak{a}^*$ and $T \in [V_\kappa \otimes V_\gamma]^K$. We form the corresponding elements ψ_T, δ_T in $H^{\kappa,i\lambda}_{\pm\infty} \otimes V_\gamma$. Then we have by Corollary 4.12

$$\left(q_\gamma \circ \tau_{\mu_\kappa}(P'(i\lambda)) + J_P(i\lambda)\right)(R(\psi_T) \otimes \delta_T) = 0 \ .$$

Now we apply formula (4.7).

5 The relation of the microlocal and representation theoretic lifts

5.1 The discussion in the present section is completely parallel to [7], Sec. 5.4. Its goal is to provide the link between the microlocal and the representation theoretic lifts. The main result is Corollary 5.2.

5.2 Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Then we can write the cotangent bundle of $M := \Gamma\backslash G/K$ as $T^*M = \Gamma\backslash G \times_K \mathfrak{p}^*$. We let $\pi : \Gamma\backslash G \times \mathfrak{p}^* \to T^*M$ denote the projection.

The Killing form of $\mathfrak{g}$ restricts to a metric on $\mathfrak{p}$ which is K-invariant. It induces a Riemannian metrics on TM and T^*M. We further get an orthogonal decomposition $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. This induces an embedding $\mathfrak{a}^* \hookrightarrow \mathfrak{p}^*$.

Let $SM \subset T^*M$ denote the unit cosphere bundle. Then π restricts to a map

$$q : \Gamma\backslash G \times S(\mathfrak{a}^*) \to SM \ .$$

5.3 We now consider a unitary representation $\gamma \in \hat{K}$. It gives rise to a bundle $V(\gamma) := \Gamma\backslash G \times_K V_\gamma$ over M. Let $p : SM \to M$ be the projection. We consider the identification

$$\Gamma\backslash G \times S(\mathfrak{a}^*) \times V_\gamma \to q^* \circ p^* V(\gamma)$$

defined such that $(\Gamma g, \lambda, v)$ corresponds to the point $[\Gamma g, v] \in V(\gamma)$ in the fibre of $q^* \circ p^* V(\gamma)$ over $(\Gamma g, \lambda)$.

In a similar manner we obtain an identification

$$\Gamma\backslash G \times S(\mathfrak{a}^*) \times \mathtt{End}(V_\gamma) \to q^* \circ p^* \mathtt{End}(V(\gamma)) \ .$$

5.4 Let $f \in C(SM, p^*\mathtt{End}(V_\gamma))$. Then $q^* f \in C(\Gamma\backslash G \times S(\mathfrak{a}^*)) \otimes \mathtt{End}(V_\gamma)$. For $\lambda \in S(\mathfrak{a}^*)$ we define $\tilde{f}_\lambda \in A_\gamma$ to be the restriction of $q^* f$ to $\Gamma\backslash G \times \{\lambda\}$. The map $f \mapsto \tilde{f}_\lambda$ is a homomorphism of C^*-algebras

$$I_\lambda : C(SM, p^*\mathtt{End}(V_\gamma)) \to A_\gamma \ .$$

5.5 We now consider a regular element $l \in L(\kappa) \subset S(\mathfrak{a}^*)$ and $T \in [V_\kappa \otimes V_\gamma]^M$ with $\|T\| = 1$. Let ξ_n be an l-conveniently arranged sequence giving rise to a representation theoretic lift $\sigma \in V(l, \kappa, \gamma, T)$.

We have a sequence of normalized vectors $\xi_n(\psi_T) \in L^2(\Gamma \backslash G \times_K V_\gamma)$. These sections are in fact smooth. They give rise to functionals $\sigma_{\xi_n(\psi_T)}$ on $C(SM, p^*\mathtt{End}(V_\gamma))$ (see 2.4). After taking a subsequence we can and will assume that the sequence $\sigma_{\xi_n(\psi_T)}$ converges weakly to some limit state, which we denote by σ_{micro} here. It is the microlocal lift associated with the family of eigensections $\xi_n(\psi_T) \in L^2(\Gamma \backslash G \times_K V_\gamma)$ considered in Section 2.

On the other hand we have functionals $\sigma^{\xi_n}_{\psi_T, \delta_T}$ on $A_{\gamma,K}$ defined in (3.1). In fact, the same discussion as in 3.11 shows that for K-finite f and ϕ one can extend $\sigma^{\xi}_{\phi,\psi}(f)$ to distributions ψ.

5.6 Let $o(1)$ denote a quantity which tends to zero as n tends to infinity.

Theorem 5.1 *Assume that $f \in C(SM, p^*\mathtt{End}(V_\gamma))$ is such that $\tilde{f}_\lambda$ is K-finite. Then we have*

$$\sigma_{\xi_n(\psi_T)}(f) = \sigma^{\xi_n}_{\psi_T, \delta_T}(\tilde{f}_l) + o(1) .$$

Corollary 5.2 *We have $\sigma_{micro} = I_l^*(\sigma)$. In particular, σ_{micro} is supported on $q(\Gamma \backslash G \times \{l\})$.*

5.7 The idea of the proof is to verify the theorem on the symbols of pseudodifferential operators $D(d, U, b)$ defined below. This is the contents of Proposition 5.3. Then we show σ_{micro} is supported on $q(\Gamma \backslash G \times \{l\})$ (Lemma 5.6). Finally we use that these symbols span a dense subspace of $C(q(\Gamma \backslash G \times \{l\}), p^*\mathtt{End}(V_\gamma))$ (Lemma 5.7).

5.8 We start with the construction of the family $D(d, U, b)$ of zero order pseudodifferential operators on $C^\infty(M, V(\gamma))$, where $U \in \mathcal{U}(\mathfrak{g})^{\leq d}$ and $b \in C^{\infty.K}(\Gamma \backslash G) \otimes \mathtt{End}(V_\gamma)$. Let $\Omega \in \mathcal{Z}(\mathfrak{g})$ be the Casimir operator. Note that $\Omega + i$ is invertible on $L^2(M, V(\gamma))$ so that we can define $(\Omega + i)^{-d/2}$ by the spectral theorem. We identify $L^2(M, V(\gamma))$ with the subspace of K-invariants $[L^2(\Gamma \backslash G) \otimes V_\gamma]^K$. Let $I_K : L^2(\Gamma \backslash G) \otimes V_\gamma \to L^2(M, V(\gamma))$ denote the orthogonal projection. It is given by

$$I_K(f)(\Gamma g) = \int_K \gamma(k) f(\Gamma g k) .$$

Then we define

$$D(d,U,b) := I_K \circ M_b \circ (R(U) \otimes 1) \circ (\Omega + i)^{-d/2} .$$

Here $R(U)$ denotes the right-regular action of U on $C^\infty(\Gamma\backslash G)$, and M_b is the multiplication by b, i.e. $(M_b f)\Gamma g) = b(\Gamma g) f(\Gamma g)$.

The composition $I_K \circ M_b \circ (R(U) \otimes 1)$ is a differential operator of order $\le d$. Since $(\Omega + i)^{-d/2}$ is a pseudodifferential operator of order $-d$ we conclude that $D(d,U,b)$ is a pseudodifferential operator of order zero.

5.9 In this paragraph we compute the symbol of $D(d,U,b)$. We consider the symmetrization map $sym : Sym(\mathfrak{p}) \to U(\mathfrak{g})$ defined on the degree r-subspace by

$$sym(X_1 \otimes \cdots \otimes X_r) = \frac{1}{r!} \sum_{\sigma \in S_r} X_{\sigma(1)} \ldots X_{\sigma(r)} .$$

It extends to an isomorphism of vector spaces

$$\Phi := mult(sym \otimes \mathtt{id}) : Sym(\mathfrak{p}) \otimes U(\mathfrak{k}) \to U(\mathfrak{g}) .$$

This map preserves the filtrations by degree on both sides. There exists a uniquely determined $u \in S(\mathfrak{p})^d$ and $r \in S(\mathfrak{p})^{\le d-1} \otimes U(\mathfrak{k})$ such that $\Phi(u \otimes 1 + r) = U$. Note that $U - \Phi(u \otimes 1)$ acts as a differential operator of order $\le d - 1$.

We can now compute the symbol of $I_K \circ M_b \circ (R(U) \otimes 1)$. At the point $[\Gamma g, \lambda] \in SM = \Gamma\backslash G \times_K S(\mathfrak{p}^*)$ it is given by

$$s(D(d,U,b)(\Gamma g, \lambda)) = \int_K u(\lambda^{k^{-1}}) \gamma(k) b(\Gamma g k) \gamma(k)^{-1} \in \mathtt{End}(V_\gamma) .$$

Note that $s(\widetilde{D(d,U},b))_\lambda$ is K-finite.

Proposition 5.3 *The assertion of Theorem 5.1 holds true for the functions of the form* $s(D(d,U,b)) \in C^\infty(SM, p^*\mathtt{End}(V_\gamma))$.

5.10 We know already that

$$< \xi_n(\psi_T), D(d,U,b) \xi_n(\psi_T) >= \sigma_{\xi_n(\psi_T)}\big(s(D(d,U,b))\big) + o(1) .$$

In order to prove the proposition we rewrite the left hand side. Note that

$$\xi_n(\psi_T)(\Gamma g\Omega) = (\lambda_n^2 + c(\kappa))\xi_n(\psi_T)(\Gamma g) ,$$

where $\xi_n \in H_{-\infty}^{\kappa,i\lambda_n}$ determines λ_n, and $c(\kappa) \in \mathbb{C}$ is some constant independent of n. Therefore we have

$$(\Omega + i)^{-d/2}\xi_n(\psi_T) = (\lambda_n^2 + c(\kappa) + i)^{-d/2}\xi_n(\psi_T) .$$

We thus get

$$\begin{aligned}
&< \xi_n(\psi_T), D(d,U,b)\xi_n(\psi_T) > \\
&= (\lambda_n^2 + c(\kappa) + i)^{-d/2} \int_{\Gamma\backslash G}\int_K < \xi_n(\psi_T)(\Gamma g), \gamma(k)b(\Gamma gk)\xi_n(\psi_T)(\Gamma gkU) > \\
&= (\lambda_n^2 + c(\kappa) + i)^{-d/2} \int_{\Gamma\backslash G}\int_K < \xi_n(\psi_T)(\Gamma g), \gamma(k)b(\Gamma gk)\gamma(k)^{-1}\xi_n\left(\pi(U^k)\otimes 1)\psi_T\right)(\Gamma g) > ,
\end{aligned}$$

where $U^k := \mathtt{Ad}(k)(U)$ and $\pi := \pi^{\kappa,i\lambda_n}$.

5.11 We now use that $\int_K(\pi(k)\otimes\gamma(k))\delta_T = \psi_T$ in order to write

$$\begin{aligned}
(\pi(U^k)\otimes 1)\psi_T &= (\pi(U^k)\otimes 1)\int_K(\pi(h)\otimes\gamma(h))\delta_T \\
&= \int_K(\pi(h)\otimes\gamma(h))(\pi(U^{h^{-1}k})\otimes 1)\delta_T
\end{aligned}$$

Now we use that

$$(\pi(U^k)\otimes 1)\delta_T = u(\lambda_n^{k^{-1}})\delta_T + \|\lambda_n\|^d o(1) .$$

It follows that

$$(\pi(U^k)\otimes 1)\psi_T = \int_K u(\lambda_n^{k^{-1}h})(\pi(h)\otimes\gamma(h))\delta_T + \|\lambda_n\|^d o(1) .$$

Our final rewriting is

$$\begin{aligned}
&< \xi_n(\psi_T), D(d,U,b)\xi_n(\psi_T) > \qquad (5.4)\\
&= \int_{\Gamma\backslash G}\int_K\int_K < \xi_n(\psi_T)(\Gamma g), \gamma(k)^{-1}b(\Gamma gk^{-1})\gamma(k)u(l^{k^{-1}h})\xi_n((\pi(h)\otimes\gamma(h))\delta_T)(\Gamma g) > +o(1) .
\end{aligned}$$

5.12 We now consider the right-hand side of the equation 5.1. We have

$$\begin{aligned}
&\sigma^{\xi_n}_{\psi_T,\delta_T}(\widetilde{D}(d,U,b)_l)\\
&= \int_{\Gamma\backslash G} < \xi_n(\psi_T)(\Gamma g), s(D,U,b)(\Gamma g,l)\xi_n(\delta_T)(\Gamma g) >\\
&= \int_{\Gamma\backslash G}\int_K\int_K < \gamma(k)\xi_n(\psi_T)(\Gamma gk), \gamma(h)b(\Gamma gh)\gamma(h)^{-1}u(l^{h^{-1}})\xi_n(\delta_T)(\Gamma g) >\\
&= \int_{\Gamma\backslash G}\int_K\int_K < \xi_n(\psi_T)(\Gamma g), \gamma(k^{-1}h)b(\Gamma gk^{-1}h)\gamma(k^{-1}h)^{-1}u(l^{h^{-1}})\xi_n((\pi(k^{-1})\otimes\gamma(k^{-1}))\delta_T)(\Gamma g) >\\
&= \int_{\Gamma\backslash G}\int_K\int_K < \xi_n(\psi_T)(\Gamma g), \gamma(k)^{-1}b(\Gamma gk)\gamma(k^{-1})u(l^{k^{-1}h})\xi_n((\pi(h)\otimes\gamma(h))\delta_T)(\Gamma g) > \qquad (5.5)
\end{aligned}$$

The Proposition 5.3 now follows from the comparison of (5.4) and (5.5). □

5.13

Lemma 5.6 *We have* $\mathtt{supp}(\sigma_{micro}) \subset q(\Gamma\backslash G\times\{l\})$.

Proof. Let $f\in C(SM, p^*\mathtt{End}(V(\gamma))$ be such that $\tilde{f}_l = 0$. We must show that $\sigma_{\xi_n(\psi_T)}(f) = o(1)$.

Let $D(G,\gamma)$ denote the algebra of G-invariant differential operators on $G\times_K V_\gamma$. Note that the operators of $D(G,\gamma)$ descent to $\Gamma\backslash G\times_K V_\gamma$. The right-regular representation induces a homomorphism $\tau : U(\mathfrak{g})^K \to D(G,\gamma)$ such that $(\tau(U)f)(g) := f(gU)$. If we compose τ with the symmetrization map (see 5.9), then we get a linear map

$$D : Sym(\mathfrak{p})^K \to D(G,\gamma) \; .$$

Let $p\in Sym^d(\mathfrak{p})^K$. Then the symbol of the corresponding degree-d differential operator is given by the function $s(D(p))\in C(SM, p^*\mathtt{End}(V_\gamma))$, $s(D(p))([\Gamma g,\lambda]) = p(\lambda)$. There is a Harish-Chandra homomorphism $\Phi_\gamma : D(G,\gamma)\to U(\mathfrak{a})\otimes\mathtt{End}_M(V_\gamma)$ (see [4]). We identify $U(\mathfrak{a})\cong Sym(\mathfrak{a})$ naturally. For $D\in D(G,\gamma)$ we have ([4], Lemma 2.13)

$$D\xi_n(\psi_T) = \xi_n(\psi_{\Phi_\gamma(D)(\lambda_n)\circ T}) \; .$$

In addition, the $\mathtt{End}_M(V_\gamma)$-valued polynomial $\mathfrak{a}^*_{\mathbb{C}}\ni\lambda\mapsto p(\lambda)\mathtt{id}-\Phi_\gamma(D)(\lambda)$ has degree at most $d-1$ ([4], Lemma 2.6). It follows that

$$D(p)\xi_n(\psi_T) = p(\lambda_n)\xi_n(\psi_T) + \|\lambda_n\|^d o(1) \; .$$

Let now $f \in C(SM, p^*\mathrm{End}(V(\gamma))$. Then we have

$$p(l)\sigma_{\xi_n(\psi_T)}(f) = \sigma_{\xi_n(\psi_T)}(s(D(p))f) + o(1) .$$

If $p(l) = 0$, then $\sigma_{\xi_n(\psi_T)}(s(D(p))f) = o(1)$. We now argue as in [7]. If $\tilde{f}_l = 0$, then it can be approximated by products of the form $s(D(p))h$ with $p(l) = 0$ (here one has to use the regularity of l again). This implies the lemma. □

5.14 Sending $[\Gamma g, l]$ to ΓgM identifies $q(\Gamma\backslash G \times \{l\})$ with the double quotient $\Gamma\backslash G/M$. In order to finish the proof of Theorem 5.1 it remains to verify the following lemma.

Lemma 5.7 *The symbols* $s(D(d,U,b))$ *with* $d \geq 0$, $b \in C^K(\Gamma\backslash G) \otimes \mathrm{End}(V_\gamma)$, *and* $U \in \mathcal{U}^{\leq d}(\mathfrak{g})$, *span a dense subspace of* $C(q(\Gamma\backslash G \times \{l\}), p^*\mathrm{End}(V(\gamma))) \cong C(\Gamma\backslash G \times_M \mathrm{End}(V_\gamma))$.

Proof. We have a K-bundle

$$\Gamma\backslash G \times K/M \times \mathrm{End}(V_\gamma) \to \Gamma\backslash G \times_M \mathrm{End}(V_\gamma)$$

given by $(\Gamma g, kM, \Phi) \mapsto [\Gamma gk, \gamma(k^{-1})\Phi\gamma(k)]$. Since l is regular, the functions $k \mapsto u(l^{k^{-1}})$, $u \in S(\mathfrak{p})$, span a dense subspace of $C(K/M)$ (see the proof of Lemma 3.6). Therefore the functions $(\Gamma g, kM) \mapsto b(\Gamma g)u(l^{k^{-1}})$ span a dense subspace of $C(\Gamma\backslash G \times K/M) \otimes \mathrm{End}(V_\gamma)$. It follows that the K-averages $\Gamma g \mapsto \int_K \gamma(k)b(\Gamma gk)\gamma(k)^{-1}u(l^{k^{-1}})$ span a dense subspace of $C(\Gamma\backslash G \times_M \mathrm{End}(V_\gamma))$. □

References

[1] Y. Colin de Verdiere. Ergodicité et fonctions propres du laplacien. *Commun. Math. Phys.* 102 (1985), 497–502. 2.9

[2] E. Lindenstrauss. On quantum unique ergodicity for $\Gamma\backslash\mathbb{H} \times \mathbb{H}$. *Internat. Math. Res. Notices* (2001), 913-933. 1.9

[3] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. To appear in *Ann. of Math.*. 1.3

[4] M. Olbrich. Die Poisson-Transformation für homogene Vektorbündel. Dissertation, Humboldt-Universität zu Berlin, 1995. 5.13

[5] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Commun. Math. Phys.* 161 (1994), 195-213. 1.3

[6] A. Shnirel'man. Ergodic properties of eigenfunctions. *Usp. Math. Nauk* 29 (1974), 181-182. 2.9

[7] L. Silberman and A. Venkatesh. On quantum unique ergodicity for locally symmetric spaces I. Preprint arXiv:math.RT/0407413 (document), 1.3, 1.5, 1.6, 1.7, 4.5, 4.8, 4.11, 5.1, 5.13

[8] M. Taylor. *Pseudodifferential operators*. Princeton University Press, 1981. 2.2

[9] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.* 55 (1987), 919-941. 2.9