Abelian Varieties and the Fourier Mukai transformations (Foschungsseminar 2005)

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Contents

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1 Abelian varieties

1.1 Basic definitions

1.1.1 We consider an algebraically closed field k.

Definition 1.1 *An abelian variety is a group object (X, m, i, e) in the category of algebraic varieties over k such that X is in addition complete.*

Here $m : X \times X \to X$ is a group law, $i : X \to X$ is the inverse, and $e : \mathtt{spec}(k) \to X$ is the identity.

1.1.2 In scheme theoretic terms, X is an integral, separated scheme of finite type over $\mathtt{spec}(k)$ such that the structure morphism $X \to \mathtt{spec}(k)$ is proper (what in this case is the addition condition of being universally closed).

1.1.3 It suffices to require the existence of m and a two sided identity.

1.1.4 The group law is automatically commutative. This follows from the following observation. We consider the conjugation map $C : X \times X \to X$ given (in the language of points by $C(x, y) := m(x, m(m(y, i(x)))))$. Let $\tilde{C}_n : X \to \mathtt{End}(\mathcal{O}_e/m_e^n)$ be the induced map on the nth infinitesimal neighborhood of the identity. Now $\mathtt{End}(\mathcal{O}_e/m_e^n)$ is affine, and any map from a complete variety to an affine variety is locally constant. From this we conclude that $\tilde{C}_n(x) = \mathtt{id}$ for all n and $x \in X(k)$. This then implies that $C = \mathtt{pr}_2$ since X is connected.

1.1.5 X is automatically smooth. In fact, using left translation on shows that the sheaf of relative differentials $\Omega_{X/\mathtt{spec}(k)}$ is locally free.

1.2 Examples of abelian varieties

1.2.1 Let $k = \mathbb{C}$ and consider a k-vector space V together with a lattice U. Then $X = V/U$ is an analytic abelian group. In general it does not come from an algebraic variety.

Theorem 1.2 *X comes from an algebraic variety iff there exists a positive definite hermitean form H on V such that its imaginary part $\mathtt{im}(H)$ is integral on $U \times U$.*

1.2.2 The idea of the proof is as follows. On constructs a line bundle on X with Chern form $\mathtt{im}(H)$. It is fixed by an Appel-Humbert datum (α, H). The pull-back of this bundle to V is trivial. One tries to construct sections downstairs by averaging sections upstairs. The positive definiteness of H implies the convergence of the theta series. They provide enough sections of L for an embedding of X into a projective space.

1.2.3 Let C be a compact Riemann surface of genus g. We consider the space $V := H^0(C, \Omega^1_C)$ of holomorphic one forms on C. It has $\dim(V) = g$. We define a hermitean form on V by

$$H(\omega_1, \omega_2) := 2i \int_C \omega_1 \wedge \bar{\omega}_2 \ .$$

Let $U \subset V$ be the image of $H^1(C, \mathbb{Z})$ under the natural injective map

$$H^1(C, \mathbb{Z}) \to H^1(C, \mathbb{C}) \to H^1(C, \Omega^1_C) \ .$$

Then the imaginary part of H is the intersection form on U and therefore integral.

Let V^* be the $\mathbb{C}$-linear dual of V, and $U^* \subset V$ be the dual of U. The complex torus V^*/U^* comes from an abelian variety $J(C)$, the Jacobian of C.

1.2.4 Let us fix a base point c_0 of C. Then we define the map $\Phi : C \to J(C)$ such that $\Phi(c)$ is the class of the linear map $V \ni \omega \mapsto \int_\gamma \omega \in C$, where γ is a path from c_0 to c. The map does depend on the choice of the path, but the class is independent. We have an induced map

$$\tilde{\Phi} : \underbrace{C \times \cdots \times C}_{g\times} \to J(C)$$

given by $\tilde{\Phi}(c_1, \dots, c_g) = \Phi(c_1) + \cdots + \dots \Phi(c_g)$. This map has degree $g!$ and represents $J(C)$ birationally as a quotient of C^g by the symmetric group S_g.

1.2.5 Let now k be an algebraically closed field. The Jacobian $J(C)$ of a complete smooth curve C over k is an abelian variety. It is birational to the quotient $\mathtt{pr} : C^g \to J(C)$ by the symmetric group. The group law depends on the choice of a positive divisor a on C of degree g. Roughly it can be described as follows. For $x \in J(C)$ the preimage $\tilde{x} := \mathtt{pr}^{-1}(x)$ can be considered as a positive divisor of degree g. Given $x, y \in J(C)$ there exists a unique positive divisor in the one-dimensional system $\tilde{x} + \tilde{y} - a$ which represents $x + y$.

1.3 Line bundles

1.3.1 Let X be an abelian variety over k. For $x \in X(k)$ we have a left translation $T_x : X \to X$.

Definition 1.3 *We define the subgroup*

$$\mathtt{Pic}^0(X) := \{L \in \mathtt{Pic}(X) \mid T_x^* L \cong L \ \forall x \in X(k)\}$$

We have an exact sequence of groups

$$0 \to \mathtt{Pic}^0(X) \to \mathtt{Pic}(X) \to NS(X) \to 0 \ ,$$

where $NS(X)$ is called the Neron-Severi group of X. We will see later that $\mathtt{Pic}^0(X)$ is the group of closed points of an abelian variety $\hat{X}$, the dual of X.

1.3.2 We shall need the following technical result.

Proposition 1.4 (Refined seesaw principle) *Let X be a complete variety, Y be a scheme, and $L \to X \times Y$ be a line bundle. Then there exists a unique closed subscheme $Y_1 \to Y$ such that*

(1) The restriction $L_{|X \times Y_1}$ is isomorphic to $\mathtt{pr}_2^* M$ for some $M \in \mathtt{Pic}(Y_1)$.

(2) If $f : Z \to Y$ is a morphism of schemes such that $(\mathtt{id} \times f)^* L \cong \mathtt{pr}_2^* K$ for some $K \in \mathtt{Pic}(Z)$, then f factors through Y_1.

1.3.3 The idea of the proof is a follows. First one reduces to the case that $Y = \mathtt{spec}(A)$. The subset $F := \{y \in Y(k) | L_{|X \times y} \text{trivial}\}$ is closed. One checks that if $L_{|X \times y}$ is not trivial, then Y_1 does not meet a neighborhood of y. Therefore F is the set of closed points of Y_1 and it remains to discuss the scheme structure.

One shows that there is an A-module M which gives $(\mathtt{pr}_2)_*(L)$ universally (after all possible base changes). For $y \in F$ we see that $\dim_k(M/m_y M) = 1$. After further localization $M \cong A/a$ for some ideal a. Finally one checks that Y_1 is the subscheme corresponding to a.

1.3.4

Theorem 1.5 (Theorem of the cube) *Let X, Y be complete varieties, Z be a connected scheme, and $L \to X \times Y \times Z$ be a line bundle such that $L_{|x_0 \times Y \times Z}$, $L_{|X \times y_0 \times Z}$, and $L_{|X \times Y \times z_0}$ are trivial for points $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Then L is trivial.*

1.3.5 The idea of the proof is the following. Let $Z' \subset Z$ be the maximal subscheme over which L is trivial. We have $z_0 \in Z'$ so that $Z' \neq \emptyset$. Since Z' is closed it suffices to show that Z' is also open. This is achieved by a local consideration. Essentially one must extend a trivialization of $L_{|X \times Y \times z_0}$ to a neighborhood of z_0. The obstruction lies in $H^1(X \times Y \times z_0, L_{|X \times Y \times z_0})$ By the Kuenneth formula, completeness of X, Y, and triviality of $L_{|X \times Y \times z_0}$ we have an injection

$$H^1(X \times Y \times z_0, L_{|X \times Y \times z_0}) \to H^1(X \times y_0 \times z_0, L_{|X \times y_0 \times z_0}) \oplus H^1(x_0 \times Y \times z_0, L_{|x_0 \times Y \times z_0}) \ .$$

By our assumptions the trivialization can be extended to $X \times y_0 \times Z$ and $x_0 \times Y \times Z$. This implies that the obstruction vanishes.

1.3.6 Let $f, g, h : Y \to X$ be maps from a variety to an abelian variety and $L \in \mathtt{Pic}(X)$.

Corollary 1.6 *We have*

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1} \ .$$

1.3.7 In order to show this we apply the theorem of the cube to

$$M = m^*L \otimes m_{12}^*L^{-1} \otimes m_{13}^*L^{-1} \otimes m_{23}^*L^{-1} \otimes \mathtt{pr}_1^*L \otimes \mathtt{pr}_2^*L \otimes \mathtt{pr}_3^*L$$

over $X \times X \times X$.

1.3.8

Theorem 1.7 (Theorem of the square) *Let $L \in \mathtt{Pic}(X)$. Then we have for all $x, y \in X(k)$ that*

$$T_{x+y}^*L \otimes L \cong T_x^*L \otimes T_y^*L \ .$$

1.3.9 The idea of the proof is as follows. We apply 1.6 to $X = Y$ and $f := \mathtt{const}_x$, $g := \mathtt{const}_y$ and $h := \mathtt{id}$.

1.3.10 Let $L \in \mathtt{Pic}^0(X)$.

Lemma 1.8 *On $X \times X$ we have $m^*L \cong \mathtt{pr}_1^*L \otimes \mathtt{pr}_2^*L$.*

1.3.11 This identity holds on $x \times X$ and $X \times x$ for all $x \in X(k)$. Then we apply the seesaw principle.

1.3.12 Let $L \in \mathtt{Pic}(X)$.

Definition 1.9 *We define $\phi_L : X \to \mathtt{Pic}(X)$ by*

$$\phi_L(x) = T_x^*L \otimes L^{-1} \ .$$

1.3.13 One checks that ϕ_L factors over $\mathtt{Pic}^0(X) \subset \mathtt{Pic}(X)$.

1.3.14 We have $\phi_{L\otimes M} = \phi_L + \phi_M$.

1.3.15 The map $\phi_L : X(k) \to \mathtt{Pic}^0(X)$ is a homomorphism.

1.4 Projectivity

1.4.1 Let $L \in \mathtt{Pic}(X)$.

Definition 1.10 *We define*

$$K(L) := \ker(\phi_L)\ .$$

1.4.2 $K(L) \subset X$ is Zariski closed.

1.4.3 Let D be an effective divisor on X and $L = \mathcal{O}(D)$.

Theorem 1.11 *The following assertions are equivalent.*

(1) $H(D) := \{x \in X(k)\,|T_x^*(D) = D\}$ is finite (the equality is equality of divisors).

(2) $K(L)$ is finite.

(3) The linear system $|2D|$ has no base points and defines a finite morphism $X \to \mathbf{P}^N$.

(4) L is ample.

1.4.4 As an illustration we discuss the implication (4) $\to$ (2). If $K(L)$ is not finite, then let $Y \subset K(L)$ be the connected component of 0. It is an abelian variety of positive dimension, and $L_Y := L_{|Y}$ is ample. We have $T_y^* L_Y \cong L_Y$ for all $y \in Y$. It follows that $m^* L_Y \otimes \mathtt{pr}_1^* L_Y^{-1} \otimes \mathtt{pr}_2^* L_Y^{-1}$ is trivial. We pull-back by $(\mathtt{id}, i) : Y \to Y \times Y$ and obtain that $L_Y \otimes i^* L_Y$ is trivial. Now L_Y and $i^* L_Y \cong L_Y^{-1}$ are ample. This is impossible.

1.4.5 Let X be an abelian variety.

Theorem 1.12 *X is projective.*

The idea of the proof is the following. Let $U \subset X$ be an open affine subset containing 0 and $D = X \setminus U$. Then $H(D)$ is a closed subgroup and U is stable under $H(D)$. It follows that $H \subset U$. Since $H(D)$ is complete and U is affine it follows that $H(D)$ is finite. We now apply the conclusion (1) $\to$ (4) of the Theorem 1.11.

1.4.6 Let X be an abelian variety. It admits an ample line bundle L. In this case $K(L)$ is finite.

Definition 1.13 *A polarized abelian variety is a pair (X, L) of an abelian variety and an ample line bundle.*

Any abelian variety admits a polarization.

1.5 The dual variety and the Poincare bundle

1.5.1 Let (X, L) be a polarized abelian variety. Then we define

$$M := m^*L \otimes \mathtt{pr}_1^*(L)^{-1} \otimes \mathtt{pr}_2^*(L)^{-1} .$$

There exists a maximal subscheme $\mathcal{K}(L) \subset X$ such that $M_{|\mathcal{K}(L)\times X} = \mathtt{pr}_1^*(P)$ for some $p \in \mathtt{Pic}(\mathcal{K}(L))$. Maximal means, that for any morphism $f : Z \to X$ such that $(f \times \mathtt{id}_X)^*M = \mathtt{pr}_1^*P$ for some $P \in \mathtt{Pic}(\mathcal{K})$ the map f factors over $\mathcal{K}(L)$.

1.5.2 $\mathcal{K}(L)$ is a subgroup scheme. Therefore it acts freely on X. We have $\mathcal{K}(L)(k) = K(L)$ as groups.

1.5.3

Definition 1.14 *The dual abelian variety $\hat{X}$ is defined by the scheme theoretic quotient*

$$\hat{X} := X/\mathcal{K}(L) \ .$$

The canonical morphism $\pi : X \to \hat{X}$ is finite, surjective and flat.

1.5.4 Here are some details of the construction of the quotient. The construction is done locally. So let G be an affine group scheme $G = \mathtt{spec}(R)$ over k, which acts on $V := \mathtt{spec}(A)$. The action is given by $\mu^* : A \to R \otimes_k A$. We consider

$$A^G := \{a \in A | \mu^*(a) = 1 \otimes a\} \ .$$

Then one verifies that $\mathtt{spec}(A) \to \mathtt{spec}(A^G)$ has the required properties of a quotient.

1.5.5 If $\mathtt{char}(k) = 0$, then $\mathcal{K}(L)$ is the group scheme corresponding to the finite group $K(L)$. In particular $\hat{X} = X/K(L)$.

1.5.6 If $k = \mathbb{C}$, $X = V/U$ and L corresponds to a Hermitan form H with $\mathtt{im}(H)$ integral, then we have $\hat{X} = V/U^\perp$, where $U^\perp = \{v \in V | \mathtt{im}(H)(v, U) \subset \mathbb{Z}\}$. Then we have $U \subset U^\perp$ and $K(L) \cong U^\perp/U$.

1.5.7 We want to define the Poincare bundle $P \in \mathtt{Pic}(\hat{X} \times X)$ such that $\pi^* P = M$.

Since $\mathcal{K}(L) \times \{0\} \subset X \times X$ it acts freely with quotient $\hat{X} \times X$ there is a one-to one correspondence of $\mathcal{K}(L) \times \{0\}$-equivariant sheaves on X and sheaves on $\hat{X} \times X$.

We must lift the action of $\mathcal{K}(L) \times \{0\}$ to M. We use the language of S-valued points of $\mathcal{K}(L)$. Let the subscript $_S$ denote objects obtained by base extension. For $x \in \mathcal{K}(L)(S)$ we have a $\tilde{P} \in \mathtt{Pic}(S)$ such that $(M_S)_{x \times X_S} \cong \mathtt{pr}_1^* \tilde{P}$. This is equivalent to $T_x^* L_S \cong L_S \otimes \tilde{P}$

We calculate

$$T^*_{(x,0)} M_S \cong m_S^* T_x^*(L_S) \otimes \mathtt{pr}_1^* T_x^*(L_S)^{-1} \otimes \mathtt{pr}_2^*(L_S)^{-1} \cong m_S^* \tilde{P} \otimes \mathtt{pr}_1^*(\tilde{P})^{-1} \otimes M_S \cong M_S \ .$$

This isomorphism is uniquely fixed by its restriction to $X_S \times_S 0_S$. We have a canonical isomorphism

$$(M_S)_{|X_S \times 0_S} \cong L_S \otimes L_S^{-1} \otimes V_S \cong V_S \ ,$$

(identifying $X_S \cong X_S \times_S 0_S$ on the right-hand side), where $V := \mathcal{O}_X \otimes_k \mathcal{O}_0/m_0$. Note that T_x acts canonically on the trivial bundle V_S. We define the isomorphism $T^*_{(x,0)} M_S \cong M_S$ such that it induces this canonical action on V_S.

Definition 1.15 *We define the Poicaré bundle $P \to \hat{X} \times X$ as the quotient of M by the action of $\mathcal{K}(L) \times 0$ constructed above.*

1.6 The universal property

1.6.1 The universal property of the Poincaré bundle can formally be phrased as follows. We consider the so-called Picard functor B on the category of schemes S over k which associates to each S the set $B(S)$ of isomorphism classes of line bundles L over $X \times S$ such that $L_{|0 \times S}$ is trivial and $L_{|X \times s} \in \mathtt{Pic}^0(X)$ for all $s \in S(k)$.

Theorem 1.16 *The dual abelian variety represents the functor B, and the Poicaré bundle $P \to X \times \hat{X}$ induces a natural isomorphism $\mathtt{Map}(\dots, \hat{X}) \cong B(\dots)$.*

1.6.2 We indicate the proof. We consider $M := \mathtt{pr}^*_{13} P \otimes \mathtt{pr}^*_{12}(L)^{-1}$ over $X \times S \times \hat{X}$. Then we let $\Gamma_S \subset S \times \hat{X}$ be the maximal subscheme on which M is the pull-back of a bundle over Γ_S. Then we show that Γ_S is the graph of a well-defined map $f : S \to \hat{X}$. Then $(\mathtt{id}_X \times f)^* P \cong L$. The uniqueness assertion of the universal property follows from the construction. To show that Γ_S is a graph we must show that $\mathtt{pr}_2 : \Gamma_S \to S$ is an isomorphism. This is a longer argument.

2 The Fourier-Mukai transformation

2.1 Cohomology of Line bundles

2.1.1 Let X be an abelian variety and let $L \in \mathtt{Pic}^0(X)$.

Proposition 2.1 *If for some $p \geq 0$ we have $H^p(X, L) \neq 0$, then L is trivial.*

2.1.2 The idea of the proof is the following. First one shows that $H^0(X, L) \neq 0$ implies that $L = \mathcal{O}_X$. To this end we employ that then also $H^0(X, i^*L) \neq 0$ and $i^*L = L^{-1}$ (i is the inversion on X).

Assume now that L is not trivial. Then we consider the composition

$$X \stackrel{x \mapsto (x,0)}{\to} X \times X \stackrel{m}{\to} X \ .$$

It induces the identity on cohomology of line bundles. On the other hand by the Kuenneth formula since $m^*L \cong \mathtt{pr}_1^*L \otimes \mathtt{pr}_2^*L$ we have

$$H^k(X \times X, m^*L) \cong \sum_{i+j=k} H^i(X, L) \otimes H^j(X, L) \ .$$

This shows that the higher cohomology vanishes, too

2.2 The cohomology of the Poincaré bundle

2.2.1 We consider the Poincaré bundle $P \to X \times \hat{X}$. Let $g = \dim(X)$.

Theorem 2.2 *We have*

$$R(\mathtt{pr}_2)_* P = k(\hat{0})[-g] \ ,$$

the skyscraper sheaf at $\hat{0}$ with fibre $\mathcal{O}_{\hat{X},\hat{0}}/m_{\hat{0}}[-g]$.

2.2.2 Here is the idea of the proof. Since $P_{|X \times \hat{x}}$ is trivial if and only if $\hat{x} = \hat{0}$ we see first that $P_{X \times \hat{x}} \in \mathtt{Pic}^0(X)$ for all $\hat{x} \in \hat{X}$, and second that $R(\mathtt{pr}_2)_* P$ concentrated at $\hat{0}$. We perform the base change for $\mathtt{spec}(\mathcal{O}_{\hat{X},\hat{0}}) \to \hat{X}$. Then $R^i(\mathtt{pr}_2)_* P_{\hat{0}}$ has finite length over $\mathcal{O}_{\hat{X},\hat{0}}$ and is calculated by a complex

$$0 \to K_0 \to \cdots \to K_g \to 0$$

of free finitely generated $\mathcal{O}_{\hat{X},\hat{0}}$-modules.

It is then a general fact (since $\mathcal{O}_{\hat{X},\hat{0}}$ is local regular, of dimension g), that $H^i(K^\cdot) \cong 0$ for $0 \le i < g$.

Finally one calculates $H^g(K^\cdot)$. Since this is the essential calculation for this presentation we give more details then elsewhere. We have an exact sequence

$$0 \to K_0 \to \cdots \to K_g \to N \to 0$$

of $\mathcal{O}_{\hat{X},\hat{0}}$-modules with $N = R^g(\mathtt{pr}_2)_* P_{\hat{0}}$. The same argument as above gives the exact sequence

$$0 \to \hat{K}^g \to \cdots \to \hat{K}_0 \to K \to 0 \ ,$$

where $\hat{K}_i = \mathtt{Hom}_{\mathcal{O}_{\hat{X},\hat{0}}}(K_i, \mathcal{O}_{\hat{X},\hat{0}})$. Since $P_{|X\times\hat{0}}$ is trivial we have

$$0 \to k \to K_0/m_{\hat{0}} \to K_1/m_{\hat{0}} \to \ .$$

It follows that $K/m_{\hat{0}} \cong k$ and $K \cong \mathcal{O}_{\hat{X},\hat{0}}/a$ for some ideal $a \subset m_{\hat{0}}$.

We shall see that $a = m_{\hat{0}}$. To this end we show that $P_{|X\times V(a)}$ is trivial and use the fact that the closed point $\hat{0} \subset \hat{X}$ is the largest subscheme on which P is trivial by the construction of $\hat{X}$. We have

$$H^0(X \times V(a), P_{|X\times V(a)}) \cong \ker(K_0/a \to K_1/a) \cong \mathtt{Hom}_{\mathcal{O}_{\hat{X},\hat{0}}}(K, \mathcal{O}_{\hat{X},\hat{0}}/a) \cong \mathcal{O}_{\hat{X},\hat{0}}/a \ .$$

We see that the restriction

$$H^0(X \times V(a), P_{|X\times V(a)}) \to H^0(X \times V(a), P_{|X\times\hat{0}}) \cong k$$

is surjective ($P_{|X\times\hat{0}}$ is trivial). Let $s \in H^0(X \times V(a), P_{|X\times V(a)})$ be the section which maps to a non-trivial constant section of $P_{|X\times\hat{0}}$. It induces a trivialization of $P_{|X\times V(a)}$.

We can write $N \cong \mathtt{Ext}^g_{\mathcal{O}_{\hat{X},\hat{0}}}(k, \mathcal{O}_{\hat{X},\hat{0}})$. This extension can also be calculated using the Koszul resolution, since $\mathcal{O}_{\hat{X},\hat{0}}$ is regular local, of dimension g. It follows that

$$\dim_k \mathtt{Ext}^l_{\mathcal{O}_{\hat{X},\hat{0}}}(k, \mathcal{O}_{\hat{X},\hat{0}}) = \frac{g!}{(l-g)!l!} \ ,$$

and in particular, $N \cong k$.

2.3 The Fourier-Mukai transformation

2.3.1 Let $\mathtt{pr}_i$ denote the projections from $X \times \hat{X}$ to the factors.

Definition 2.3 *We define*

$$\mathcal{S} : D(\hat{X}) \to D(X) \ , \mathcal{S}(\dots) := (\mathtt{pr}_1)_*(\mathcal{P} \otimes \mathtt{pr}_2^*(\dots)) \ .$$

We define $\hat{\mathcal{S}} : D(X) \to D(\hat{X})$ by an analogous construction.

2.3.2

Theorem 2.4

$$\mathcal{S} \circ \hat{\mathcal{S}} \cong i^*[-g]$$

2.3.3 The composition is given by $(\mathtt{pr}_1)_*(P * P \otimes \mathtt{pr}_2^*(\dots))$, where $P * P \in D(X \times X)$ is given by $(\mathtt{pr}_{12})_*(\mathtt{pr}_{13}^* P \otimes \mathtt{pr}_{23}^* P)$, and the projections map $X \times X \times \hat{X}$ to the corresponding factors.

We now observe, using the Theorem of the cube, that $\mathtt{pr}_{13}^* P \otimes \mathtt{pr}_{23}^* P \cong (m \times \mathtt{id})^* P$. Then we have

$$(\mathtt{pr}_{12})_*(\mathtt{pr}_{13}^* P \otimes \mathtt{pr}_{23}^* P) = m^*(\mathtt{pr}_2)_* P = m^* k(0)[-g] \ .$$

It follows that $P * P = \mathcal{O}_{\Gamma_i}[-g]$, where Γ_i is the graph of i.

2.3.4 We have $\hat{\mathcal{S}} \circ \mathcal{S} \cong i^*[-g]$.

Corollary 2.5 $\mathcal{S}$ *and* $\hat{\mathcal{S}}$ *are isomorphisms of triangulated categories.*

2.3.5

Theorem 2.6 *We have*

$$\begin{aligned} \mathcal{S} \circ T_{\hat{x}}^* &\cong (\otimes P_{-\hat{x}}) \circ \mathcal{S} \\ \mathcal{S} \circ (\otimes P_x) &\cong T_x^* \circ \mathcal{S} \end{aligned}$$

This is checked by a direct calculation.

2.3.6 Let X be an abelian variety.

Definition 2.7 *We define the convolution* $D(X) \times D(X) \to D(X)$ *by*

$$(\dots) * (\dots) := m_*(\mathtt{pr}_1^*(\dots) \otimes \mathtt{pr}_2^*(\dots)) \ .$$

2.3.7 The Fourier-Mukai transformations is compatible with the tensor product in the following sense.

Theorem 2.8 *There exist natural equivalences of functors*

$$\begin{aligned} \mathcal{S} \circ ((\dots) * (\dots)) &\cong \mathcal{S}(\dots) \otimes \mathcal{S}(\dots) \\ \mathcal{S} \circ ((\dots) \otimes (\dots)) &\cong \mathcal{S}(\dots) * \mathcal{S}(\dots) \end{aligned}$$

This is again checked by a direct calculation.

2.4 Principally polarized abelian varieties and $\widetilde{SL(2,\mathbb{Z})}$-action

2.4.1 Let (X, L) be a polarized abelian variety and $\phi_L : X \to \hat{X}$.

Lemma 2.9 *We have*

$$\deg(\phi_L) = \chi(L)^2 \ .$$

2.4.2 We have

$$(1 \times \phi_L)^* P \cong m^* L \otimes \mathtt{pr}_1^* L^{-1} \otimes \mathtt{pr}_2^* L^{-1} \ .$$

$R\mathtt{pr}_1^*(1 \times \phi_L)^* P$ is supported in the finite subset $K(L) \subset \hat{X}$. Thus

$$R^i(\mathtt{pr}_1)_*(1 \times \phi_L)^* P \cong R^i(\mathtt{pr}_1)_*(m^* L \otimes \mathtt{pr}_2^* L^{-1}) \ .$$

It follows that

$$\chi((1\times\phi_L)^*P) = \chi(m^*L\otimes \mathtt{pr}_2^*L^{-1})\ .$$

We know by 2.2 that $\chi(P)=(-1)^g$. Now $(m,\mathtt{pr}_2): X\times X\to X\times X$ is an isomorphism Thus

$$\chi((1\times\phi_L)^*P) = \chi(\mathtt{pr}_1^*L\otimes \mathtt{pr}_2^*L^{-1}) = \chi(L)\chi(L^{-1}) = (-1)^g\chi(L)^2\ .$$

It follows that

$$(-1)^g\deg(\phi_L) = \deg(\Phi)\chi(P) = \chi((1\times\phi_L)^*P) = (-1)^g\chi(L)^2\ .$$

2.4.3

Definition 2.10 *We say that (X,L) is principally polarized if $\chi(L)=1$.*

Corollary 2.11 *If X admits a principal polarization, then $\hat{X}\cong X$.*

2.4.4 Let (X,L) be principally polarized.

Lemma 2.12 *We have $\mathcal{S}(L)\cong L^{-1}$.*

2.4.5 We have $\mathcal{S}(L)\cong(\mathtt{pr}_1)_*(m^*L)\otimes L^{-1}$. Thus we must show that $(\mathtt{pr}_1)_*(m^*L)\cong\mathcal{O}_X$. We have $H^0(X,L)\cong k$ (higher cohomology vanishes). Let s be a generator. m^*s is then a section of m^*L. $m^*s_{|x\times X}$ is a section of T_x^*L, which is again a principal polarization. $H^0(x\times X, m^*L_{|x\times X})$ is generated by $m^*s_{|x\times X}$. It follows that $(\mathtt{pr}_1)_*(m^*L)\cong\mathcal{O}_X$.

2.4.6 Let (X,L) be principally polarized.

Lemma 2.13 *We have a isomorphism of functors*

$$(\dots)*L\cong L\otimes\mathcal{S}(i^*(\dots)\otimes L)\ .$$

2.4.7 Here is the trick. Let $\xi : X \times X \to X \times X$ and $d : X \times X \to X$ be given by $\xi : (x, y) \mapsto (x, x + y)$ and $d : (x, y) \mapsto y - x$. The first step is

$$(\dots) * L \cong (\mathtt{pr}_2)_*(\mathtt{pr}_1^*(\dots) \otimes d^*L) \ .$$

The second step uses

$$d^*L \cong \mathtt{pr}_1^* i^* L \otimes \mathtt{pr}_2^* L \otimes P \ .$$

2.4.8 Let (X, L) be principally polarized. Then we have $\mathcal{S} : D(X) \to D(X)$. It follows $\mathcal{S}^2 \cong i^*[-g]$ and hence $\mathcal{S}^4 \cong [-2g]$.

2.4.9

Lemma 2.14 *We have*

$$(L \otimes \mathcal{S})^3(\dots) = (\dots)[-g] \ .$$

This is by a calculation using the Lemmas above.

2.4.10 Let (X, L) be principally polarized. The group $SL(2, \mathbb{Z})$ is generated by S, T with relations $(TS)^3 = 1$, $S^4 = 1$. We define a cental extension

$$0 \to \mathbb{Z} \to \widetilde{SL(2, \mathbb{Z})} \to SL(2, \mathbb{Z}) \to 0$$

by $(TS)^3 = -g$, $S^3 = -2g$. This group acts on $D(X)$ such that S acts as $\mathcal{S}$, T acts as $L \otimes (\dots)$, and the center acts by shifts.

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