# Lecture course on coarse geometry

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## Contents



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# 1 Bornological spaces

In this section we introduce the notion of a bornological space. We show that the category of bornological spaces and proper maps is cocomplete and almost cocomplete. We explain how bornologies can be constructed and how they are applied.

Let $X$ be a set. By $\mathcal{P}_X$ we denote the power set of $X$. Let $\mathcal{B}$ be a subset of $\mathcal{P}_X$.

**Definition 1.1.** *$\mathcal{B}$ is called a bornology if it has the following properties:*

1. *$\mathcal{B}$ is closed under taking subsets.*
2. *$\mathcal{B}$ is closed under forming finite unions.*
3. $\bigcup_{B\in\mathcal{B}} B = X$.

The elements of the bornology are called the bounded subsets of $X$. 2 in Definition 1.1

**Definition 1.2.** *$\mathcal{B}$ is a generalized bornology if it satisfies the Conditions 1 and 2 in Definition 1.1.*

Thus we we get the notion of a generalized bornology by dropping Condition 3 in Definition 1.1.

**Remark 1.3.** Let $x$ be in $X$. Then the singleton $\{x\}$ belongs to any bornology. Indeed, by Condition 3 there exists an element $B$ in $\mathcal{B}$ such that $x \in B$. Then $\{x\} \subseteq B$ and hence $\{x\} \in \mathcal{B}$ by Condition 1.

If $\mathcal{B}$ is a generalized bornology, then a point $x$ in $X$ is called bounded if $\{x\} \in \mathcal{B}$. Otherwise it is called unbounded. A generalized bornology is a bornology if and only all points of $X$ are bounded.

If $X$ is a generalized bornological space, then we have a disjoint decomposition $X = X_b \sqcup X_u$ into the subsets of bounded and unbounded points. Then $\mathcal{B}$ becomes a bornology on $X_b$. □

**Remark 1.4.** Let $A$ be an abelian group and consider the abelian group $A^X$ of functions from $X$ to $A$. For $f$ in $A^X$ we let

$$\mathtt{supp}(f) := \{x \in X \mid f(x) \neq 0\}$$

be the support of $f$. We have $\mathtt{supp}(f + f') \subseteq \sup(f) \cup \mathtt{supp}(f')$.

If $\mathcal{B}$ is a generalized bornology on $X$, then we can consider the subset

$$C_{\mathcal{B}}(X, A) := \{f \in A^X \mid \mathtt{supp}(f) \in \mathcal{B}\} \ .$$

We shall see that this is a subgroup of $A^X$. Indeed, if $f, f'$ are in $C_\mathcal{B}(X,A)$, then $\mathtt{supp}(f)$ and $\mathtt{supp}(f)$ belong to $\mathcal{B}$. Since $\mathtt{supp}(f+f') \subseteq \mathtt{supp}(f) \cup \mathtt{supp}(f')$ we see that also $\mathtt{supp}(f+f')$ belongs to $\mathcal{B}$ so that $f+f'$ is in $C_\mathcal{B}(X,A)$. Similarly, $\mathtt{supp}(-f) = \mathtt{supp}(f)$ so that with $f$ also $-f$ belongs to $C_\mathcal{B}(X,A)$.

In this way $\mathcal{B}$ determines a subgroup of $A^X$ of functions with bounded support.

If $A$ is a ring, then $A^X$ is also a ring and $C_\mathcal{B}(X,A)$ is a subring since $\mathtt{supp}(ff') \subseteq \mathtt{supp}(f) \cap \mathtt{supp}(f')$. $\square$

We now turn to examples and constructions of bornologies.

Let $X$ be a set and $(\mathcal{B}_i)_{i\in I}$ be a family of (generalized) bornologies on $X$.

**Lemma 1.5.** *The intersection $\bigcap_{i\in I}\mathcal{B}_i$ is a (generalized) bornology.*

*Proof.* Let $\mathcal{B} := \bigcap_{i\in I}\mathcal{B}_i$. Consider $B$ in $\mathcal{B}$ and let $B'$ be a subset of $B$. Then $B \in \mathcal{B}_i$ for all $i$ in $I$ and hence $B' \in \mathcal{B}_i$ for all $i$ in $I$. Hence $B' \in \mathcal{B}'$.

Assume that $B, B'$ belong to $\mathcal{B}$. Then $B \in \mathcal{B}_i$ and $B' \in \mathcal{B}_i$ for all $i$ in $I$. Hence $B \cup B' \in \mathcal{B}_i$ for all $i$ in $I$ and hence $B \cup B' \in \mathcal{B}$.

This finishes the case of generalized bornologies. For the case of bornologies we consider $x$ in $X$. Then $\{x\} \in \mathcal{B}_i$ for every $i$ in $I$ and hence $\{x\} \in \mathcal{B}$. Hence $x \in \bigcup_{B\in\mathcal{B}} B$. $\square$

Let $\mathcal{A}$ be a subset of $\mathcal{P}_X$. Then there is a smallest bornology containing $\mathcal{A}$ given by

$$\mathcal{B}\langle\mathcal{A}\rangle := \bigcap_{\mathcal{B},\ \mathcal{A}\subseteq\mathcal{B}} \mathcal{B}\ ,$$

where the intersection runs over all bornologies $\mathcal{B}$ on $X$. Similarly there is a smallest generalized bornology containing $\mathcal{A}$

$$\tilde{\mathcal{B}}\langle\mathcal{A}\rangle = \bigcap_{\mathcal{B},\ \mathcal{A}\subseteq\tilde{\mathcal{B}}} \tilde{\mathcal{B}}\ ,$$

where the intersection runs over all generalized bornologies $\mathcal{B}$ on $X$ containing $\mathcal{A}$.

We can describe $\mathcal{B}\langle\mathcal{A}\rangle$ explicitly. We will assume that $\bigcup_{A\in\mathcal{A}} A = X$. Since singletons belong to every bornology (see Example 1.3) we can enlarge $\mathcal{A}$ by singletons without changing $\mathcal{B}\langle\mathcal{A}\rangle$ in order to ensure this condition.

**Lemma 1.6.** *Assume that $\bigcup_{A\in\mathcal{A}} A = X$. Then a subset $B$ in $\mathcal{P}_X$ belongs to $\mathcal{B}\langle\mathcal{A}\rangle$ if and only if of there exists a finite family $(A_i)_{i\in I}$ in $\mathcal{A}$ such that $B \subseteq \bigcup_{i\in I} A_i$.*

*Proof.* We consider the subset $\mathcal{B}'$ of $\mathcal{P}$ of subsets $B$ of $X$ such that there exists a finite family $(A_i)_{i\in I}$ in $\mathcal{A}$ with $B \subseteq \bigcup_{i\in I} A_i$.

One checks that $\mathcal{B}'$ is a bornology which contains $\mathcal{A}$. Therefore $\mathcal{B}\langle\mathcal{A}\rangle \subseteq \mathcal{B}'$. On the other hand, $\mathcal{B}'$ is contained in any other bornology which contains $\mathcal{A}$. This implies that $\mathcal{B}' \subseteq \mathcal{B}\langle\mathcal{A}\rangle$. □

**Remark 1.7.** The generalized bornology $\tilde{\mathcal{B}}\langle\mathcal{A}\rangle$ has a similar description. We let $X_b := \bigcup_{A\in\mathcal{A}} A$. Then $\mathcal{A}$ generates a bornology $\mathcal{B}'$ on $X_b$ whose elements are described as in Lemma 1.6. We have $\tilde{\mathcal{B}}\langle\mathcal{A}\rangle = \mathcal{B}'$.

**Example 1.8.** If $X$ is a set, then it has the minimal bornology $\mathcal{B}_{min}$ of finite subsets and the maximal bornology $\mathcal{B}_{max}$ of all subsets.

It has the empty generalized bornology. □

**Example 1.9.** Let $X$ be a topological space. The set $\mathcal{B}_{qc}$ of subsets of quasi-compact subsets of $X$ is a bornology.

The set $\mathcal{B}_{rc}$ of relatively quasi compact subsets (subsets whose closures are quasi compact) is a generalized bornology. It might happen that the closure of a point is not quasi compact. This can happen of $X$ is not Hausdorff.

If we replace the condition "quasi compact" by "compact", then in general we do not get a bornology since the union of two compact subsets is not necessarily compact. Again this can happen since compactness includes the condition of being Hausdorff and a union of two Hausdorff subsets need not be Hausdorff. □

**Example 1.10.** Let $d$ be a quasi-metric (infinite distances are allowed) on $X$. Then the metrically bounded subsets generate a bornology $\mathcal{B}_d$. Note that we must say "generate" since the union of two bounded sets might be unbounded.

But if $d$ is a metric, then the bounded sets form a bornology since Condition 2 follows from the triangle inequality for the metric.

The bornology $\mathcal{B}_d$ is generated by the set of balls $\{B(x,r) \mid x \in X \;\; r \in [0,\infty)\}$. □

**Example 1.11.** Let $Y$ be a topological space, $A$ be a subset, and $X := Y \setminus A$.

Let $\mathcal{B} := \{Z \subseteq X \mid \bar{Z} \cap A = \emptyset\}$. This is a generalized bornology. If $Y$ is Hausdorff then it is a bornology. □

**Example 1.12.** Recall that a filter on a set $X$ is a subset $\mathcal{F}$ of $\mathcal{P}_X$ with the following properties:

1. $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

2. $\mathcal{F}$ is closed under forming finite intersections.

3. $\mathcal{F}$ is closed under taking supersets.

If $\mathcal{F}$ is a filter on $X$, then the set of complements

$$\mathcal{F}^c := \{X \setminus F \mid F \in \mathcal{F}\}$$

is a generalized bornology on $X$. Vice versa the complements of a generalized bornology $\mathcal{B}$ is a filter if and only if $X \notin \mathcal{B}$, i.e., $\mathcal{B}$ is not the maximal bornology.

The generalized bornology $\mathcal{F}^c$ is a bornology if and only if the filter is free, i.e., if and only if $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Hence, upon taking complements, non-maximal bornologies correspond to free filters on $X$. □

**Example 1.13.** Let $\mathcal{F}$ be a subset of $\mathcal{P}_X$. Then

$$\mathcal{F}^\perp := \{Y \subseteq X \mid (\forall L \in \mathcal{F} \mid |L \cap Y| < \infty\}$$

is a bornology. It is called the dual bornology to $\mathcal{F}$. We have an obvious inclusion $\mathcal{F} \subseteq (\mathcal{F}^\perp)^\perp$. □

**Definition 1.14.** *A (generalized) bornological space is a pair $(X, \mathcal{B})$ of a set with a (generalized) bornology.*

Usually we write $X$ for generalized bornological spaces and let $\mathcal{B}_X$ denote its generalized bornology.

**Example 1.15.** For a set $X$ we write $X_{min}$ and $X_{max}$ for $X$ equipped with the minimal and maximal bornology. We write $X_\emptyset$ for the generalized bornological space with the empty bornology. □

**Example 1.16.** We consider a bornological space $X$ and a subset $L$:

**Definition 1.17.** *$L$ is called locally finite if $|L \cap B| < \infty$ for all $B$ in $\mathcal{B}_X$.*

We let $\mathcal{LF}(X)$ denote the set of locally finite subsets of $X$. By Example 1.13, the subset $\mathcal{LF}(X)$ of $\mathcal{P}_X$ is a bornology on $X$.

We write $(X, \mathcal{LF}(X)) =: X^\perp$. With this notation we have $(X_{min})^\perp = X_{max}$ and $(X_{max})^\perp = X_{min}$. □

Let $f : X \to Y$ be a map between the underlying sets of generalized bornological spaces.

**Definition 1.18.** *$f$ is called:*

1. *proper if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$.*

2. *bornological, if $f(\mathcal{B}_X) \subseteq \mathcal{B}_Y$.*

Let $\mathcal{A}$ be a subset of $\mathcal{P}_Y$. By the following lemma we can check properness of a map on generators.

**Lemma 1.19.**

1. *We assume that* $Y = \bigcup_{A \in \mathcal{A}} A$ *and* $\mathcal{B}_Y = \mathcal{B}\langle \mathcal{A} \rangle$. *Then the map* $f$ *is proper if and only if* $f^{-1}(A) \in \mathcal{B}_X$ *for all* $A$ *in* $\mathcal{A}$.

2. *We assume that* $\mathcal{B}_Y := \tilde{\mathcal{B}}\langle \mathcal{A} \rangle$. *Then the map* $f$ *is proper if and only if* $f^{-1}(A) \in \mathcal{B}_X$ *for all* $A$ *in* $\mathcal{A}$.

*Proof.* We show Assertion 1. If $f$ is proper, then $f^{-1}(A) \in \mathcal{B}_X$ for all $A$ in $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{B}_Y$.

For the converse we assume that $B$ is in $\mathcal{B}_Y$. Then by Lemma 1.6 there exists a finite family $(A_i)_{i \in I}$ in $\mathcal{A}$ such that $B \subseteq \bigcup_{i \in I} A_i$. Then $f^{-1}(B) \subseteq f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i) \in \mathcal{B}_X$ since $f^{-1}(A_i) \in \mathcal{B}_X$ for every $i$ in $I$ and $\mathcal{B}_X$ is closed under forming finite unions.

For Assertion 2 we argue similarly using Remark 1.7. We do not have to assume that $\mathcal{A}$ covers $Y$. □

**Example 1.20.** Let $f : X \to Y$ be a proper map between bornological spaces. Let $A$ be a group. Then the pull-back $f^* : A^Y \to A^X$ restricts to a homomorphism $f^* : C_{\mathcal{B}_Y}(Y, A) \to C_{\mathcal{B}_X}(X, A)$. This follows from the relation $\mathtt{supp}(f^*\phi) \subseteq f^{-1}(\mathtt{supp}(\phi))$ for all $\phi$ in $A^Y$. □

**Example 1.21.** Let $f : X \to Y$ be a map between the underlying sets of bornological spaces. If $f : X \to Y$ is proper, then $f : X^\perp \to Y^\perp$ is bornological. Indeed, let $L$ be in $\mathcal{LF}(X)$ and $B$ be in $\mathcal{B}_Y$. Then $f(L) \cap B = f(L \cap f^{-1}(B))$ is the image under $f$ of a finite subset of $X$ and hence finite. We see hat $f(L) \in \mathcal{LF}(B)$. □

**Example 1.22.** Let $X$ be a bornological space, $Y$ be a set, and $f : Y \to X$ be a map. We set

$$f^{-1}(\mathcal{B}_X) := \{f^{-1}(B) \mid B \in \mathcal{B}_X\} .$$

Then $\mathcal{B}\langle f^{-1}(\mathcal{B}_X) \rangle$ is called the induced bornology. It is the minimal bornology on $Y$ such that $f : X \to Y$ becomes a proper map of bornological spaces.

Similarly one can define the induced generalized bornology $\tilde{\mathcal{B}}\langle f^{-1}(\mathcal{B}_X) \rangle$. It is the minimal generalized bornology such that $f$ becomes a proper map of generalied bornological spaces. If $Y$ contains unbounded points, then the inclusion $\tilde{\mathcal{B}}\langle f^{-1}(\mathcal{B}_X) \rangle \subseteq \mathcal{B}\langle f^{-1}(\mathcal{B}_X) \rangle$ might be proper. □

**Example 1.23.** Let $X$ be a (generalized) bornological space and $f : X \to Y$ be a map of sets. Then we can consider the maximal (generalized) bornology on $Y$ such that $f$ becomes a proper map. This bornology is given by

$$f_*\mathcal{B}_X := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{B}_X\} .$$

We call this bornology the coinduced bornology. □

We let **Born** denote the category of bornological spaces and proper maps. Furthermore, we write $\widetilde{\mathbf{Born}}$ for the category of generalized bornological spaces and proper maps. Our next goal is to study these categories and various functors relating them with the category of sets.

We have a forgetful functors $S : \mathbf{Born} \to \mathbf{Set}$ and $S : \widetilde{\mathbf{Born}} \to \mathbf{Set}$.

**Proposition 1.24.** *We have an adjunctions*

$$(X \mapsto X_{max}) : \mathbf{Set} \leftrightarrows \mathbf{Born} : S \ , \quad (X \mapsto X_{max}) : \mathbf{Set} \leftrightarrows \widetilde{\mathbf{Born}} : S \ .$$

*Proof.* For the case of **Born** on checks the equality

$$\mathtt{Hom}_{\mathbf{Born}}(X_{max}, Y) = \mathtt{Hom}_{\mathbf{Set}}(X, S(Y))$$

for all sets $X$ and bornological spaces $Y$.

The argument for $\widetilde{\mathbf{Born}}$ is similar. □

**Proposition 1.25.** *We have an adjunction*

$$S : \widetilde{\mathbf{Born}} \leftrightarrows \mathbf{Set} : (X \mapsto X_{\emptyset}) \ .$$

*Proof.* One checks the equality

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(S(Y), X) = \mathtt{Hom}_{\mathbf{Set}}(Y, X_{\emptyset})$$

for all sets $X$ and generalized bornological spaces $Y$. □

Note that there is no such adjunction in the case of **Born**.

**Proposition 1.26.** *The categories* **Born** *and* $\widetilde{\mathbf{Born}}$ *are cocomplete.*

*Proof.* Let $X : \mathbf{I} \to \mathbf{Born}$ be a small diagram. We then define the set $Y := \mathtt{colim}_{\mathbf{I}} S(X)$. It comes with a family $(e_i : S(X) \to Y)_{i \in \mathbf{I}}$ of maps of sets exhibiting $Y$ as a colimit of $S(X)$ in **Set**. We equip $Y$ with the intersection of the coinduced bornologies

$$\mathcal{B}_Y := \bigcap_{i \in \mathbf{I}} e_{i,*} \mathcal{B}_{X_i} \ .$$

In other words a subset $A$ of $Y$ belongs to $\mathcal{B}_Y$ if and only $e_i^{-1}(A)$ is bounded for all $i$ in $\mathbf{I}$.

From now on $Y$ denotes the bornological space $(Y, \mathcal{B}_Y)$. Then $e_i : X \to Y$ are morphisms in **Born**. We now check that $(Y, (e_i)_{i \in \mathbf{I}})$ is a colimit of $X$ in **Born**. Let $T$ be in **Born** arbitrary. By construction the family of maps $(e_i)_{i \in \mathbf{I}}$ induces a bijection

$$\mathtt{Hom}_{\mathbf{Set}}(S(Y), S(T)) \stackrel{\cong}{\to} \lim_{\mathbf{I}^{op}} \mathtt{Hom}_{\mathbf{Set}}(S(X), S(T)) \ , \quad g \mapsto (g \circ e_i)_{i \in \mathbf{I}}$$

Since $e_i$ is a morphisms **Born** for every $i$ in $\mathbf{I}$ one first observes that this bijection restricts to a (necessarily injective) map

$$\mathtt{Hom}_{\mathbf{Born}}(Y,S) \to \lim_{\mathbf{I}^{op}} \mathtt{Hom}_{\mathbf{Born}}(X,T) \ .$$

In order to show surjectivity assume that $(f_i : X \to T)_{i\in\mathbf{I}}$ represents an element in $\mathtt{lim}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Born}}(X,T)$ and let $g : S(Y) \to S(T)$ be the corresponding map of underlying sets. We must show that $g \in \mathtt{Hom}_{\mathbf{Born}}(Y,S)$, i.e., that $g$ is proper. If $B$ is in $\mathcal{B}_T$, then $f_i^{-1}(B) = e_i^{-1}(g^{-1}(B))$ is in $\mathcal{B}_{X_i}$ for all $i$ in $I$. By the definition of $\mathcal{B}_Y$ we conclude that $g^{-1}(B) \in \mathcal{B}_Y$. Hence $g$ is proper.

The same argument works for generalized bornological spaces. □

**Example 1.27.** Let $(X_i)_{i\in I}$ be a family of bornological spaces. Then we can describe its coproduct $X := \coprod_{i\in I}$ in **Born** explicitly. The underlying set is the disjoint union $S(X) := \bigsqcup_{i\in I} S(X_i)$ of the underlying sets of the $X_i$. We consider the sets $X_i$ as subsets of $X$. A subset $B$ of $X$ is bounded if and only if $B \cap X_i$ is bounded in $X_i$ for all $i$ in $I$.

For example, for a set $Y$ we have $Y_{max} \cong \bigsqcup_{y\in Y}\{y\}$.

**Proposition 1.28.** *The category* $\widetilde{\mathbf{Born}}$ *is complete.*

*Proof.* Let $X : \mathbf{I} \to \widetilde{\mathbf{Born}}$ be a diagram. We define the set $Y := \mathtt{lim}_{\mathbf{I}} S(X)$. It comes with a family of maps $(p_i : Y \to S(X))_{i\in\mathbf{I}}$ exhibiting $Y$ as a limit of $S(X)$. We equip $Y$ with the generalized bornology

$$\mathcal{B}_Y := \tilde{\mathcal{B}}\langle \bigcup_{i\in\mathbf{I}} p_i^{-1}(\mathcal{B}_{X_i})\rangle \ .$$

Then $p_i : Y \to X$ becomes a morphism in $\widetilde{\mathbf{Born}}$ for every $i$ in $\mathbf{I}$. One now checks that $(Y,(p_i)_{i\in\mathbf{I}})$ is a limit of $X$. Let $T$ be in $\widetilde{\mathbf{Born}}$. By construction the family $(p_i)_{i\in\mathbf{I}}$ induces a bijection

$$\mathtt{Hom}_{\mathbf{Set}}(S(T),S(Y)) \xrightarrow{\cong} \lim_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Set}}(S(T),S(X)) \ , \quad g \mapsto (p_i \circ g)_{i\in\mathbf{I}}$$

Since $p_i$ is a morphism in $\widetilde{\mathbf{Born}}$ for every $i$ in $\mathbf{I}$ it restricts to a (necessarily injective) map

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T,Y) \to \lim_{\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T,X) \ .$$

In order to show surjectivity we consider a family $(f_i : T \to X_i)_{i\in\mathbf{I}}$ in $\mathtt{lim}_{\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T,X)$ and let $g : S(T) \to S(Y)$ be the corresponding map of underlying sets.

We must show that $g$ is proper. We use Lemma 1.19 in order to check properness on generators. Fix $i$ in $\mathbf{I}$ and assume that $B$ is bounded in $X_i$. Then $p_i^{-1}(B)$ is a typical enerator of the generalized bornology of $Y$. Then $g^{-1}(p_i^{-1}(B)) = f_i^{-1}(B)$ is bounded in $T$ since $f_i$ is proper.

We conclude that $g$ is proper. □

The empty limit in $\widetilde{\mathbf{Born}}$ is a final object $*_\emptyset$. It does not belong to **Born**.

**Proposition 1.29.** *The category* **Born** *admits all non-empty limits.*

*Proof.* The same argument as for Proposition 1.28 works. In this case $\mathcal{B}_{X_i}$ is a bornology for every $i$ in $\mathbf{I}$. The condition $\mathbf{I} \neq \emptyset$ ensures that every point in $Y$ belongs to $\tilde{\mathcal{B}}\langle\bigcup_{i\in\mathbf{I}} p_i^{-1}(\mathcal{B}_{X_i})\rangle$ so that this generalized bornology is a bornology. □

**Example 1.30.** Let $(X_i)_{i\in I}$ be a family of bornological spaces. Then we can describe its cartesian product $X := \prod_{i\in I} X_i$ explicitly. The underlying set of the product is the cartesian product $S(X) := \prod_{i\in I} S(X_i)$ of underlying sets. The bornology on $X$ is generated by the cylinder sets $p_i^{-1}(B)$ for all $i$ in $I$ and $B$ in $\mathcal{B}_{X_i}$. □

The categories **Born** and $\widetilde{\mathbf{Born}}$ have a symmetric monoidal structure which will be denoted by $\otimes$. It will be obtained from the cartesian product of the underlying sets by equipping the products with bornology specified as follows:

**Definition 1.31.** *We define the functor*

$$- \otimes - : \widetilde{\mathbf{Born}} \times \widetilde{\mathbf{Born}} \to \widetilde{\mathbf{Born}}$$

*such that it sends $X, X'$ in $\widetilde{\mathbf{Born}}$ to the set $X \times X'$ with the bornology generated by $B \times B'$ for all $B$ in $\mathcal{B}_X$ and $B'$ in $\mathcal{B}_{X'}$.*

If $f : X \to Y$ and $f' : X' \to Y'$ are proper, then $f \otimes f' : X \otimes X' \to Y \otimes Y'$ is again proper since it is obvious that preimages of generators are again generators. The space $\{*\}_{\max}$ is the tensor unit of this structure.

The tensor structure on $\widetilde{\mathbf{Born}}$ restricts to a structure on **Born**.

We have a morphism $X \times X' \to X \otimes X'$ given by the identity of the underlying sets, but this map is general not a morphism except if both $X'$ and $X'$ are bounded.

**Example 1.32.** For sets $X, Y$ we have $X_{min} \otimes Y_{min} \cong (X \times Y)_{min}$.

We have $X \otimes \{*\}_\emptyset \cong S(X)_\emptyset$ which shows that $\{*\}_\emptyset$ does not act as a tensor unit. □

# 2 Coarse spaces

In this section we introduce the category of coarse spaces and proper map. We show that it is complete and cocomplete. We explain various ways how coarse structures can appear, and how they are used to define subalgebras of matrix algebras.

Let $X$ be a set. We call subsets $U$ of $X \times X$ entourages. The diagonal $\mathtt{diag}(X)$ is an example of an entourage. For an entourage $U$ of $X$ we define its inverse by

$$U^{-1} := \{(y,x) \in X \times X \mid (x,y) \in U\} \ .$$

If $V$ is a second entourage, then we define the composition of $V$ and $U$ by

$$V \circ U := \{(x,z) \in X \times X \mid (\exists y \in X \mid (x,y) \in V \text{ and } (y,z) \in U\} \ .$$

We have the relations

$$\mathtt{diag}(X) \circ U = U \circ \mathtt{diag}(X) = U \ , \quad (W \circ V) \circ U = W \circ (V \circ U) \ .$$

Let $U$ be an entourage of $X$ and $B$ be a subset of $X$. Then we call the subset

$$U[B] := \{x \in X \mid (\exists b \in B \mid (x,b) \in U)\}$$

of $X$ the $U$-thickening of $B$.

Write $x \sim_U y$ if $(x,y) \in U$. So $U[Y]$ consists of all points $x$ in $X$ such that $x \sim_U y$.

Let $Y, Z$ be subsets of $X$ and $U$ be an entourage on $X$.

**Definition 2.1.** *We say that $Y$ is $U$-separated from $Z$ if $Y \cap U[Z] = \emptyset$.*

This means that there is no pair of points $y$ in $Y$ and $z$ in $Z$ such that $y \sim_U z$.

**Example 2.2.** Let $X$ be a set with subsets $Y$ and $Z$. Let $U$ and $V$ be entourages of $X$. Then we have the following assertions.

If $Y$ is $U$-separated from $Z$, then $Z$ is $U^{-1}$-separated from $Y$. Indeed $y \sim_U z$ is equivalent to $z \sim_{U^{-1}} y$.

If $Y$ is $V \circ U$-separated from $Z$, then $Y$ is $V$-separated from $U[Z]$. If $Y$ were not $V$-separated from $U[Z]$, then there exists $y$ in $Y$, $x$ in $X$ and $z$ in $Z$ such that $y \sim_V x$ and $x \sim_U z$. But then $y \sim_{V \circ U} z$. □

**Example 2.3.** Let $U$ be an entourage on $X$. Then $U$ is an equivalence relation if and only if

1. $\mathtt{diag}(X) \subseteq U$
2. $U = U^{-1}$
3. $U \circ U = U$.

In this case, for $x$ in $X$ the set $U[\{x\}]$ is the equivalence class of $x$. □

**Example 2.4.** Let $f : X \to X$ be a map. Then the graph of $f$

$$\mathrm{graph}(f) := \{(f(x), x) \mid x \in X\}$$

is an entourage. We have the relations

$$\mathrm{graph}(f' \circ f) = \mathrm{graph}(f') \circ \mathrm{graph}(f)$$

and if $f^{-1}$ exists, also

$$\mathrm{graph}(f^{-1}) = \mathrm{graph}(f)^{-1} \ .$$

Note that

$$\mathrm{graph}(\mathtt{id}_X) = \mathtt{diag}(X) \ .$$

Entourages can be considered as generalized maps, which may be multivalued and not everwhere defined. □

Let $\mathcal{C}$ be a subset of $\mathcal{P}_{X \times X}$.

**Definition 2.5.** *$\mathcal{C}$ is called a coarse structure if it has the following properties:*

1. $\mathtt{diag}(X) \in \mathcal{C}$.
2. *$\mathcal{C}$ is closed under finite unions and taking subsets.*
3. *If $U, V$ are in $\mathcal{C}$, then $V \circ U \in \mathcal{C}$.*
4. *If $U$ is in $\mathcal{C}$, then $U^{-1} \in \mathcal{C}$.*

The elements of $\mathcal{C}$ are called the coarse entourages of $X$.

**Remark 2.6.** The notion of a coarse structure and the main ideas of coarse geometry as presented here have been invented by John Roe [Roe93].

**Example 2.7.** Let $X$ be a set. Let $R$ be a ring, and consider the $R$-module $R[X]$. We let $[x]$ denote the basis element corresponding to $x$ in $X$. For every subset $Y$ of $X$ we can consider a projection $\mu(Y)$ in $\mathtt{End}(R[X])$ determined by the condition that

$$\mu(y)[x] := \left\{ \begin{array}{cc} [x] & x \in Y \\ 0 & x \notin Y \end{array} \right. \ .$$

Let $A$ be in $\mathtt{End}(R[X])$ and $U$ be an entourage of $X$.

**Definition 2.8.** *We say that $A$ is $U$-controlled if for all pairs of subsets $Y, Z$ of $X$ such that $Y$ is $U$-separated from $Z$ we have $\mu(Y) A \mu(Z) = 0$.*

Let now $\mathcal{C}$ be a coarse structure on $X$.

We consider the subset

$$\mathtt{End}^{\mathcal{C}}(R[X]) := \{A \in \mathtt{End}(R[X]) \mid (\exists U \in \mathcal{C} \mid A \text{ is } U\text{-controlled})\} .$$

The first three axioms of a coarse structure for $\mathcal{C}$ imply that $\mathtt{End}^{\mathcal{C}}(R[X])$ is a subalgebra of $\mathtt{End}(R[X])$.

1. The identity $1_{R[X]}$ is $\mathtt{diag}(X)$-controlled and belongs to $\mathtt{End}^{\mathcal{C}}(R[X])$.

2. If $A, B$ are in $\mathtt{End}^{\mathcal{C}}(R[X])$ and $A$ is $U$-controlled and $B$ is $V$-controlled, then $A+B$ is $U \cup V$ controlled. Indeed, if $Y$ is $U \cup V$-separated from $Z$, then it is $U$- and $V$-separated from $Z$. Hence

$$\mu(Y)(A+B)\mu(Z) = \mu(Y)A\mu(Z) + \mu(Y)B\mu(Z) = 0 + 0 = 0 .$$

3. If $A, B$ are in $\mathtt{End}^{\mathcal{C}}(R[X])$ and $A$ is $U$-controlled and $B$ is $V$-controlled, then $A \circ B$ is $U \circ V$-controlled. Assume that $Y$ is $U \circ V$-separated from $Z$. Let $W := V[Z]$. Then $Y$ is still $U$-separated from $W$, and $X \setminus W$ is $V$-separated from $Z$. Hence using $1_{R[X]} = \mu(W) + \mu(X \setminus W)$ we get

$$\mu(Y)(A \circ B)\mu(Z) = \mu(Y)A\mu(W)B\mu(Z) + \mu(Y)A\mu(X \setminus W)B\mu(Z) = 0 + 0 = 0 .$$

□

**Example 2.9.** Note that the Example 2.7 does not yet motivate the fourth condition that a coarse structure is stable under taking inverses. This is achieved with the following related example.

Let $X$ be a set. We equip $X$ with the counting measure. Then we consider the Hilbert space $L^2(X)$. We have an orthonormal basis $([x])_{x \in X}$. For a subset $Y$ of $X$ we can define the orthogonal projections $\mu(Y)$ in $B(L^2(X))$ as before such that

$$\mu(y)[x] := \begin{cases} [x] & x \in Y \\ 0 & x \notin Y \end{cases} .$$

We define the notion of $U$-control as in Definition 2.8. Let now $\mathcal{C}$ be a coarse structure. We let

$$B^{\mathcal{C}}(L^2(X)) := \{A \in B(L^2(X)) \mid (\exists U \in \mathcal{C} \mid A \text{ is } U\text{-controlled})\} .$$

We claim that $B^{\mathcal{C}}(L^2(X))$ is a $*$-subalgebra. It is a subalgebra by the same argument as in Example 2.7. Furthermore, it is closed under taking adjoints since if $A$ is $U$-controlled, then $A^*$ is $U^{-1}$-controlled. This follows from the fact that for subsets $Y, Z$ of $X$ we have: $Y$ is $U$-separated from $Z$ if and only if $Z$ is $U^{-1}$-separated from $Y$.

Note that in general $B^{\mathcal{C}}(L^2(X))$ is not (topologically) closed, i.e., a $C^*$-algebra. But it is so if $\mathcal{C}$ has a maximal entourage, i.e., if it is generated by an equivalence relation.

**Definition 2.10.** *The $C^*$-algebra $C_u^*(X, \mathcal{C}) := \overline{B^{\mathcal{C}}(L^2(X))}$ obtained by forming the closure of $B^{\mathcal{C}}(L^2(X))$ in $B(L^2(X))$ is called the uniform Roe algebra of the coarse space $(X, \mathcal{C})$.*

□

We now turn to examples and constructions of coarse structures. Let $X$ be a set and $(\mathcal{C}_i)_{i \in I}$ be a family of coarse structures on $X$.

**Lemma 2.11.** *The intersection $\bigcap_{i \in I} \mathcal{C}_i$ is a coarse structure.*

*Proof.* We set $\mathcal{C} := \bigcap_{i \in I} \mathcal{C}_i$.

Since $\mathtt{diag}(X) \in \mathcal{C}_i$ for every $i$ in $I$ we conclude that $\mathtt{diag}(X) \in \mathcal{C}$.

Assume that $U$ and $V$ are in $\mathcal{C}$, and that $W$ is a subset of $U$. Then for every $i$ in $I$ we have $U \in \mathcal{C}_i$ and $V \in \mathcal{C}_i$. This implies $W$ in $\mathcal{C}_i$, $U \cup V \in \mathcal{C}_i$, $V \circ U \in \mathcal{C}_i$ and $U^{-1} \in \mathcal{C}_i$ for every $i$ in $I$. Hence $W$ in $\mathcal{C}$, $U \cup V \in \mathcal{C}$, $V \circ U \in \mathcal{C}$ and $U^{-1} \in \mathcal{C}$. □

Let $\mathcal{A}$ be a subset of $\mathcal{P}_{X \times X}$. Then there is a smallest coarse structure containing $\mathcal{A}$ is given by

$$\mathcal{C}\langle \mathcal{A} \rangle = \bigcap_{\mathcal{C},\, \mathcal{A} \subseteq \mathcal{C}} \mathcal{C} \ ,$$

where the intersections runs over the coarse structures on $X$ containing $\mathcal{A}$.

Let $X$ be a set an $\mathcal{A}$ be a subset of $\mathcal{P}_{X \times X}$. We can describe the elements of $\mathcal{C}\langle \mathcal{A} \rangle$ explicitly. Since any coarse structure contains $\mathtt{diag}(X)$ we can add $\mathtt{diag}(X)$ to $\mathcal{A}$ without changing the coarse structure generated by $\mathcal{A}$. Furthermore with $U$ in $\mathcal{A}$ we have $U \cup U^{-1} \in \mathcal{C}\langle \mathcal{A} \rangle$. So by enlarging the generators without changing $\mathcal{C}\langle \mathcal{A} \rangle$ we can assume that all elements in $\mathcal{A}$ are symmetric.

**Lemma 2.12.** *For simplicity we assume that $\mathtt{diag}(X) \in \mathcal{A}$ and that $\mathcal{A}$ consists of symmetric entourages. An entourage $V$ of $X$ belongs to $\mathcal{C}\langle \mathcal{A} \rangle$ if and only if there exists a finite family of families $((U_{j,i})_{i \in 1, \ldots, n_j})_{j \in J}$ of elements of $\mathcal{A}$ such that*

$$V \subseteq \bigcup_{j \in J} U_{j,1} \circ \cdots \circ U_{j,n_j} \ .$$

*Proof.* Let $\mathcal{C}'$ be the subset of $\mathcal{P}_{X \times X}$ of entourages $V$ such that there exists a finite family of families $((U_{j,i})_{i \in 1, \ldots, n_j})_{j \in J}$ of elements of $\mathcal{A}$ such that

$$V \subseteq \bigcup_{j \in J} U_{j,1} \circ \cdots \circ U_{j,n_j} \ .$$

Then $\mathcal{C}'$ is a coarse structure. Indeed, $\mathtt{diag}(X) \in \mathcal{C}'$ since $\mathtt{diag}(X) \in \mathcal{A}$. Furthermore, by construction $\mathcal{C}'$ is closed under taking subsets and finite unions.

If $V \subseteq \bigcup_{j\in J} U_{j,1} \circ \cdots \circ U_{j,n_j}$ and $V' \subseteq \bigcup_{j\in J} U_{j',1} \circ \cdots \circ U_{j',n_j}$ for families $((U_{j,i})_{i\in 1,\ldots,n_j})_{j\in J}$ and $((U'_{j',i})_{i\in 1,\ldots,n'_{j'}})_{j'\in J'}$, then

$$V \circ V' \subseteq \bigcup_{j\in J, j'\in J'} U_{j,1} \circ \cdots \circ U_{j,n_j} \circ U'_{j',1} \circ \cdots \circ U'_{j',n'_{j'}} \ .$$

It is clear that $\mathcal{C}'$ contains $\mathcal{A}$ and therefore $\mathcal{C}\langle\mathcal{A}\rangle \subseteq \mathcal{C}'$. On the other hand $\mathcal{C}'$ is contained in every coarse structure containing $\mathcal{A}$, hence $\mathcal{C}' \subseteq \mathcal{C}\langle\mathcal{A}\rangle$. □

**Example 2.13.** Coarse structure generated by a single entourage are particularly easy to describe. Let $U$ be an entourage. We set $U^0 := \mathtt{diag}(X)$ and

$$U^n := \underbrace{U \circ \cdots \circ U}_{n\times}$$

for all positive integers. Then we consider the coarse structure $\mathcal{C}_U := \mathcal{C}\langle\{U\}\rangle$.

Assume that $U = U^{-1}$. Then an entourage $V$ belongs to $\mathcal{C}\langle\{U\}\rangle$ if and only if there exists $n$ in $\mathbb{N}$ such that $V \subseteq U^n$.

If $U$ is an equivalence relation, then $\mathcal{C}_U = \{V \in \mathcal{P}_{X\times X} \mid V \subseteq U\}$. □

**Example 2.14.** Let $X$ be a set. It has a minimal coarse structure $\mathcal{C}_{min}$ consisting of all subsets of $\mathtt{diag}(X)$. We have $\mathcal{C}_{min} = \mathcal{C}\langle\emptyset\rangle$.

The set $\mathcal{C}_{max} := \mathcal{P}_{X\times X}$ is the maximal coarse structure. □

**Example 2.15.** Let $(X, d)$ be a quasi-metric space. Then for $r$ in $[0, \infty)$ we define the metric entourage

$$U_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\}$$

of width $r$.

The coarse structure $\mathcal{C}_d := \mathcal{C}\langle(U_r)_{r\in[0,\infty)}\rangle$ is called the metric coarse structure on $X$.

The triangle inequality for the quasi-metric implies that $U_r \circ U_s \subseteq U_{r+s}$. In view of Lemma 2.12 an entourage $V$ of $X$ belongs to $\mathcal{C}_d$ if and only if there exists an $r$ in $[0, \infty)$ such that $V \subseteq U_r$.

The $U_r$-thickening of a point is given by the $r$-ball centered at this point: $U_r[\{x\}] = B(x, r)$. □

**Example 2.16.** Let $U$ be an entourage of $X$ with $U = U^{-1}$. Then we can define a quasi-metric on $X$ as follows:

$$d_U(x, y) := \inf\{n \in \mathbb{N} \mid (x, y) \in U^n\} \ .$$

It can happen that $d_U(x,y) = \infty$, namly if the argument of inf above is empty. We verify the axioms of a distance.

1. $d_U(x,x) = 0$ since $(x,x) \in U^0 = \mathtt{diag}(X)$.

2. $d_U(x,y) = d_U(y,x)$ since $U^n = (U^n)^{-1}$ for every $n$ in $\mathbb{N}$.

3. If $d_U(x,y) = m$ and $d_U(y,z) = n$, then $(x,y) \in U^m$ and $(y,z) \in U^n$. Hence $(x,z) \in U^m \circ U^n = U^{m+n}$. Hence $d_U(x,y) \leq m+n = d_U(x,y) + d_U(y,z)$.

We then have the relation

$$\mathcal{C}_U = \mathcal{C}_{d_U} \ .$$

□

**Example 2.17.** The space $\mathbb{R}^n$ is a coarse space with the coarse structure induced by the usual metric. If not said differently we will consider subsets of $\mathbb{R}^n$ like $\mathbb{Z}^n$, $\mathbb{R}^n_+$ or $\mathbb{Q}^n$ as coarse spaces with respect to the induced metric. □

**Example 2.18.** Let $\bar{X}$ be a topological space and $Y$ be a subset of $\bar{X}$. We consider $X := \bar{X} \setminus Y$.

A subset $U$ of $X \times X$ is called continuously controlled (w.r.t $(\bar{X}, Y)$) if for every net $((x_i, x_i'))_{i\in I}$ in $U$ the condition $x_i \to y \in Y$ implies that $x_i' \to y$ and $U^{-1}$ satisfies the same condition.

The set of continuously controlled entourages forms a coarse structure called the continuously controlled coarse structure. We verify the axioms:

1. It is clear that the diagonal is continuously controlled.

2. It is also clear that if $U$ is continuously controlled and $V$ is a subset of $U$, then $V$ is continuously controlled. Assume that $U$ and $V$ are continuously controlled. We must show that $U \cup V$ is continuously controlled. Let $((x_i, x_i'))_{i\in I}$ be a net in $U \cup V$ such that $x_i \to y \in Y$. We can find subsets $I_U$ and $I_V$ of $I$ such that $I = I_U \cup I_V$ and $(x_i, x_i') \in V$ for $i$ in $I_V$ and $(x_i, x_i') \in U$ for $i$ in $I_U$. If $I_U$ is cofinal in $I$, then we conclude that $\mathtt{lim}_{i\in I_U} x_i' = y$, and similarly, if $U_V$ is cofinal in $I$, we have $\mathtt{lim}_{i\in I_V} x_i' = y$. This implies $\mathtt{lim}_{i\in I} x_i' = y$.

3. Assume that $U$ and $V$ are continuously controlled. We show that $U \circ V$ is continuously controlled. The main argument is as follows. Let $((x_i, x_i'))_{i\in I}$ be a net in $U \circ V$ such that $\mathtt{lim}_I x_i = y \in Y$. For every $i$ in $I$ we find $x_i''$ such that $(x_i, x_i'') \in U$ and $(x_i'', x_i') \in V$. We first conclude that $\mathtt{lim}_{i\in I} x_i'' = y$ since $U$ is continuously controlled, and then $\mathtt{lim}_{i\in I} x_i' = y$ since $V$ is continuously controlled.

4. If $U$ is continuously controlled, then $U^{-1}$ is continuously controlled by definition.

□

**Example 2.19.** Let $\mathcal{C}$ be a coarse structure on $X$. Then

$$R := \bigcup_{U \in \mathcal{C}} U$$

is an equivalence relation on $X$. For $x, y$ in $X$ we have $x \sim y$ if and only of $(x, y) \in \mathcal{C}$.

**Definition 2.20.** *The equivalences classes for $R$ are called the coarse components of $X$.*

We let $\pi_0^{\mathcal{C}}(X)$ denote the set of coarse components. □

**Definition 2.21.** *A coarse space is a pair $(X, \mathcal{C})$ of a set with a coarse structure.*

We usually write $X$ for a coarse space and $\mathcal{C}_X$ for the corresponding coarse structrure.

**Example 2.22.** For a set $X$ we write $X_{min} := (X, \mathcal{C}_{min})$ and $X_{max} := (X, \mathcal{P}_X)$.

Let $X, Y$ be coarse spaces and $f : X \to Y$ be a map between the underlying sets. We write $f(U) := (f \times f)(U)$.

**Definition 2.23.** *The map $f$ is called controlled if $f(\mathcal{C}_X) \subseteq \mathcal{C}_Y$.*

In details this means that for every coarse entourage $U$ of $X$ the set $f(U)$ is a coarse entourage of $Y$. We obtain the category **Coarse** of coarse spaces and controlled maps.

Let $\mathcal{A}$ be a family in $\mathcal{P}_{X \times X}$ and assume that $\mathcal{C}_X = \mathcal{C}\langle\mathcal{A}\rangle$.

**Lemma 2.24.** *Then the map $f : X \to Y$ is controlled if and only if $f(A) \in \mathcal{C}_Y$ for all $A$ in $\mathcal{A}$.*

*Proof.* If $f$ is controlled, then $f(A) \in \mathcal{C}_Y$ for all $A$ in $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{C}_X$.

We now consider the converse. Since $f(\mathtt{diag}(X)) \subseteq \mathtt{diag}(Y)$ we know that $f(\mathtt{diag}(X)) \in \mathcal{C}_Y$. Furthermore $f(U \cup U^{-1}) \subseteq f(U) \cup f(U^{-1})$. Hence we can assume that $\mathcal{A}$ contains $\mathtt{diag}(X)$ and consists of symmetric entourages.

We consider $V$ in $\mathcal{C}_X$. By Lemma 2.12 there exists a finite family of families $((U_{j,i})_{i \in 1,\dots,n_j})_{j \in J}$ of elements of $\mathcal{A}$ such that

$$V \subseteq \bigcup_{j \in J} U_{j,1} \circ \dots \circ U_{j,n_j} \ .$$

Then

$$f(V) \subseteq f(\bigcup_{j \in J} U_{j,1} \circ \dots \circ U_{j,n_j}) \subseteq \bigcup_{j \in J} f(U_{j,1}) \circ \dots \circ f(U_{j,n_j})$$

belongs to $\mathcal{C}_Y$.

Here we used the relations $f(U \circ U') \subseteq f(U) \circ f(U')$ and $f(U \cup U') = f(U) \cup f(U')$. □

**Example 2.25.** Assume that $X$ and $Y$ are metric spaces and have the metric coarse structures.

**Lemma 2.26.** *A map $f : X \to Y$ is controlled if and only if for all $S$ in $[0,\infty)$ there exist $R$ in $[0,\infty)$ such that $d_X(x,x') \leq S$ implies $d_Y(f(x),f(x')) \leq R$.*

*Proof.* Assume that $f$ is controlled. If $S$ is in $[0,\infty)$, then $U_{X,S} \in \mathcal{C}_X$ and therefore $f(U_{X,S}) \in \mathcal{C}_Y$. As explained in Example 2.15 there exists $R$ in $[0,\infty)$ such that $f(U_S) \subseteq U_{Y,R}$. This inclusion is equivalent to the assertion that $d_X(x,x') \leq S$ implies $d_Y(f(x),f(x')) \leq R$.

We now consider the converse. Let $V$ be in $\mathcal{C}_X$. Again by Example 2.15 there exists $S$ in $[0,\infty)$ such that $V \subseteq U_{X,S}$. The condition on $f$ says that there exist $R$ in $[0,\infty)$ with $f(U_{X,S}) \subseteq U_{Y,R}$. Since $U_{Y,R} \in \mathcal{C}_Y$ and $f(V) \subseteq f(U_{X,S}) \subseteq U_{Y,R}$ we conclude that $f(V) \in \mathcal{C}_Y$. □

If $f$ is Lipschitz with Lipschitz constant $C$, then we can take $R := CS$. In particular, Lipshitz maps are controlled. More generally maps which satisfy

$$d(f(x),f(x')) \leq Cd(x,x') + C'$$

(quasi-Lipschitz) for some $C, C'$ and all $x, x'$ in $X$ are controlled.

The map

$$\mathbb{R} \to \mathbb{Z}\ , \quad t \mapsto \text{nearest integer to } t$$

satisfy this with $C' = 1$ and $C = 1$.

The map $x \mapsto x + 1$ on $\mathbb{R}$ is controlled.

The map $x \mapsto -x$ on $\mathbb{R}$ is controlled.

The map

$$\mathbb{N} \to \mathbb{N}\ , \quad n \mapsto n^2$$

is not controlled. □

**Example 2.27.** Let $X$ be a coarse space and $f : Y \to X$ be a map of sets. Then $\mathcal{C}\langle f^{-1}(\mathcal{C}_X)\rangle$ is the induced coarse structure on $Y$. It is the largest coarse structure on $Y$ for which $f$ becomes controlled map. □

**Example 2.28.** Let $X$ be a coarse space and $f : X \to Y$ be a map of sets. Then $\mathcal{C}\langle f(\mathcal{C}_X)\rangle$ is the coinduced coarse structure on $Y$. It is the smallest coarse structure on $Y$ such that $f$ becomes a controlled map. □

We now study the category **Coarse** and its relation with the category **Set** of sets.

Let $S$ : **Coarse** $\to$ **Set** be the forgetful functor.

**Lemma 2.29.** *We have adjunctions*

$$S : \mathbf{Coarse} \leftrightarrows \mathbf{Set} : (X \mapsto X_{max})$$

*and*

$$(X \mapsto X_{min}) : \mathbf{Set} \leftrightarrows \mathbf{Coarse} : S \ .$$

*Proof.* We have equalities for all sets $X$ and coarse spaces $Y$

$$\mathtt{Hom}_{\mathbf{Coarse}}(X_{min}, Y) = \mathtt{Hom}_{\mathbf{Set}}(X, S(Y))$$

and

$$\mathtt{Hom}_{\mathbf{Coarse}}(Y, X_{max}) = \mathtt{Hom}_{\mathbf{Set}}(S(Y), X) \ .$$

□

By Lemma 2.29 the underlying set of a limit or colimit is the limit or colimit of the underlying sets.

**Proposition 2.30.** *The category* **Coarse** *admits all limits and colimits.*

*Proof.* We start with colimits. Let $X : \mathbf{I} \to \mathbf{Coarse}$ be a diagram. We consider the colimit of sets $Y := \mathtt{colim}_{\mathbf{I}} S(Y)$ with the family of structure maps $(e_i : X_i \to Y)_{i \in \mathbf{I}}$. We equip $Y$ with the smallest coarse structure such that $e_i$ is controlled for all $i$, i.e., with $\mathcal{C}_Y := \mathcal{C}\langle \bigcup_{i \in I} e_i(\mathcal{C}_{X_i}) \rangle$.

We claim that $(Y, (e_i)_{i \in \mathbf{I}})$ is the colimit of the diagram $X$. Let $T$ be a coarse space. Then we have a bijection

$$\mathtt{Hom}(S(Y), S(T)) \stackrel{\cong}{\to} \lim_{\mathbf{I}^{op}} \mathtt{Hom}(S(X), S(T)) \ .$$

It restricts to an injective map

$$\mathtt{Hom}_{\mathbf{Coarse}}(Y, T) \to \lim_{\mathbf{I}^{op}} \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

We must show that it is surjective. Let $(f_i : X \to T)_{i \in I}$ represent an element in the limit and assume that $g : Y \to T$ is the corresponding map of underlying sets. We must show that $g$ is controlled. It is enough to consider generating entourages. Let $U$ be an entourage of $Y$. Let $U_i$ be an entourage of $X_i$ and $e_i(U_i)$ be the corresponding generating entourage of $Y$. Then $g(e_i(U)) = f_i(U_i)$ is a coarse entourage of $T$ since $f_i$ is controlled.

We now consider limits. We form the limit $Z := \mathtt{lim}_{\mathbf{I}} S(X)$ of underlying sets with canonical projections $p_i : Z \to S(X_i)$. We equip $Z$ with the maximal coarse structure such that all $p_i$ become controlled. This is the intersection of the coarse structures induced by the $p_i$. We claim that $(Z, (p_i)_{i\in \mathbf{I}})$ has the required universal property of a limit of $X$. For $T$ in **Coarse** we have a bijection

$$\mathtt{Hom}(S(T), S(Z)) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}(S(T), S(Z)) \ .$$

It restricts to an injective map

$$\mathtt{Hom}_{\mathbf{Coarse}}(T, Z) \to \lim_{\mathbf{I}} \mathtt{Hom}(T, X) \ .$$

In order to show that this map is surjective we consider a family $(f_i : T \to X_i)_{i\in\mathbf{I}}$ in the limit and the corresponding map of underlying sets $g : T \to Z$. We must show that $g$ is controlled. Let $U$ be an entourage of $T$. Then $f_i(U) = p_i(g(U))$ is a coarse entourage of $X_i$ for every $i$ in $\mathbf{I}$. This implies by construction of the coarse structure of $Y$ that $g(U)$ is a coarse entourage of $Z$. □

**Example 2.31.** Let $(X_i)_{i\in I}$ be a family of coarse spaces. Then we can describe its coproduct $\coprod_{i\in I} X_i$ explicitly. The underlying set of the coproduct is the disjoint union $S(X) := \bigsqcup_{i\in I} S(X_i)$ union of the underlying sets. An entourage $U$ of $X$ is coarse in $X$ for all $i$ in $I$, and if $U \cap (X_i \times X_i)$ is coarse in $X_i$ and $U \cap (X_i \times X_i) = \mathtt{diag}(X_i)$ for all but finitely many $i$ in $I$. □

**Example 2.32.** Let $(X_i)_{i\in I}$ be a family of coarse spaces. Then we can describe its cartesian product $\prod_{i\in I} X_i$ explicitly. The underlying set of the product is the cartesian product $S(X) := \prod_{i\in I} S(X_i)$ union of the underlying sets. The coarse structure on $X$ is generated by entourages $(U_i)_{i\in I}$ of $X$ where $U_i$ is a coarse entourage of $X_i$ for every $i$ in $I$. □

In the following we will introduce various concepts of coarse geometry.

Let $f, g : X \to Y$ be two maps between sets and $U$ be an entourage of $Y$.

**Definition 2.33.** *We say that $f$ and $g$ are $U$-close to each other if $(f, g)(\mathtt{diag}(X)) \subseteq U$.*

We also write $f \sim_U g$.

Let $f, g : X \to Y$ be two maps into a coarse space.

**Definition 2.34.** *$f$ and $g$ are close to each other if $f \sim_U g$ for some $U$ in $\mathcal{C}_Y$.*

**Remark 2.35.** Let $X$ be a set and $Y$ be a coarse space. The condition that $f, g : X \to Y$ are close to each other is equivalent to the condition that map $h : \{0, 1\}_{max} \times X_{min} \to Y$ with $h(0, x) := f(x)$ and $h(1, x) := g(x)$ is a morphism of coarse spaces. □

**Example 2.36.** The map

$$\mathbb{R} \to \mathbb{R} \ , \quad x \mapsto x + 1$$

is close to $\mathtt{id}_{\mathbb{R}}$. The map

$$\mathbb{R} \to \mathbb{R} \ , \quad x \mapsto -x$$

is not close to $\mathtt{id}_{\mathbb{R}}$. □

**Lemma 2.37.**

1. *Closeness is an equivalence relation on morphisms of* **Coarse**.

2. *Closeness is compatible with composition.*

*Proof.* We use the symbol $\sim$ in order to denote the relation of closeness. We consider maps from $X$ to $Y$. We first show that closeness is an equivalence relation.

1. Since $(f,f)(\mathtt{diag}(X)) \subseteq \mathtt{diag}(Y)$ it is clear that $f \sim f$.

2. We have $(f,g)(\mathtt{diag}(X)) = (g,f)(\mathtt{diag}(X))^{-1}$. Then the coarse structure $\mathcal{C}_Y$ is closed under taking inverses conclude that $f \sim g$ if and only $g \sim f$.

3. For three maps $f, g, h$ from $X$ to $Y$ we have
$$(f,h)(\mathtt{diag}(X)) \subseteq (f,g)(\mathtt{diag}(X)) \circ (g,h)(\mathtt{diag}(X)) \ .$$
Since the coarse structure $\mathcal{C}_Y$ is closed under forming compositions we conclude that $f \sim g$ and $g \sim h$ implies $f \sim h$.

We next show that closeness is compatible with compositions. Let $h : Y \to Z$ be a morphism. If $f \sim g$, then using that $h$ is controlled we see that
$$(h \circ f, h \circ g))(\mathtt{diag}(X)) = h((f,g)(\mathtt{diag}(X)))$$
is a coarse entourage of $Z$. Hence $h \circ f \sim h \circ g$.

Let $l : W \to X$ be a morphism. Then
$$(f \circ l, g \circ l)(\mathtt{diag}(W)) \subseteq (g,f)(l(\mathtt{diag}(W)) \subseteq (f,g)(\mathtt{diag}(X))$$
is a coarse entouarge of $Y$. Hence $f \circ l \sim g \circ l$. □

We can form the category $\overline{\textbf{Coarse}}$ with the same objects as **Coarse** and closeness classes of maps.

**Definition 2.38.** *A morphism $f : X \to Y$ in* **Coarse** *is a coarse equivalence if it is invertible in* $\overline{\textbf{Coarse}}$. *Two coarse spaces are called coarsely equivalent if they are isomorphic in* $\overline{\textbf{Coarse}}$.

**Remark 2.39.** Explicitly, $f : X \to Y$ is a coarse equivalence if and only if there exists a morphism $g : Y \to X$ such that $f \circ g \sim \mathtt{id}_Y$ and $g \circ f \sim \mathtt{id}_X$. □

It is an overall idea in coarse geometry that one should study coarse spaces up to coarse equivalence.

**Example 2.40.** The embedding $\mathbb{Z} \to \mathbb{R}$ (where $\mathbb{Z}$ has the induced coarse structure) is a coarse equivalence. An inverse up to closeness is the map $x \mapsto [x]$ (integer part of $x$) $\square$

**Example 2.41.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be a map between the underlying sets.

**Definition 2.42.** *The map $f$ is a quasi-isometry if there are constants $C, C'$ in $(0, \infty)$ such that*

$$C^{-1} d_X(x', x) - C' \leq d_Y(f(x), f(x')) \leq Cd(x, x') + C' \tag{2.1}$$

*for all $x, x'$ in $X$.*

We have already seen in Example 2.26 that a quasi-isometry induces a morphism $f : X_d \to Y_d$ in **Coarse**. This only uses the second inequality in (2.1). Let $r$ be in $(0, \infty)$.

**Definition 2.43.** *We say that $f(X)$ is $r$-dense if $\bigcup_{x \in X} B(f(x), r) = Y$.*

**Lemma 2.44.** *If $f$ is a quasi-isometry and $f(X)$ is $r$-dense for some $r$ in $(0, \infty)$, then $f$ is a coarse equivalence.*

*Proof.* We define $g : Y \to X$ by choosing for every $y$ in $Y$ some $x$ in $X$ such that $y$ in $B(f(x), y)$. Then $g$ is controlled. Note that $d_Y(f(g(y)), y) \leq r$ for all $y$ in $Y$. By the first inequality in (2.1) we have

$$\begin{aligned} d_X(g(y), g(y')) &\leq Cd_Y(f(g(y)), f(g(y'))) + CC' \\ &\leq Cd_Y(y, y) + C(2r + C') \end{aligned}$$

for all $y, y'$.

Furthermore $f \circ g$ is close to $\texttt{id}_Y$ since $d_Y(f(g(x)), y) \leq r$ for all $y$ in $Y$ and $g \circ f$ is close to $\texttt{id}_X$ since

$$d_X(g(f(x)), x) \leq Cd_Y(f(g(f(x), f(x)) + CC' \leq Cr + CC'$$

for all $x$ in $X$. $\square$

Let $(X, d)$ be a metric space. Then we can define a new metric by

$$d'(x, y) := \ln(1 + d(x, y)) \ .$$

The identity of the underlying sets is a coarse equivalence between $X_d$ and $X_{d'}$. But if $X$ is unbounded, then it is not a quasi-isometry provided $X$.

So in general the condition of being quasi-isometric is stronger than the condition of being coarsely equivalent. $\square$

**Example 2.45.** The embedding $\mathbb{Z}^n \to \mathbb{R}^n$ is an isometry, hence in particular a quasi isometry. Furthermore, $\mathbb{Z}^n$ is $\sqrt{n/2}$-dense in $\mathbb{R}^n$. □

The coarse spaces $\mathbb{Z}$ and $\mathbb{R}$ are equivalent, but $\mathbb{Z}$ is much smaller. It is often convenient to represent coarse spaces up to equivalence but small models. To this end the notion of a dense subset of a coarse space is useful. This notion extends the notion of $r$-density in the metric case.

Let $X$ be a set and $L$ be a subset. Let $U$ be an entourage of $X$.

**Definition 2.46.** *$L$ is $U$-dense if $U[L] = X$.*

Explicitly this means that for every point $x$ of $X$ there exists a point $l$ in $L$ such that $(x, l) \in U$.

Let $X$ be a coarse space and $L$ be a subset of $X$.

**Definition 2.47.** *$L$ is dense in $X$ if it is $U$-dense for some coarse entourage $U$ of $X$.*

**Example 2.48.** The subset $\mathbb{Z}$ is $U_1$-dense in $\mathbb{R}$.

The subset $\{n^2 \mid n \in \mathbb{N}\}$ is not dense in $\mathbb{N}$. □

Let $X$ be a coarse space and $L$ be a subset equipped with the induced coarse structure.

**Lemma 2.49.** *If $L$ is dense, then the inclusion $f : L \to X$ is a coarse equivalence.*

*Proof.* Let $U$ be a coarse entourage of $X$ such that $L$ is $U$-dense. In order to construct an inverse define a map $g : X \to L$ by choosing for every $x$ in $X$ the point $g(x)$ in $L$ such that $x \in U[\{g(x)\}]$. Then $f \circ g$ and $\mathtt{id}_X$ are $U$-close and $g \circ f$ and $\mathtt{id}_L$ are $U \cap (L \times L)$-close to each other.

It remains to check that $g$ is controlled. Let $W$ be in $\mathcal{C}_X$. If $(x, y)$ is in $W$, then $(g(x), g(y)) \in U^{-1} \circ W \circ U$ since $g(x) \sim_{U^{-1}} x \sim_W y \sim_U g(y)$. Hence $g(W) \subseteq (U^{-1} \circ W \circ U) \cap (L \times L)$.

This shows that $g(W) \in \mathcal{C}_L$. □

Let $X$ be a set, $U$ be an entourage of $X$ containing the diagonal and $L$ be a subset.

**Definition 2.50.** *We say that $L$ is $U$-separated, if for every $l, l'$ with $l \neq l'$ we have $l' \notin U[\{l\}]$.*

In other words, we have $U \cap (L \times L) \subseteq \mathtt{diag}(L)$.

**Lemma 2.51.** *If $U$ is symmetric, then $X$ admits a $U$-dense and $U$-separated subset.*

*Proof.* We consider the poset of $U$-separated subsets with respect to inclusion. We check that any totally ordered chain $(L_\alpha)_{\alpha\in A}$ in this poset is bounded by $\bigcup_{\alpha\in A} L_\alpha$. Indeed, this subset is also $U$-separated.

By the lemma of Zorn there exists a maximal $U$-separated subset $\bar{L}$. We claim that it is $U$-dense. Assume that it is not. Then there exists $x$ in $X$ with $x \notin U[\bar{L}]$. But then $\bar{L} \cup \{x\}$ is still $U$-separated. In order to see that $U[\{x\}] \cap L = \emptyset$ we use that $U$ is symmmetric. We obtain a contradiction to the maximality of $\bar{L}$. □

Let $X$ be a coarse space.

**Definition 2.52.**

1. *$X$ is uniformly locally finite if for every entourage $U$ of $X$ we have $\sup_{x\in X} |U[\{x\}]| < \infty$.*
2. *$X$ has bounded geometry if it is coarsely equivalent to a uniformly locally finite coarse space.*

Bounded geometry is preserved under coarse equivalences by definition. This is not true for uniform local finiteness.

**Example 2.53.** The inclusion $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. $\mathbb{Z}$ is uniformly locally finite, but $\mathbb{R}$ is not.

Let $X$ be a set and $U$ be in $\mathcal{P}_{X\times X}$. Let $B$ be a subset of $X$.

**Definition 2.54.** *The subset $B$ is called $U$-bounded if $B \times B \subseteq U$.*

**Example 2.55.** If $X$ is a metric space, then $B$ is $U_r$-bounded if and only if $\mathrm{diam}(B) \le r$. □

**Example 2.56.** For a set $X$ let $P(X)$ denote the space of finitely supported functions $\mu : X \to [0,1]$ such that $\sum_{x\in X} \mu(x) = 1$. If $f : X \to X'$ is a map of sets, then we define the map $P(f) : P(X) \to P(X')$ by

$$P(f)(\mu(x')) := \sum_{x\in f^{-1}(x')} \mu(x) \ .$$

We have

$$P(X) \cong \operatorname*{\mathtt{colim}}_{F\subseteq X} P(F)$$

as sets where $F$ runs over all finite subsets of $X$. Note that the structure maps are injective so that this can also be interpreted as forming a union of subets.

As a topological space we identify $P(F)$ with a subspace of $\mathbb{R}^F$. Then we equip $P(X)$ with the topology of the colimit. The map $P(f) : P(X) \to P(X')$ is then continuous.

Note that $P(F)$ is a standard simplex of dimension $|F| - 1$, and $P(X)$ is a simplicial complex. The map $P(f)$ is a morphism of simplicial complexes.

Let now $U$ be an entourage of $X$. Then we consider the closed subspace $P_U(X)$ of $P(X)$ of functions which have $U$-bounded support. Note that $P_U(X)$ is a subcomplex of $P(X)$. If $X_U$ has bounded geomety, then $P_U(X)$ is finite-dimensional. If $U'$ is an entourage of $X'$ and $f(U) \subseteq U'$, then by restriction we get a continuous map $P(f) : P_U(X) \to P_{U'}(X')$. Indeed, if $(x', y')$ are in $\mathtt{supp} f_*(\mu)$, then there exist $x, y$ in $X$ such that $f(x) = x'$ and $f(y) = y'$. Then $(x, y) \in U$ and hence $(x', y') = (f(x), f(y)) \in f(U) \subseteq U'$.

This construction is functorial on the category of pairs $(X, U)$ (the Grothendieck construction of the functor **Coarse** $\to$ **Set**, $X \mapsto \mathcal{C}_X$).

Assume now that $f, g : X \to X'$ are morphisms such that $f(U) \subseteq U'$, $g(U) \subseteq U'$ and $f$ and $g$ are $V'$-close to each other for some further entourage $V'$. Then $P(f)$ is homotopic to $P(g)$ as maps from $P_U(X)$ to $P_{W'}(X')$ for any entourage $W'$ of $X'$ such that $U' \subseteq W'$ and $VU'^{-1} \subseteq W'$. The homotopy is given by convex interpolation:

$$h(u, \mu) := (1 - u)P(f)(\mu) + uP(g)(\mu) \ .$$

One first checks continuity by observing that for every finite subset $F$ of $X$ the restriction of this map to $P_U(F)$ factors over the obviously continuous map $P_U(F) \to P(f(F) \cup g(F))$ given by the same formula. It remains to check that the support of $h(u, \mu)$ is $U'$-bounded for every $u$ in $[0, 1]$.

If $x', y'$ are in $\mathtt{supp}(h(u, \mu))$ then we have one of the following cases:

1. $x', y'$ are in $\mathtt{supp} P(f)(\mu)$: In this case $(x', y') \in U'$ since the support of $P(f)(\mu)$ is $U'$-bounded.

2. $x', y'$ are in $\mathtt{supp} P(g)(\mu)$: In this case $(x', y') \in U'$ since the support of $P(g)(\mu)$ is $U'$-bounded.

3. $x'$ is in $\mathtt{supp} P(f)(\mu)$ and $y'$ is $P(g)(\mu)$: Then $x' = f(x)$ and $y' = g(y)$ for some $x, y$ in $\mathtt{supp}(\mu)$. Then $(x, y) \in U$. This implies that $(x', y') \in V \circ U'$. To this end we consider the chain $x' = f(x) \sim_V g(x) \sim_{U'} g(y) = y'$

4. $x'$ is in $\mathtt{supp} P(g)(\mu)$ and $y'$ is $P(g)(\mu)$: This is analoguous to the previous case.

Let $F : \mathbf{Top} \to \mathbf{M}$ be any functor to a cocomplete target. Then we can define a functor

$$F\mathbf{P} : \mathbf{Coarse} \to \mathbf{M} \ , \quad F\mathbf{P}(X) := \operatorname*{\mathtt{colim}}_{U \in \mathcal{C}_X} F(P_U(X)) \ .$$

If $F$ is homotopy invariant, then $F\mathbf{P}$ is coarsely invariant and hence factorizes over a functor

$$\overline{F\mathbf{P}} : \overline{\mathbf{Coarse}} \to \mathbf{M} .$$

For example, we can consider $\pi_0 : \mathbf{Top} \to \mathbf{Set}$. Then

$$\pi_0^{coarse}(X) \cong \pi_0 \mathbf{P}(X) .$$

Another example of a homotopy invariant functor is $H_n^{\mathtt{sing}}(-) : \mathbf{Top} \to \mathbf{Ab}$. But observe (Excercise!) that for $n \geq 1$ we have $H_n^{\mathtt{sing}}\mathbf{P}(X) \cong 0$ for every coarse space $X$. We will learn later how to modify the functor $H_n^{\mathtt{sing}}$ to get non-trivial answers.

Using this construction we can use homotopy invariant functors from algebraic topology to get invariants of coarse spaces up to equivalence.

**Example 2.57.** This example is the combinatorial version of Example 2.56. Let $X$ be a coarse space. Let $U$ be an entourage containing the diagonal. Then we can define a simplicial set $P_U^\bullet(X)$ as follows. A point $(x_0, \ldots, x_n)$ in $\prod_{i=0}^n X$ is called $U$-bounded if $(x_i, x_{i'}) \in U$ for all $i, i'$ in $\{0, \ldots, n\}$.

We consider the simplicial set $X^\bullet$ with the set of $n$-simplices $X^n = \prod_{i=0}^n X$. The faces are the projections, and the degenerations are diagonal insertions. Thus

$$\partial_i(x_0, \ldots, x_n) := (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

and

$$s_i(x_0, \ldots, x_n) := (x_0, \ldots, x_i, x_i, \ldots x_n) .$$

The complex $P_U^\bullet(X)$ is the simplicial subset of $X^\bullet$ consisting of all $U$-bounded simplices. It is clear that it is preserved by the faces and degenerations.

**Definition 2.58.** *The simplicial set $P_U^\bullet(X)$ is called the Rips complex of $X$ for $U$.*

If $f : X \to X'$ is a map and $f(U) \subseteq U'$, then we get an induced map $P(f) : P_U^\bullet(X) \to P_{U'}^\bullet(X')$.

The dimension of a simplicial set is the supremum of the dimensions of its non-degenerated simplices.

If $X$ has uniformly locally bounded geometry, then $P_U^\bullet(X)$ is finite-dimensional for every coarse entourage $U$ containing $\mathtt{diag}(X)$.

Seien $f, g : X \to X'$ be morphisms of coarse spaces such that $f(U) \subseteq U'$ and $g(U) \subseteq U'$ and $f$ and $g$ are $V'$-close. Let $W'$ be an entourage of $X'$ such that $U' \subseteq W'$ and $U'V' \subseteq W'$.

**Lemma 2.59.** *$P^\bullet(f)$ and $P^\bullet(g)$ are simplicially homotopic as morphisms $P^\bullet_U(X) \to P^\bullet_{W'}(X')$.*

*Proof.* For all $n$ in $\mathbb{N}$ and $i$ in $\{0,\ldots,n\}$ we define maps $h_i : P^n_U(X) \to P^{n+1}_{W'}(Y)$ as follows:

$$h_i(x_0,\ldots,x_n) := (f(x_0),\ldots,f(x_i),g(x_i),\ldots,g(x_n)) \ .$$

One checks the following defining relations:

$$d_i h_j = \begin{cases} h_{j-1}d_i & i<j \\ d_i h_{i-1} & i=j\neq 0 \\ h_j d_{i-1} & i>j+1 \end{cases}$$

and

$$s_i h_j = \begin{cases} h_{j+1}s_i & i\le j \\ h_j s_{i-1} & i>j \end{cases} \ .$$

□

Consider a functor $F : \mathbf{sSet} \to \mathbf{M}$ to some target. Then we can consider

$$F\mathbf{P}^\bullet(X) := \operatorname*{colim}_{U\in\mathcal{C}_X} F(P^\bullet_U(X)) \ .$$

Then $X \mapsto F\mathbf{P}^\bullet(X)$ is a functor

$$F\mathbf{P}^\bullet : \mathbf{Coarse} \to \mathbf{M} \ .$$

If $F$ is homotopy invariant, then $F\mathbf{P}^\bullet$ sends coarse equivalences to equivalences (equalities) and hence factorizes over a functor

$$\overline{F\mathbf{P}^\bullet} : \overline{\mathbf{Coarse}} \to \mathbf{M} \ .$$

# 3 Bornological coarse spaces

Let $X$ be a set with a coarse structure $\mathcal{C}$ and a (generalized) bornological structure $\mathcal{B}$.

**Definition 3.1.** *$\mathcal{C}$ and $\mathcal{B}$ are compatible if for every $B$ in $\mathcal{B}$ and $U$ in $\mathcal{C}$ we have $U[B]\in\mathcal{B}$.*

Compatibility means that the bornology $\mathcal{B}$ is stable under $U$-thickening for all coarse entourages $U$ of $X$.

Let $X$ be a coarse space. Let $\mathcal{B}$ be a subset of $\mathcal{P}_X$.

**Lemma 3.2.** *The following are equivalent.*

1. *$\mathcal{B}$ is the minimal bornology compatible with $\mathcal{C}_X$.*

2. *$\mathcal{B} = \mathcal{B}\langle\{U[\{x\}] \mid x \in X \text{ and } U \in \mathcal{C}_X\}\rangle$.*

3. *$\mathcal{B} = \mathcal{B}\langle\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}\rangle$*

*Proof.* $1 \Leftrightarrow 2$ : Let $\mathcal{B}$ as in 1. and $\mathcal{B}' := \mathcal{B}\langle\{U[\{x\}] \mid x \in X \& U \in \mathcal{C}_X\}\rangle$. Since $\{x\}$ is bounded in any bornology we have $\{x\} \in \mathcal{B}$. Since $\mathcal{B}$ is compatible with $\mathcal{C}_X$ we have $U[\{x\}] \in \mathcal{B}$ for all $U$ in $\mathcal{C}_X$ and $x$ in $X$. Hence $\mathcal{B}' \subseteq \mathcal{B}$. We show that $\mathcal{B}'$ is compatible with $\mathcal{C}_X$ and conclude $\mathcal{B}' = \mathcal{B}$ by minimality of $\mathcal{B}$. Let $A$ be in $\mathcal{B}'$ and $U \in \mathcal{C}_X$. Then there exist finite families $(U_i)_{i\in I}$ in $\mathcal{C}_X$ and $(x_i)_{i\in I}$ such that $A \subseteq \bigcup_{i\in I} U_i[x_i]$ (see Example 1.6). We conclude that $U[A] \subseteq \bigcup_{i\in I}(U \circ U_i)[x_i]$. Since $U \circ U_i \in \mathcal{C}_X$ for all $i$ in $I$ this implies that $U[A] \in \mathcal{B}'$.

$2 \Leftrightarrow 3$ : Let $\mathcal{B}'' := \mathcal{B}\langle\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}$

We first show that $\mathcal{B}' \subseteq \mathcal{B}''$. Let $x$ be in $X$ and $U$ be in $\mathcal{C}_X$. Then we have $[U\{x\}] \times U[\{x\}] \subseteq U \circ U^{-1}$. Since for $U$ in $\mathcal{C}_X$ also $U \circ U^{-1} \in \mathcal{C}_X$ we conclude that $[U\{x\}] \times U[\{x\}]$ is bounded by a coarse entourage of $X$. Hence $[U\{x\}] \times U[\{x\}] \in \mathcal{B}''$. This implies that $\mathcal{B}' \subseteq \mathcal{B}''$.

We now show that $\mathcal{B}'' \subseteq \mathcal{B}'$. Let $B$ be in $\mathcal{B}''$ and not empty. Then there exists a finite family $(B_i)_{i\in I}$ of non-empty subsets of $X$ such that $B = \bigcup_{i\in I} B_i$ and $U_i := B_i \times B_i \in \mathcal{C}_X$. Then for every $i$ in $I$ we have $B_i = U_i[\{b_i\}]$ for some point $b_i$ of $B_i$. This implies $B_i \in \mathcal{B}'$ for all $i$ in $I$ and hence $B \in \mathcal{B}'$. □

**Example 3.3.** The minimal generalized bornology compatible with a coarse structure is the empty bornology.

Every (generalized) bornology is compatible with the minimal coarse structure.

The maximal bornology is compatible with any coarse structure. □

**Example 3.4.** Show by example that in general $\{B \mid B \text{ is } U\text{-bounded for some } U \text{ in } \mathcal{C}_X\}$ is not a bornology. □

**Remark 3.5.** Let $X$ be a bornological space and $\mathcal{A}$ be a subset of $\mathcal{P}_{X\times X}$. In order to check that $\mathcal{C}_X := \mathcal{C}\langle\mathcal{A}\rangle$ is compatible with $\mathcal{B}_X$ it suffices to show that $A[B] \in \mathcal{B}_X$ and $A^{-1}[B] \in \mathcal{B}_X$ for all $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}_X$.

In order to see this we will use the notation from Lemma 2.12. If $V$ is in $\mathcal{C}_X$, then $V \subseteq \bigcup_{j\in J}\bigcup_{i=1,\dots,n_j} A_{j,i}$. We then use that $(W \circ W')[B] = W[W'[B]]$ and $(W \cup W')[B] \subseteq W[B] \cup W'[B]$ for any two entourages $W$ and $W'$ of $X$. □

**Definition 3.6.** *A bornological coarse space is a triple $(X, \mathcal{C}, \mathcal{B})$ of a set with a coarse and a bornological structure which are compatible.*

**Example 3.7.** This example generalizes 1.4 and 2.7 at the same time. Let $X$ be a bornological coarse space. We assume that $\mathcal{B}_{min}$ is compatible with $\mathcal{C}_X$. This means that $U[\{x\}]$ is finite for every $x$ in $X$ and $U$ in $\mathcal{C}_X$.

We consider the $R$-module

$$\mathbf{A}(X) := \{A \in R^{X \times X} \mid \mathtt{supp}(A) \in \mathcal{C}_X\} \ .$$

This $R$-module has an associative algebra structure defined by matrix multiplication:

$$A''(x,y) := (A' \circ A)(x,y) := \sum_{z \in X} A'(x,z)A(z,y) \ .$$

Indeed, the sum runs over the finite set $\mathtt{supp}(A')^{-1}[\{x\}] \cap \mathtt{supp}(A)[\{y\}]$. Note that $\mathtt{supp}(A) \subseteq \mathtt{supp}(A') \circ \mathtt{supp}(A)$.

We consider the $R$-module $C_{\mathrm{lf}}(X,R)$ of functions $f : X \to R$ whose support is locally finite (Definition 1.17). Thus for $f$ in $C_{\mathrm{lf}}(X,R)$ we have $|\mathtt{supp}(f) \cap B|$ is finite for every $B$ in $\mathcal{B}_X$.

We can define an action of $\mathbf{A}(X)$ on $C_{\mathrm{lf}}(X,R)$ as follows:

$$(Af)(x) = \sum_{y \in X} A(x,y)f(y) \ .$$

This sum is finite. Let $B$ be bounded in $X$. Then $B \cap \mathtt{supp}(Af) \subseteq A[\mathtt{supp}(A)^{-1}[B] \cap \mathtt{supp}(f)]$. Since $[\mathtt{supp}(A)^{-1}[B]$ is bounded the intersection with $\mathtt{supp}(f)$ is finite, and hence $A[\mathtt{supp}(A)^{-1}[B] \cap \mathtt{supp}(f)]$ is finite. This shows that $Af$ belongs to $C_{\mathrm{lf}}(X,R)$. □

**Example 3.8.** If $(X,d)$ is a metric space, then $\mathcal{B}_d$ and $\mathcal{C}_d$ are compatible. We get a bornological coarse space $X_d := (X, \mathcal{C}_d, \mathcal{B}_d)$.

**Example 3.9.** Let $X$ be a Hausdorff space and $A$ be a subset. We set $X := X \setminus A$. Then the continuously controlled coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$ of subsets $B$ with $\bar{B} \cap A = \emptyset$ are compatible.

We check the compatibility. Let $B$ be in $\mathcal{B}$ and $U$ in $\mathcal{C}$. Assume that $U[B] \notin \mathcal{B}$. Then $\overline{U[B]} \cap A$ contains a point $a$. Then there exists a net $(x_i, b_i)_{i \in I}$ in $U$ such that $\mathtt{lim}_{i \in I}\, x_i = a$. Then also $\mathtt{lim}_{i \in I}\, b_i = a$ and hence $\bar{B} \cap A \neq \emptyset$. This is a contradiction.

We call the structure $(\mathcal{C}, \mathcal{B})$ the continuously controlled bornological coarse structure on $X$.

If we omit the Hausdorff assumption then the same works for generalized bornologies. □

By **BornCoarse** we denote the category of bornological coarse spaces and proper and controlled maps. We can apply the above definitions to generalized bornologies and obtain the notion of a generalized bornological coarse space. We get the category $\widetilde{\mathbf{BornCoarse}}$ of generalized bornological coarse spaces and a fully faithful inclusion

$$\mathbf{BornCoarse} \to \widetilde{\mathbf{BornCoarse}} .$$

**Example 3.10.** If $X$ is a generalized bornological coarse space, then we have a canonical coarsely disjoint decomposition $X = X_b \sqcup X_u$ into the subsets of bounded and unbounded points. Assume that $b$ is in $X_b$ and $u$ is in $X_u$ and $b$ and $u$ belong to same coarse component, then $\{(u, b)\}$ would be a coarse entourage of $X$. But then $u \in U[\{b\}]$ and hence $\{u\}$ would be bounded, which is a contradiction. □

**Example 3.11.** Let $f : X \to Y$ be a map of sets and assume that $Y$ has a bornological coarse structure $(\mathcal{C}_Y, \mathcal{B}_Y)$. Then we can equip $X$ with the maximal coarse structure $\mathcal{C}_X$ such that $f : X \to Y$ is controlled and the minimal bornology $\mathcal{B}_X$ such that $f : X \to Y$ is proper.

We check that these structures are compatible. Let $U$ be in $\mathcal{C}_X$ and $B$ be in $\mathcal{B}_X$. Then there exists a finite family $(B_i)_{i\in I}$ in $\mathcal{B}_Y$ such that $B \subseteq \bigcup_{i\in I} f^{-1}(B_i)$. We then have $U[B] \subseteq \bigcup_{i\in I} U[f^{-1}(B_i)]$. We now check that $U[f^{-1}(B_i)] \subseteq f^{-1}((f \times f)(U)[B_i])$. By construction of $\mathcal{C}_X$ we know that $(f \times f)(U)$ is controlled in $Y$ and hence $(f \times f)(U)[B_i]$ is bounded in $Y$. But then $f^{-1}((f \times f)(U)[B_i])$ is bounded in $X$ for all $i$ in $I$ and hence also $U[B]$ is bounded.

We call the bornological coarse structure on $X$ the induced structure. Note that $f : X \to Y$ is then a morphism in **BornCoarse**. □

**Example 3.12.** Let $G$ be a group. we equip $G$ with the minimal bornology $\mathcal{B}_{min}$ consisting of the finite subsets. Furthermore we consider the canonical coarse structure $\mathcal{C}_{can} := \mathcal{C}\langle\{G(B \times B) \mid B \in \mathcal{B}_{min}\}\rangle$. Then $\mathcal{C}_{can}$ and $\mathcal{B}_{min}$ are compatible. Indeed, if $A$ is in $\mathcal{B}_{min}$ and $U = G(B \times B)$, then

$$U[A] \subseteq \bigcap_{\{g\in G \mid gB\cap A\neq\emptyset\}} gB \cap A .$$

This set is finite since $\{g \in G \mid gB \cap A \neq \emptyset\}$ is finite and $gB \cap A$ is finite for every $g$ in $G$.

**Definition 3.13.** *We write $G_{can,min}$ for the bornological coarse space $G$ with the structures $\mathcal{C}_{can}$ and $\mathcal{B}_{min}$.*

Note that $G$ acts on $G_{can,min}$ by automorphisms of bornological coarse spaces from the left. Furthermore the set of $G$-invariant entourages $\mathcal{C}^G_{can}$ is cofinal in $\mathcal{C}_{can}$. This condition characterizes $G$-bornological coarse spaces among bornological coarse spaces with $G$-action.

Many groups admit a finite description by generators and relations $G \cong \langle S|R\rangle$. Going over to $G_{can,min}$ we obtain a description of interesting bornological coarse spaces (with high symmetry) in finite terms.

Note that $\mathbb{Z}^n_{can,min}$ is equivalent to $\mathbb{Z}^n$ with the metric structures from $\mathbb{R}^n$. The group $\mathbb{Q}^n_{can,min}$ is completely different from $\mathbb{Q}^n$ with the metric structure. □

Let $X, Y$ be in **BornCoarse**.

**Definition 3.14.** *A morphism $f : X \to Y$ is called an equivalence if there exists a morphism $g : Y \to X$ such that $f \circ g$ is close to $\mathtt{id}_Y$ and $g \circ f$ is close to $\mathtt{id}_X$.*

**Example 3.15.** We consider $\mathbb{R}$ with the standard bornological coarse structure from the metric. We consider $\mathbb{Z}$ with the bornological coarse structure induced from $\mathbb{R}$. Then $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. If we equip $\mathbb{Z}$ with the maximal bornology, then this map is no longer an equivalence in **BornCoarse** since any potential inverse $\mathbb{R} \to \mathbb{Z}$ is not proper. □

The category **BornCoarse** has a symmetric monoidal structure $\otimes$. Let $X$ and $Y$ be in **BornCoarse**.

**Definition 3.16.**

*We define $X \otimes Y$ as follows:*

1. *The underlying set of $X \otimes Y$ is $X \times Y$.*
2. *The coarse structure on $X \otimes Y$ is generated by the entourages $U \times V$ for all $U$ in $\mathcal{C}_X$ and $V \in \mathcal{C}_Y$.*
3. *The bornology of $X \otimes Y$ is generated by the subsets $A \times B$, where $A$ is in $\mathcal{B}_X$ and $B$ is in $\mathcal{B}_Y$.*

One checks that $\mathcal{B}_X$ is compatible with $\mathcal{C}_X$. The unit, associativity and symmetry constraints are induced from the cartesian symmetric monoidal structure on **Set**.

**Example 3.17.** The underlying coarse space of $X \otimes Y$ is the cartesian product of the underlying coarse spaces. The underlying bornological space of $X \otimes Y$ is the bornological tensor product from Definition 1.31. □

**Example 3.18.** We have an equivalence $\mathbb{R}^{n+m} \simeq \mathbb{R}^n \otimes \mathbb{R}^m$. □

**Example 3.19.** There are functors

$$(-)_{min,max}, (-)_{max,max} : \mathbf{Set} \to \mathbf{BornCoarse}$$

which send a set $X$ to the bornological coarse space $X_{min,max}$ (or $X_{max,max}$) with the minimal (or maximal) coarse and the maximal bornological structure.

There are no such functors which equip $X$ with the minimal bornology. □

**Lemma 3.20.** *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_{min}, \mathcal{B}_X)) : \mathbf{Born} \leftrightarrows \mathbf{BornCoarse} : \mathrm{forget} \ .$$

*Proof.* We write $X_{min,\mathcal{B}_X} := (X, \mathcal{C}_{min}, \mathcal{B}_X)$. For every $T$ in **BornCoarse** we have an equality

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(X_{min,\mathcal{B}_X}, T) = \mathtt{Hom}_{\mathbf{Born}}(X, T) \ .$$

□

The same argument gives:

**Lemma 3.21.** *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_{min}, \mathcal{B}_X)) : \widetilde{\mathbf{Born}} \leftrightarrows \widetilde{\mathbf{BornCoarse}} : \mathrm{forget} \ .$$

**Remark 3.22.** Note that there is no adjunction

$$\mathrm{forget} : \widetilde{\mathbf{BornCoarse}} \leftrightarrows \widetilde{\mathbf{Born}} : (X \mapsto (X, \mathcal{C}_{max}, \mathcal{B}_X))$$

generalizing the adjunction from Lemma 2.29. The problem is that $\mathcal{C}_{\mathrm{max}}$ is not compatible with a general bornology.

**Lemma 3.23.** *We have an adjunction*

$$(X \mapsto (X, \mathcal{C}_X, \mathcal{B}_{max})) : \mathbf{Coarse} \leftrightarrows \mathbf{BornCoarse} : \mathrm{forget} \ .$$

*Proof.* We write $X_{\mathcal{C}_X,max} := (X, \mathcal{C}_X, \mathcal{B}_{max})$. For every $T$ in **BornCoarse** we have an equality

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(X_{\mathcal{C}_X,max}, T) = \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

□

**Lemma 3.24.** *We have an adjunctions*

$$(X \mapsto (X, \mathcal{C}_X, \mathcal{B}_{max})) : \mathbf{Coarse} \leftrightarrows \widetilde{\mathbf{BornCoarse}} : \mathrm{forget}$$

*and*

$$\mathrm{forget} : \widetilde{\mathbf{BornCoarse}} \leftrightarrows \mathbf{Coarse} : (X \mapsto (X, \mathcal{C}_X, \emptyset))$$

*Proof.* The first case is as in Lemma 3.23. For the second we write $X_{\mathcal{C}_X,\emptyset} := (X, \mathcal{C}_X, \emptyset)$. For every $T$ in $\widetilde{\mathbf{BornCoarse}}$ we have an equality

$$\mathtt{Hom}_{\widetilde{\mathbf{BornCoarse}}}(T, X_{\mathcal{C}_X,\emptyset}) = \mathtt{Hom}_{\mathbf{Coarse}}(X, T) \ .$$

□

**Proposition 3.25.** *The category* $\widetilde{\mathbf{BornCoarse}}$ *is complete and cocomplete.*

*Proof.* It is clear from the adjunctions above that the underlying coarse spaces of colimits and limits in $\widetilde{\mathbf{BornCoarse}}$ are the limits and colimits of the underlying diagrams in **Coarse**. The same applies to the underlying bornological space of a limit. As suggested by the observation in Remark 3.22 the bornology of a colimit is more complicated.

Let $X : \mathbf{I} \to \widetilde{\mathbf{BornCoarse}}$ be a diagram. Then we equip the limit $Y$ of the diagram of underlying sets with the coarse structure and generalized bornology such that the resulting coarse and generalized bornological space represent the limit in **Coarse** and $\widetilde{\mathbf{Born}}$. We first check that the coarse structure and the generalized bornology on $Y$ are compatible so that $Y$ becomes an object of $\widetilde{\mathbf{BornCoarse}}$. We check the compatibility on the generators of the coarse structure with the bornology.

For $i$ in $\mathbf{I}$ let $p_i : Y \to X_i$ be the canonical projection. By construction $p_i$ is a morphism in $\widetilde{\mathbf{BornCoarse}}$ for every $i$ in $\mathbf{I}$. An entourage $U$ of $X$ is coarse if $p_i(U)$ is coarse in $X_i$ for all $i$ in $\mathbf{I}$. The generators of the bornology of $X$ are the subsets $p_i^{-1}(B)$ for $i$ in $\mathbf{I}$ and bounded subsets $B$ in $X_i$. For such a generator we have $U[p^{-1}(B)] \subseteq p_i^{-1}(p_i(U)[B])$. Since $p_i(U)[B]$ is again bounded by the compatibility of structures on $X_i$ we conclude that $U[p^{-1}(B)]$ is bounded in $X$.

We now show that $(Y, (p_i)_{i\in\mathbf{I}})$ is a limit of the diagram $X$. Let $T$ be in $\widetilde{\mathbf{BornCoarse}}$. By construction we have bijections

$$\mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T,Y) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{Born}}}(T,X) \ , \quad \mathtt{Hom}_{\mathbf{Coarse}}(T,Y) \stackrel{\cong}{\to} \lim_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Coarse}}(T,X)$$

which immediately implies $\mathtt{Hom}_{\widetilde{\mathbf{BornCoarse}}}(T,Y) \to \mathtt{lim}_{i\mathbf{I}} \mathtt{Hom}_{\widetilde{\mathbf{BornCoarse}}}(T,X)$.

For cocompleteness we show the existence of coproducts and coequalizers.

Let $(X_i)_{i\in I}$ be a family in $\widetilde{\mathbf{BornCoarse}}$. We consider the coproduct of sets $X := \coprod_{i\in I} X_i$ and the embeddings $e_i : X_i \to X$. We equip $X$ with the minimal coarse structure such that $e_i$ is controlled for all $i$ in $I$ and the maximal generalized bornology such that $e_i$ is proper for all $i$ in $I$. This the coarse structure generated by the entourages $e_i(U)$ for $i$ in $I$ and coarse entourages $U$ of $X_i$, and a subset $B$ is bounded if $e_i^{-1}(B)$ is bounded for every $i$ in $I$. We check compatibility on generators. We have

$$e_i^{-1}(e_j(U)[B]) \subseteq \left\{ \begin{array}{cc} \emptyset & i \neq j \\ U[e_i^{-1}(B)] & i = j \end{array} \right. .$$

This set is bounded for every $i$ by the compatibility of structures on $X_i$. Hence $e_j(U)[B]$ is bounded in $X$.

We claim that the generalized bornological coarse space $X$ with the family of embeddings $(e_i)_{i\in I}$ represents the coproduct in $\widetilde{\mathbf{BornCoarse}}$. Let $T$ be in $\widetilde{\mathbf{BornCoarse}}$ together with a family of morphisms $t_i : X_i \to T_i$. There exists a unique map of sets $t : X \to T$ such that $t \circ e_i = t_i$ for all $i$ in $I$. This map is controlled and proper. Indeed, for a bounded $B$ in $T$ the set $e_i^{-1}(t^{-1}(B)) = t_i^{-1}(B)$ is bounded for every $i$ in $I$. Hence $t^{-1}(B)$ is bounded in $X$. For a coarse entourage $U$ of $X_i$ the entourage $t(e_i(U)) = t_i(U)$ is coarse in $X_i$. This implies that $t$ is controlled.

We now show the existence of coequalizers. Let $f, g : X \to Y$ be two morphisms in **BornCoarse**. Then we consider the coequalizer $q : Y \to Q$ of the two maps on the level of underlying sets. We equip $Y$ with the minimal coarse structure structure such that this map is controlled and the maximal compatible genberalized bornology such that this map is proper. If $U$ is an entourage of $Y$ and $B$ is a subset of $Q$, then we have $U(q^{-1}[B]) \subseteq q^{-1}(q(U)[B])$. Even if $q^{-1}(B)$ is bounded this does in general not imply that $q^{-1}(q(U)[B])$ is bounded. The generalized bornology of $Q$ is given by subsets $B$ such that $q^{-1}(q(U)[B])$ is bounded for all $U$ in $\mathcal{C}_Y$.

We claim that $Y \to X$ represents the coequalizer of $f$ and $g$. Let $t : Y \to T$ be a morphism such that $t \circ f = t \circ g$. Then there is a unique factorization over a map of sets $c : Q \to T$. One checks that this map is a morphism. The coarse structure of $Q$ is generated by the entourages $q(U)$ for coarse entouragesq of $Y$. Since $c(q(U)) = t(U)$ we conclude that $t$ is controlled. Let now $B$ be bounded in $T$. Then $t(U)[B]$ is also bounded for every coarse entourage $U$ of $Y$. This implies that $t^{-1}(t(U)[B])$ is bounded. Now

$$q^{-1}(q(U)[c^{-1}(B)]) \subseteq q^{-1}c^{-1}(c(q(U))[B]) = t^{-1}(t(U)[B])$$

implies that $q^{-1}(q(U)[c^{-1}(B)])$ is bounded. We conclude that $c^{-1}(B)$ is bounded in $Q$.

□

**Example 3.26.** Consider the two projections $\mathtt{pr}_0, \mathtt{pr}_1 (X \times X)_{min,max} \to X_{min,min}$. If $X$ is infinite, then the coequalizer of this diagram is the unbounded point. □

Note that **BornCoarse** is a full subcategory of $\widetilde{\mathbf{BornCoarse}}$. It therefore inherits all limits and colimits taken in $\widetilde{\mathbf{BornCoarse}}$ of diagrams in **BornCoarse** which are represented by objects of **BornCoarse**.

**Proposition 3.27.** *The category* **BornCoarse** *has all non-empty limits.*

*Proof.* If $X_- : \mathbf{I} \to \mathbf{BornCoarse}$ is a diagram such that $\mathbf{I}$ is non-empty, then $\mathtt{lim}_{\mathbf{I}} X_i$ consists of bounded points. Indeed let $x$ be such a point. Then $p_i(x)$ is bounded for every $i$ in $I$. Hence $x$ is a point in the bounded subset $\bigcap_{i \in I} p_i^{-1}(\{x\})$. □

**Proposition 3.28.** *The category* **BornCoarse** *has all coproducts.*

*Proof.* Exercise. □

Example 3.26 shows that **BornCoarse** does not have all colimits.

**Example 3.29.** The square

$$\begin{array}{ccc} \{*\} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ -\mathbb{N} & \longrightarrow & \mathbb{Z} \end{array}$$

is a push-out in **BornCoarse**. □

**Example 3.30.** We consider subsets of $\mathbb{N}$ given by

$$Y := \bigcup_{n\in\mathbb{N}}[(2n)^2,(2n+1)^2]\ , \quad X := \bigcup_{n\in\mathbb{N}}[(2n+1)^2,(2n+2)^2]\ .$$

The square

$$\begin{array}{ccc} Y\cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbb{N} \end{array}$$

(all subspaces have the induced structure) is not a push-out. The coarse structure of $\mathbb{N}$ is generated by a single entourage $U_1$ and $|\pi_0^{coarse}(\mathbb{N})| = 1$. But the coarse structure of the push-out is not generated by a single entourage. Indeed, for any entourage $U$ of the push-out structure we have $|\pi_0^{coarse}(\mathbb{N}_U)| = \infty$. □

# 4 Coarse homology theories

Let $E$ : **BornCoarse** $\to$ **M** be a functor.

**Definition 4.1.** *$E$ is coarsely invariant if $E$ sends coarse equivalences to equivalences (isomorphisms).*

**Lemma 4.2.** *The following are equivalent.*

1. *$E$ is coarsely invariant.*

2. *$E$ sends pairs of close map to pairs of equivalent (equal) maps.*

3. *$E(\{0,1\}_{max,max}\otimes X)\to X$ is an equivalence (isomorphism) for all $X$ in **C**.*

*Proof.*

$1\Rightarrow 2$

We observe that $p : \{0,1\}_{max,max}\otimes X \to X$ is coarse equivalence. Inverses are the inclusion $i_0,i_1 : X\to\{0,1\}_{max,max}\otimes X$ given by $i_0(x) := (0,x)$ and $i_1(x) := (1,x)$. Indeed,

$p \circ i_0 = \mathtt{id}_X$ and $i_0 \circ p$ is close to $\mathtt{id}$ since $(i_0 \circ p, \mathtt{id})(x,x) \in \{0,1\} \times \{0,1\} \times \mathtt{diag}(X)$ for all $x$ in $X$, and $\{0,1\} \times \{0,1\} \times \mathtt{diag}(X)$ is a coarse entourage of $\{0,1\}_{max,max} \otimes X$. A similar argument applies to $i_1$.

Now assume that $f, g : X \to Y$ are morphisms which are close to each other. Then we define

$$h : \{0,1\} \otimes X \to Y \ , \quad h(i,x) := \begin{cases} f(x) & i = 0 \\ g(x) & i = 1 \end{cases} \ .$$

This is a morphism. If $B$ is bounded in $Y$, then $h^{-1}(B) \subseteq \{0,1\} \times (f^{-1}(B) \cup g^{-1}(B))$. This shows that $h$ is proper.

Assume that $f \sim_V g$ for some entourage $V$ of $Y$. Let $U$ be an entourage of $X$ and set $W := \{0,1\} \times \{0,1\} \times U$. Then $h(W) \subseteq V \cup f(U) \cup g(U)$.

Since $E$ is coarsely invariant we have $E(i_0) = E(i_1)$ since both are inverse to $E(p)$. We have $f = h \circ i_0$, $g = h \circ i_1$. By functoriality, $E(f) = E(h)E(i_0) = E(h)E(i_1) = E(g)$.

$2 \Rightarrow 1$

Let $f : X \to Y$ be a coarse equivalence. Then there exists a morphism $g : Y \to X$ such that $f \circ g \sim \mathtt{id}_Y$ and $g \circ f \sim \mathtt{id}_X$. We have $E(f) \circ E(g) = E(\mathtt{id}_Y) = \mathtt{id}_{E(Y)}$ and $E(g) \circ E(f) = E(\mathtt{id}_X) = \mathtt{id}_{E(X)}$.

$1 \Rightarrow 3$

We have already seen that $p$ is a coarse equivalence. Hence $E(p) : E(\{0,1\} \otimes X) \to X$ is an equivalence.

$3 \Rightarrow 2$

Let $f, g : X \to Y$ be two morphisms which are close to each other. Then we form $h : \{0,1\}_{max,max} \otimes X \to Y$ as above. Since $E(p)$ is an equivalence and $p \circ i_0 = p \circ i_1 = \mathtt{id}_X$ we conclude that $E(i_0) = E(i_1)$ since both are right inverse to $E(p)$. Then we calculate, using functoriality, that $E(f) = E(h)E(i_0) = E(h)E(i_1) = E(g)$

□

**Example 4.3.** The functor $X \mapsto \pi_0^{coarse}(X)$ is coarsely invariant. Indeed

$$\pi_0^{coarse}(\{0,1\}_{max,max} \otimes X) \to \pi_0^{coarse}(X)$$

is isomorphism. In order to see this note that $[i,x] \simeq [j,y]$ in $\pi_0^{coarse}(\{0,1\}_{max,max} \otimes X)$ if and only if $[x] = [y]$ in $\pi_0^{coarse}(X)$. □ □

**Example 4.4.** For every $Y$ in **BornCoarse** the functor

$$X \mapsto \mathtt{Hom}_{\overline{\mathbf{Coarse}}}(Y, X)$$

is coarsely invariant. □

**Definition 4.5.** *A bornological coarse space is flasque if it admits an endomorphisms $f : X \to X$ satisfying:*

1. *$f$ is close to $\mathtt{id}_X$.*

2. *For every $U$ in $\mathcal{C}_X$ we have $\bigcup_{n\in\mathbb{N}} f^n(U) \in \mathcal{C}_X$.*

3. *For every $B$ in $\mathcal{B}_X$ there exists $n$ in $\mathbb{N}$ such that $f^n(X) \cap B = \emptyset$.*

We say that $f$ witnesses flasqueness of $X$.

**Example 4.6.** For $X$ in **BornCoarse** the space $\mathbb{N}\otimes X$ is flasque with $f : \mathbb{N}\otimes X \to X$given by $f(n, x) := (n + 1, x)$.

We check the axioms.

1. We have $(\mathtt{id}, f)(\mathtt{diag}(\mathbb{N} \otimes X) \subseteq U_1 \times \mathtt{diag}(X)$.

2. We have for an entourage $U_r \times V$ of $\mathbb{N} \otimes X$ that $\bigcup_{n\in\mathbb{N}} f^n(U_r \times V) \subseteq U_r \times V$.

3. Finally, if $B$ is bounded in $\mathbb{N} \times X$, then $B \subseteq [0, n] \otimes X$ for some $n$ in $\mathbb{N}$. Since $f^{n+1}(\mathbb{N} \times X) \subseteq [n + 1, \infty) \times X$ we conclude that $f^{n+1}(\mathbb{N} \times X) \cap B = \emptyset$.

Note that also $[0, \infty) \otimes X$ is flasque.

On the other hand $\mathbb{Z} \times X$ is not flasque. The map $(n, x) \mapsto (n + 1, x)$ does not work since the third axiom is not fullfilled. We will use coarse homology theories to see that there is no other map implementing flasqueness.

□

**Definition 4.7.** *$X$ is flasque in the generalized sense if $X \to \mathbb{N} \otimes X$, $x \mapsto (0, x)$ has a retract $r : \mathbb{N} \otimes X \to X$.*

**Remark 4.8.** The retract in Definition 4.7 is the datum of a family $(f_n)_{n\in\mathbb{N}}$ of maps $f_n : \mathbb{N} \to X$ such that

1. $f_0 = \mathtt{id}_X$ (retract property).

2. There exists a coarse entourage $V$ of $X$ such that $f_n \sim_V f_{n+1}$ for all $n$ in $\mathbb{N}$, and every coarse entourage $U$ of $X$ the entourage $\bigcup_{n\in\mathbb{N}} f^n(U)$ is coarse ($r$ is controlled).

3. For every bounded subset $B$ in $X$ there exists $n_0$ in $\mathbb{N}$ such that $f_n(X) \cap B = \emptyset$ for all $n$ in $\mathbb{N}$ with $n \geq n_0$ ($r$ is proper).

□

**Lemma 4.9.** *If $X$ is flasque, then it is flasque in the generalized sense.*

*Proof.* Let $f : X \to X$ witnesses flasqueness of $X$. We define $r : \mathbb{N} \otimes X \to X$ by $r(n, x) := f^n(x)$. The family of functions $(f^n)_{n \in \mathbb{N}}$ satisfies the conditions listed in Remark 4.8. □

A pointed category is a category in which initial and final objects coincide. We write 0 for such objects. A morphism in a pointed category is a zero morphism if it admits a factorization over a zero object. The composition of a zero morphism with any morphism is again a zero morphism. Between any two objects there exists a unique zero morphism.

**Example 4.10.** The category of pointed sets $\mathbf{Set}_*$ and base-point preserving maps is pointed with zero object $*$.

**Ab** is pointed by 0. □

Let $E : \mathbf{C} \to \mathbf{M}$ be a functor to a pointed category.

**Definition 4.11.** *$E$ vanishes in flasques if for every flasque $X$ the canonical map $0 \to E(X)$ is an equivalence (isomorphism).*

**Lemma 4.12.** *The following assertions are equivalent.*

1. *$E$ vanishes on generalized flasques.*

2. *$E$ vanishes in flasques.*

3. *$0 \to E(\mathbb{N} \otimes X)$ is an equivalence for every $X$ in* **BornCoarse**.

*Proof.*

$1 \Rightarrow 2$

This is clear since clear since flasques are generalized flasques by Lemma 4.9.

$2 \Rightarrow 3$

This is clear since since $\mathbb{N} \otimes X$ is flasque by Example 4.6.

$3 \Rightarrow 1$

By assumption we have a retract $r : \mathbb{N} \otimes X \to X$ of $i_0 : X \to \mathbb{N} \otimes X$. Then $E(\mathtt{id}_X) = E(r) \circ E(i_0)$. $E(X)$ is retract of 0 and hence 0 since $\mathbb{N}$

□

Recall that for $X$ in **BornCoarse** and $U$ in $\mathcal{C}_X$ we write $X_U$ for the bornological space $X$ equipped with the coarse structure $\mathcal{C}_U := \mathcal{C}\langle\{U\}\rangle$. Since $\mathcal{C}_U \subseteq \mathcal{C}_X$ this structure is compatible with the bornology.

If $U'$ is in $\mathcal{C}_X$ and $U \subseteq U'$, then the identity of $X$ induces a map $X_U \to X_{U'}$. We have a map $X_U \to X$.

**Definition 4.13.** *We say that $E$ is u-continuous if the canonical map $\mathtt{colim}_{U \in \mathcal{C}_X} E(X_U) \to E(X)$ is an equivalence.*

**Example 4.14.** The functor

$$P : \mathbf{C} \to \mathbf{sSet} \ , \quad X \mapsto \mathop{\mathtt{colim}}_{U \in \mathcal{C}_X} P_U(X)$$

is $u$-continuous. The composition of $P$ with any filtered colimit-preserving functor is again $u$-continuous, e.g. $\mathbb{Z}[P] : \mathbf{BornCoarse} \to \mathbf{sAb}$. □

**Example 4.15.** The functor $X \mapsto \pi_0^{coarse}(X)$ is $u$-continuous. It is clear that the canonical maps $\pi_0^{coarse}(X_U) \to \pi_0^{coarse}(X_{U'}) \to \pi_0^{coarse}(X)$ are surjective for every $U$, $U'$ in $\mathcal{C}_X$ with $U \subseteq U'$. This implies that $\mathtt{colim}_{U \in \mathcal{C}_X} \pi_0^{coarse}(X_U) \to \pi_0^{coarse}(X)$ is surjective. In order to show injectivity let $x, x'$ be in $X$ and assume that $[x]_X = [x']_X$ in $\pi_0^{coarse}(X)$. Then $x \sim x'$ for some $U$ in $\mathcal{C}_X$. But then $[x]_{X_U} = [x']_{X_U}$ in $\pi_0^{coarse}(X_U)$. □

**Example 4.16.** Let $Y$ be in **BornCoarse** such that $\mathcal{C}_Y = \mathcal{C}_V$ for some entourage $V$ of $Y$. Then the corepresented functor

$$\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, -) : \mathbf{BornCoarse} \to \mathbf{Set}$$

is $u$-continuous. Indeed, $\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_U) \to \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X)$ is obviously the inclusion of a subset for every $U$ in $\mathcal{C}_Y$. If $f$ is in $\mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X)$, then we have $f \in \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_{f(V)})$. This shows that

$$\mathop{\mathtt{colim}}_{U \in \mathcal{C}_X} \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X_U) = \mathtt{Hom}_{\mathbf{BornCoarse}}(Y, X) \ .$$

Without the condition of the coarse structure of $Y$ this assertion is wrong in general. For example, let $X := \{n^2 \mid n \in \mathbb{N}\}$ with the structures induced from the inclusion into $\mathbb{N}$. Then $\mathtt{id}_X \notin \mathtt{colim}_{U \in \mathcal{C}_X} \mathtt{Hom}_{\mathbf{BornCoarse}}(X, X_U)$ since otherwise $X = X_U$ for some entourage $U$ of $X$. But one can show that the coarse structure on $X$ is not generated by any finite set of entourages. □

Let $X$ be in **BornCoarse**.

**Definition 4.17.** *A big family in $X$ is a filtered subposet $\mathcal{Y}$ of $\mathcal{P}_X$ such that for every member $Y$ in $\mathcal{Y}$ and $U$ in $\mathcal{C}_X$ there exists a member $Y'$ in $\mathcal{Y}$ such that we have $U[Y] \subseteq Y'$.*

We say that $\mathcal{Y}$ is complete if $Y$ in $\mathcal{Y}$ and $Y' \subseteq Y$ imply that $Y' \in \mathcal{Y}$. We can form the completion $\bar{\mathcal{Y}}$ by adding all subsets of members of $\mathcal{Y}$.

**Example 4.18.** The bornology of $X$ is a complete big family. □

**Example 4.19.** Let $Y$ be a subset of $X$ in **BornCoarse**. Then we can form the big family $\{Y\} := \{U[Y] \mid U \in \mathcal{C}_X\}$ generated by $Y$. This big family has the property that all members $U[Y]$ with $\mathtt{diag}(X) \subseteq U$ are coarsely equivalent to $Y$. In fact $Y$ is an $U$-dense subset of $U[Y]$. The inclusion $Y \to U[Y]$ is a coarse equivalence. □

For a map $f : X \to X'$ we write $f(\mathcal{Y}) = \{f(Y) \mid Y \in \mathcal{Y}\}$. In general this is not a big family.

We consider pairs $(X, \mathcal{Y})$ of $X$ in **BornCoarse** and a big family $\mathcal{Y}$ on $X$.

**Definition 4.20.** *A morphism $f : (X, \mathcal{Y}) \to (X', \mathcal{Y}')$ is a morphism $f : X \to X'$ such that $f(\mathcal{Y}) \subseteq \bar{\mathcal{Y}}'$.*

We get the category $\mathbf{BC}^{\mathrm{pair}}$ of pairs and morphisms.

We have a functor

$$\mathbf{BornCoarse} \to \mathbf{BC}^{\mathrm{pair}} , \quad X \mapsto (X, \emptyset) .$$

If $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$ is a functor, then we can restrict $E$ to a functor

$$uE : \mathbf{BornCoarse} \to \mathbf{M} , \quad X \mapsto E(X, \emptyset) ,$$

called the underlying functor. We will use the notation $uE$ in order to denote the underlying functor, but if we insert an argument, then we will omit $u$since it is clear from that fact that the argument has one entry that we mean the underlying functor. So we write $E(X)$ instead of $uE(X)$.

We have a functor

$$\mathbf{BC}^{\mathrm{pair}} \to \mathbf{BornCoarse} , \quad (X, \mathcal{Y}) \mapsto X .$$

Any functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ extends to a functor

$$E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} , \quad E(X, \mathcal{Y}) := E(X) .$$

Extending and then restricting reproduces the initial functor.

If $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$ is a functor to a target which admits filtered colimits, then we define a functor

$$\partial E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} \ , \quad \partial E(X, \mathcal{Y}) := E(\mathcal{Y}) := \mathtt{colim}_{Y \in \bar{\mathcal{Y}}} E(Y) \ .$$

On morphisms $f : (X, \mathcal{Y}) \to (X', \mathcal{Y}')$ the inclusion $f_{|Y} : E(Y) \to E(f(Y))$ induces a compatible family of morphisms

$$E(Y) \to E(f(Y)) \to \mathtt{colim}_{Y' \in \bar{\mathcal{Y}}'} uE(Y')$$

for all $Y$ in $\mathcal{Y}$. We finally get a morphism

$$E(\mathcal{Y}) = \mathtt{colim}_{Y \in \bar{\mathcal{Y}}} E(Y) \to \mathtt{colim}_{\bar{\mathcal{Y}}'} E(Y') = E(\mathcal{Y}') \ .$$

This finishes the construction of $\partial E$ up to straightforward verifications.

The inclusion morphisms $Y \to X$ for all $Y$ in $\mathcal{Y}$ induce compatible morphisms $uE(Y) \to uE(X)$ and finally a morphism

$$E(\mathcal{Y}) \to E(X) \ .$$

These morphisms for all $(X, \mathcal{Y})$ in $\mathbf{BC}^{\mathrm{pair}}$ fit into a natural transformation of functors

$$\partial E \Rightarrow uE : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M} \ .$$

Let $(X, \mathcal{Y})$ be in $\mathbf{BC}^{\mathrm{pair}}$ and $Z$ be a subset of $X$.

**Definition 4.21.** *We say that $(Z, \mathcal{Y})$ is a complementary pair on $X$ if $X \setminus Z \in \bar{\mathcal{Y}}$.*

Note that $Z \cap \mathcal{Y} = \{Z \cap Y \mid Y \in \mathcal{Y}\}$ is a big family on $Z$. This follows from the inclusion $U_Z[Y \cap Z] \subseteq U[Y] \cap Z$ for all $U$ in $\mathcal{C}_X$, where $U_Z = U \cap (Z \times Z)$. We get an canonical morphism $(Z, Z \cap \mathcal{Y}) \to (X, \mathcal{Y})$.

**Example 4.22.** We consider the space $X = \mathbb{Z}$, the subspace $Z = [0, \infty)$, and we let $\mathcal{Y}$ be the family of subsets $\{(-\infty, n] \mid n \in \mathbb{N}\}$. This family is big. Note that $U_r[(-\infty, n]] \subseteq (-\infty, n + r]$ for every $r$ in $\mathbb{N}$. The closure of this family consists of all upper bounded subsets. Note that $\bar{\mathcal{Y}} = \overline{\{(-\infty, 0]\}}$.

Note that every member of $\mathcal{Y}$ is flasque. This is not true for the members of $\bar{\mathcal{Y}}$. For example $\{-n^2 \mid n \in \mathbb{N}\}$ is not flasque.

The pair $(Z, \mathcal{Y})$ is complementary in $\mathbb{Z}$. $\square$

**Example 4.23.** Let $Y$ be a Hausdorff topological space, $A$ be a subset, and $X := Y \setminus A$ be equipped with the continuously controlled structure. We let $Z$ be any open neighbourhood of $A$ and $\mathcal{Y} := \mathcal{B}_X$. We know that $\mathcal{Y}$ is a big family by Example 4.18. Furthermore, $X \setminus Z$ belongs to $\mathcal{Y}$. Indeed, $\overline{X \setminus Z} \cap A = (X \setminus Z) \cap \emptyset$. Hence $(Z, \mathcal{Y})$ is a complementary pair. Note that every member of $\mathcal{Y}$ is equivalent to $Y_{max,max}$ and hence to a point. This will be used to reduce the study of the coarse homotopy theory $X$ to arbitrary small neighbourhoods of $A$.

$\square$

We consider functor $F : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$.

**Definition 4.24.** *The functor $F$ is called excisive if for every complementary pair $(Z, \mathcal{Y})$ on $X$ in* **BornCoarse** *the canonical functor induces an equivalence (isomorphism)*

$$E(Z, Z \cap \mathcal{Y}) \to E(X, \mathcal{Y}) \ .$$

**Example 4.25.** Let $R$ be a ring. Then we have a functor **BornCoarse** $\to \mathrm{Mod}(R)^{\mathrm{op}}$ given by

$$X \mapsto C_{\mathrm{lf}}(X, R) \ , \quad (f : X \to X') \mapsto (f^* : C_{\mathrm{lf}}(X', R) \to C_{\mathrm{lf}}(X, R)) \ .$$

Here $f^*$ preserves local finiteness since $f$ preserves locally finite subsets by properness (see Example 1.21). We have a functor

$$(X, \mathcal{Y}) \mapsto C_{\mathrm{lf}}(X, \mathcal{Y}, R) := \operatorname*{\mathtt{colim}}_{Y \in \mathcal{Y}} C_{\mathrm{lf}}(X, R)/C_{\mathrm{lf}}(Y, R) \ .$$

We claim that this functor is excisive.

The map $C_{\mathrm{lf}}(X, R) \to C_{\mathrm{lf}}(Z, R)$ is surjective since we can extend any $\phi$ in $C_{\mathrm{lf}}(Z, R)$ by zero in order to obtain a preimage in $C_{\mathrm{lf}}(X, R)$. This implies that $C_{\mathrm{lf}}(X, \mathcal{Y}, R) \to C_{\mathrm{lf}}(Z, Z \cap \mathcal{Y}, R)$ is surjective. We now show injectivity. Assume that $[\phi]$ is in $C_{\mathrm{lf}}(X, \mathcal{Y}, R)$ such that $[\phi_{|Z}] = 0$. Then $\mathtt{supp}(\phi_{|Z}) \in \overline{Z \cap \mathcal{Y}}$. Hence $\mathtt{supp}(\phi) \subseteq (X \setminus Z) \cup \mathtt{supp}(\phi_{|Z})$ belongs to $\overline{\mathcal{Y}}$. Hence $[\phi] = 0$. $\square$

**Remark 4.26.** The Definition 4.24 is adapted to its usage in connection with $\delta$-functors below. A better definition would start, as initial data, with the underlying functor $uE$ : **BornCoarse** $\to$ **M** whose target is cocomplete. Then we would define $\partial E$ by $\partial E(\mathcal{Y}) := \mathtt{colim}_{Y \in \mathcal{Y}} E(Y)$.

We call $uE$ excisive if for every complementary pair $(Z, \mathcal{Y})$ the square

$$\begin{array}{ccc} \partial E(Z \cap \mathcal{Y}) & \longrightarrow & uE(Z) \\ \downarrow & & \downarrow \\ \partial E(\mathcal{Y}) & \longrightarrow & uE(X) \end{array}$$

is a push-out square in **M**.

If **M** is pointed, then we can define $E(X, \mathcal{Y}) := \mathrm{Cofib}(E(\mathcal{Y}) \to E(X))$. If the square above is a push-out, then the indued map of cofibres of the horizontal morphisms $E(Z, Z \cap \mathcal{Y}) \to E(X, \mathcal{Y})$ is an equivalence. Under additional assumptions on **M** (e.g. stable $\infty$-category) the two conditions are equivalent. $\square$

**Example 4.27.** Let $T$ be in **BornCoarse** and consider the functor $\mathtt{Hom}_{\mathbf{BornCoarse}}(-,T) : \mathbf{BornCoarse} \to \mathbf{Set}^{\mathrm{op}}$.

**Lemma 4.28.** *The functor $\mathtt{Hom}_{\mathbf{BornCoarse}}(-,T)$ is excisive in the sense of Remark 4.26.*

*Proof.* Homework. (solution appears later) □

Let $(Z,\mathcal{Y})$ be a complementary pair on $X$ and $f : X \to T$ be a map of sets. In order to check that $f$ is a morphism it suffices to check that $f_{|Z}$ is a morphism and $f_{|Y}$ is a morphism for every member $Y$ in $\mathcal{Y}$.

It is important to work with complementary pairs. To this end consider the following example with underlying bornological coarse space $\mathbb{Z}$. We consider the entourage $V := (U_1 \cap ((-\infty,0] \times (-\infty,0])) \cup (U_1 \cap ([0,\infty) \times [0,\infty)))$. The identity map $f : \mathbb{Z} \to \mathbb{Z}_U$ is not a morphism. Note that $|\pi_0^{coarse}(\mathbb{Z}_U)| = 2$, while $|\pi_0^{coarse}(\mathbb{Z})| = 1$. But $f_{|(-\infty,0]}$ and $f_{|[1,\infty)}$ are morphisms.

We let $\mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ denote the category of $\mathbb{Z}$-graded abelian groups. Let $G$ be in $\mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$. Then we write $G_n$ for the degree-$n$-component. Let $[k] : \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ be the shift functor given by $(G[k])_n = G_{n+k}$. given by

**Definition 4.29.** *A coarse $\delta$-functor is a pair $(E,\delta)$ of a functor $E : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$ and a natural transformation $\delta : E \to \partial E[-1]$ such that for every pair $(X,\mathcal{Y})$ the sequence*

$$E(\mathcal{Y}) \to E(X) \to E(X,\mathcal{Y}) \xrightarrow{\delta} E(\mathcal{Y})[-1]$$

*is exact.*

**Definition 4.30.** *A coarse $\delta$-functor $(E,\delta)$ is a linear coarse homology theory if*

1. *$E$ is excisive.*
2. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ is coarsely invariant.*
3. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ vanishes on flasques.*
4. *The underlying functor $E : \mathbf{BornCoarse} \to \mathbf{M}$ is $u$-continuous.*

**Example 4.31.** We calculate that $E_n(\mathbb{Z}^k) \cong E_{n+k}(*)$ by induction.

The case $k = 0$ is clear.

Assume for $k-1$. Then consider pair $(Z,\mathcal{Y})$ on $\mathbb{Z}^k$ where

$$Z := \{(n_1,\ldots,n_k) \in \mathbb{Z}^k \mid n_1 \geq 0\}$$

and

$$\mathcal{Y} := \{Y_r := \{(n_1, \ldots, n_k) \in \mathbb{Z}^k \mid n_1 \le r\} \mid r \in \mathbb{N}\} \ .$$

We have $Z \cong \mathbb{N} \otimes \mathbb{Z}^{k-1}$ and also $Y_r \cong \mathbb{N} \otimes \mathbb{Z}^{k-1}$. These spaces are flasque. Furthermore $Z \cap Y_r = [0, r] \times \mathbb{Z}^{k-1}$ is coarsely equivalent to $\mathbb{Z}^{k-1}$.

The long exact sequence for $(\mathbb{Z}^k, \mathcal{Y})$ and $\partial E(\mathcal{Y}) \simeq 0$ gives the first isomorphism in

$$E(\mathbb{Z}^k) \stackrel{\cong}{\to} E(\mathbb{Z}^k, \mathcal{Y}) \stackrel{excision}{\cong} E(Z, Z \cap \mathcal{Y}) \stackrel{\cong,\delta}{\to} \partial E(Z \cap \mathcal{Y})[-1] = E(\mathbb{Z}^{k-1})[-1] \ .$$

Hence

$$E_n(\mathbb{Z}^k) \cong E_{n-1}(\mathbb{Z}^{k-1}) \ .$$

□

Let $X$ be in **BornCoarse** and $Y$, $Z$ be subsets such that $Y \cup Z = X$.

**Definition 4.32.** *The pair $(Y, Z)$ is called coarsely excisive if for every $U$ in $\mathcal{C}_X$ there exists $V$ in $\mathcal{C}_X$ such that*

$$U[Y] \cap U[Z] \subseteq W[Y \cap Z] \ .$$

**Example 4.33.** The pair $(-\mathbb{N}, \mathbb{N})$ in $\mathbb{Z}$ is coarsely excisive. Note that $\{0\} = -\mathbb{N} \cap \mathbb{N}$. If $U$ is in $\mathcal{C}_{\mathbb{Z}}$, then there exists $r$ in $\mathbb{N}$ such that $U \subset U_r$. Then

$$U[-\mathbb{N}] \cap U[\mathbb{N}] \subseteq [-r, r] = U_r[-\mathbb{N} \cap \mathbb{N}] \ .$$

The pair $(-\mathbb{N}, 1 + \mathbb{N})$ is not coarsely excisive. □

**Example 4.34.** Let $(X.d)$ be a path metric space and $(Y, Z)$ be a pair of closed subsets such that $Y \cup Z = X$.

**Lemma 4.35.** *The pair $(Y, Z)$ is coarsely excisive.*

*Proof.* If $U$ is in $\mathcal{C}_X$, then $U \subseteq U_r$ for some $r$ in $\mathbb{N}$. We claim that

$$U[Y] \cap U[Z] \subseteq U_{2r}[Y \cap Z] \ .$$

Let $x$ be a point in $U[Y] \cap U[Z]$. Then there exist points $y$ in $Y$ and $z$ in $Z$ such that $d(x, y) \le r$ and $d(x, z) \le r$. Hence $d(y, z) \le 2r$. There exists a path $\gamma$ from $y$ to $z$ of length $2r$. Since $\gamma$ is connected, $X = Y \cup Z$, and $Y$ and $Z$ are closed, there exists a point $m \in \gamma \cap Z \cap Y$. Then $d(m, x) \le r + \min(d(m, y), d(m, z)) \le 2r$. Hence $x \in U_{2r}[Y \cap Z]$. □

Let $(E, \delta)$ be a linear coarse homology theory and $(Y, Z)$ be a coarsely excisive pair on $X$.

**Proposition 4.36.** *We have a long exact Mayer-Vietoris sequence*

$$E(Y \cap Z) \stackrel{i \oplus j}{\to} E(Y) \oplus E(Z) \stackrel{k-l}{\to} E(X) \stackrel{\partial^{MV}}{\to} E(Y \cap Z)[-1] \ . \tag{4.1}$$

*Where $i : Y \cap Z \to Y$, $j : Y \cap Z \to Z$, $k : E(Y) \to E(X)$ and $l : E(Z) \to E(X)$ are induced by the embeddings.*

*Proof.* The boundary map $\partial^{MV}$ will be constructed in the proof. We have a complementary pair $(Z, \{Y\})$. This gives a map of exact sequences

$$\begin{array}{ccccccc} E(Z \cap \mathcal{Y}) & \xrightarrow{\tilde{j}} & E(Z) & \longrightarrow & E(Z, Z \cap \mathcal{Y}) & \xrightarrow{\delta_{(Z, Z \cap \mathcal{Y})}} & E(Z \cap \mathcal{Y})[-1] \\ \downarrow \tilde{i} & & \downarrow l & & b \downarrow \cong & & \downarrow \\ E(\mathcal{Y}) & \xrightarrow{\tilde{k}} & E(X) & \xrightarrow{a} & E(X, \mathcal{Y}) & \xrightarrow{\delta_{(X, \mathcal{Y})}} & E(\mathcal{Y})[-1] \end{array} \ . \tag{4.2}$$

This diagram gives the long exact sequence

$$E(Z \cap \mathcal{Y}) \stackrel{\tilde{i} \oplus \tilde{j}}{\to} E(\mathcal{Y}) \oplus E(Z) \stackrel{\tilde{k}-l}{\to} E(X) \stackrel{\tilde{\partial}^{MV}}{\to} E(Z \cap \mathcal{Y})[-1] \tag{4.3}$$

where $\tilde{\partial}^{MV} := \delta_{(Z, Z \cap \mathcal{Y})} \circ b^{-1} \circ a$.

The canonical map $E(Y) \to E(\mathcal{Y})$ is an isomorphism since the inclusion of $Y$ into every sufficiently large member of $\mathcal{Y}$ is a coarse equivalence.

We claim that $E(Y \cap Z) \to E(Z \cap \mathcal{Y})$ is also an isomorphism. Indeed, since $(Y, Z)$ is coarsely excisive, for every $U$ in $\mathcal{C}_X$ (with $\mathtt{diag}_X \subseteq X$) there exists $V$ in $\mathcal{C}_X$ such that

$$Z \cap Y \subseteq Z \cap U[Y] \subseteq V[Z \cap Y] \ .$$

This shows that these inclusion maps are coarse equivalences. In particular, we can conclude that $E(Z \cap Y) \to E(Z \cap \{Y\})$ is an isomorphism. After replacing the corresponding entries in (4.3) appropriately we obtain the exact sequence (4.1). ☐

Let $(X_i)_{i \in I}$ be a family in **BornCoarse**.

**Definition 4.37.** *We define the free union $\bigsqcup_{i \in I}^{\text{free}} X_i$ as follows:*

1. *The underlying set of the free union is $\bigsqcup_{i \in} X$.*
2. *The coarse structure is generated by the entourages $\bigcup_{i \in I} U_i$ for all $(U_i)_{i \in I}$ in $\prod_{i \in I} \mathcal{C}_{X_i}$.*
3. *The bornology is generated by $\bigcup_{i \in I} \mathcal{B}_i$.*

Note that in general the coarse structure of the free union is bigger than the coarse structure of the coproduct of the family $(X_i)_{i \in I}$. On the other hand the bornology of the free union is smaller than the bornology of the coproduct. The identity of the underlying set is a morphism

$$\coprod_{i\in I} X_i \to \bigsqcup_{i\in I}^{\text{free}} X_i \ ,$$

but in general not an isomorphism. The family subsets $(X_i)_{i\in I}$ is a coarsely disjoint decomposition of $\bigsqcup_{i\in I}^{\text{free}} X_i$.

**Example 4.38.** For a set $X$ we have $X_{min,min} \cong \bigsqcup_{x\in X}^{\text{free}} \{x\}_{min,min}$, while $X_{min,max} \cong \coprod_{i\in I} \{x\}_{min,min}$.

For every $j$ in $I$, from the Mayer-Vietoris sequence for the decomposition $(X_j, \bigsqcup_{i\in I\setminus\{j\}}^{\text{free}} X_i)$ we have a projection

$$p_j : E(\bigsqcup_{i\in I}^{\text{free}} X_i) \cong E(X_j) \oplus E(\bigsqcup_{i\in I\setminus\{j\}}^{\text{free}} X_i) \stackrel{\text{pr}}{\to} E(X_j) \ .$$

**Definition 4.39.** *$E$ is called:*

1. *strongly additive if for every family $(X_i)_{i\in I}$ the family of projections $(p_i)_{i\in I}$ induces an isomorphism $E(\bigsqcup_{i\in I}^{\text{free}} X_i) \stackrel{\cong}{\to} \prod_{i\in I} E(X_i)$.*

2. *additive, if for every set $X$ the family of projections $(p_x)_{x\in X}$ induces an isomorphism $E(X_{min,min}) \stackrel{\cong}{\to} \prod_{x\in X} E(\{x\})$.*

If $E$ is additive, then its values on spaces of the form $X_{min,min}$ for sets $X$ are determined by the value $E(*)$ on the one-point space. If $E$ is strongly additive, then it is additive.

**Example 4.40.** We consider the square numbers

$$Sq := \{n^2 \mid n \in \mathbb{N}\}$$

as a bornological coarse space with the structures induced from the embedding into $\mathbb{N}$. We assume that $(E, \delta)$ is a strongly additive coarse homology theory. Then we can calculate:

$$E(Sq) := E(*) \oplus \prod_{n\in\mathbb{N}} E(*)/\bigoplus_{n\in\mathbb{N}} E(*) \ .$$

For $r$ in $(0,\infty)$ let $n_r$ be the smallest integer such that $(n_r+1)^2 - n_r^2 > r$, i.e. $n_r > \frac{r-1}{2}$. Let $U_r$ be the metric entourage of size $r$. Then we have an isomorphism

$$Sq_{U_r} \cong (Sq \cap [0, n_r])_{max,max} \sqcup (Sq \cap (n_r, \infty))_{min,min} \ .$$

Note that $(Sq \cap [0, n_r])_{max,max} \to *$ is a coarse equivalence. From the strong additivity of $E$ we get an isomorphism

$$E(Sq_{U_r}) \cong E(*) \oplus \prod_{Sq\cap(n_r,\infty)} E(*) \ .$$

If $r'$ is in $(0,\infty)$ and $r'>r$, then the map $E(Sq_{U_r})\to E(Sq_{U_{r'}})$ is given by the map

$$E(*)\oplus\prod_{Sq\cap(n_r,\infty)}E(*)\cong E(*)\oplus\bigoplus_{Sq\cap(n_r,n_{r'}]}E(*)\oplus\prod_{Sq\cap(n_{r'},\infty)}E(*)\to E(*)\oplus\prod_{Sq\cap(n_{r'},\infty)}E(*)\ ,$$

where the second map takes the sum of the entries of the first two summands. We can identify

$$\prod_{Sq\cap(n_r,\infty)}E(*)\cong\prod_{Sq}E(*)/\bigoplus_{Sq\cap[0,n_r]}E(*)\ .$$

We now use $u$-continuity of $E$ in order to deduce that

$$E(Sq)\cong E(*)\oplus\operatorname{colim}_{r\in(0,\infty)}\prod_{Sq}E(*)/\bigoplus_{Sq\cap[0,n_r]}E(*)\cong E(*)\oplus\prod_{Sq}E(*)/\bigoplus_{Sq}E(*)\ .$$

From the long exact sequence we see that

$$E(Sq,\{0\})\cong\prod_{Sq}E(*)/\bigoplus_{Sq}E(*)\ .$$

□

# 5 Coarse ordinary homology

In this section we construct a linear coarse homology theory $(H\mathcal{X}(-),\delta)$. We first construct a coarse $\delta$-functor $(H\mathcal{X}(-),\delta)$ and then verify the axioms. This $\delta$-functor is derived from a functor which sends a pair $(X,\mathcal{Y})$ in $\mathbf{BC}^{\mathrm{pair}}$ to a short exact sequence

$$0\to C\mathcal{X}(\mathcal{Y})\to C\mathcal{X}(X)\to C\mathcal{X}(X,\mathcal{Y})\to 0$$

of chain complexes. We start with the construction of the functor

$$C\mathcal{X}(-):\mathbf{BornCoarse}\to\mathbf{Ch}\ .$$

Let $X$ be in **BornCoarse**. For $n$ in $\mathbb{N}$ we consider the abelian group

$$\hat{C}_n(X):=\mathbb{Z}^{X^{n+1}}$$

of functions from the $n+1$-fold product of $X$ with itself to $\mathbb{Z}$. Often we will consider elements in $\hat{C}_n(X)$ as infinite linear combinations of basis elements $(x_0,\dots,x_n)$.

For $c$ in $\hat{C}_n(X)$ we define its support by

$$\mathtt{supp}(c):=\{x\in X^{n+1}\mid c(x)\neq 0\}\ .$$

We now use the coarse structure and the bornology on $X$ in order to define a subgroup of $\hat{C}_n(X)$. To this end we introduce the following notions.

Let $U$ be an entourage of $X$, and let $B$ be a subset of $X$.

**Definition 5.1.**

1. *We say that a point* $(x_0, \ldots, x_n)$ *in* $X^{n+1}$ *is* $U$*-controlled if* $(x_i, x_j) \in U$ *for all* $i, j$ *in* $\{0, \ldots, n\}$.

2. *We say that a point* $(x_0, \ldots, x_n)$ *in* $X^{n+1}$ *meets* $B$ *if there exists* $i$ *in* $\{0, \ldots, n\}$ *such that* $x_i \in B$.

Let now $c$ be in $\hat{C}_n(X)$.

**Definition 5.2.**

1. *We say that* $c$ *is* $U$*-controlled if every* $x$ *in* $\mathtt{supp}(c)$ *is* $U$*-controlled.*

2. *We say that* $c$ *is controlled if it is* $U$*-controlled for some* $U$ *in* $\mathcal{C}_X$.

3. *We say that* $c$ *is locally finite if for every* $B$ *in* $\mathcal{B}_X$ *the set* $\{x \in \mathtt{supp}(c) \,|\, x \text{ meets } B\}$ *is finite.*

**Definition 5.3.** *We define* $C\mathcal{X}_n(X)$ *to be the subgroup of* $\hat{C}_n(X)$ *of all functions which are locally finite and controlled.*

We now define a differential

$$d : C\mathcal{X}_n(X) \to C\mathcal{X}_{n-1}(X)$$

as the linear extension of the map determined by

$$(x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n) \ .$$

More precisely,

$$(dc)(x_0, \ldots, x_{n-1}) := \sum_{i=0}^{n} (-1)^i \sum_{x \in X} c(x_0, \ldots, x_{i-1}, x, x_i, \ldots, x_n) \ .$$

We must show that the inner sum is finite for every $i$. Consider e.g. $i \geq 1$. The condition $c(x_0, \ldots, x, \ldots, x_n) \neq 0$ implies that $(x_0, \ldots, x, \ldots, x_n)$ meets the bounded set $\{x_0\}$. Since $c$ is locally finite there only finitely many points $x$ which can contribute non-trivially.

One checks that $d^2=0$. This indeed follows by linear extension from

$$\begin{aligned}
d^2(x_0,\dots,x_n) &= \sum_{i=0}^n (-1)^i d(x_0,\dots,\hat{x}_i,\dots,x_n)\\
&= \sum_{i=0}^n \sum_{j=0}^{i-1} (-1)^{i+j}(x_0,\dots,\hat{x}_j,\dots\hat{x}_i,\dots,x_n)\\
&\quad +\sum_{i=0}^n \sum_{j=i+1}^{n} (-1)^{i+j-1}(x_0,\dots,\hat{x}_i,\dots\hat{x}_j,\dots,x_n)\\
&= \sum_{i=0}^n \sum_{j=0}^{i-1} (-1)^{i+j}(x_0,\dots,\hat{x}_j,\dots\hat{x}_i,\dots,x_n)\\
&\quad +\sum_{j=0}^n \sum_{i=0}^{j-1} (-1)^{i+j-1}(x_0,\dots,\hat{x}_i,\dots\hat{x}_j,\dots,x_n)\\
&= 0
\end{aligned}$$

If $f:X\to X'$ is a morphism of bornological coarse spaces, then we have an induced map

$$f_*: C\mathcal{X}_n(C)\to C\mathcal{X}_n(X')\ .$$

It is the linear extension of the map

$$f_*(x_0,\dots,x_n)=(f(x_0),\dots f(x_n))\ .$$

The precise formula is

$$(f_*c)(x_0',\dots,x_n')=\sum_{x_0\in f^{-1}(\{x_0'\}),\dots,x_n\in f^{-1}(\{x_n'\})} c(x_0,\dots,x_n)\ .$$

We next argue that $f_*$ is well-defined.

Since $f$ is proper and $c$ is locally finite we see that the sum is finite. Indeed, all points contributing to this sum belong to the set $f^{-1}(\{x_0\})\cap \mathtt{supp}(c)$ which is finite since $f$ is proper and therefore $f^{-1}(\{x_0\})$ is bounded.

We now argue that $f_*c$ is again controlled and locally finite. If $U$ is in $\mathcal{C}_X$ and $(x_0,\dots,x_n)$ is $U$-controlled, then $f_*(x_0,\dots,x_n)$ is obviously $f(U)$-controlled. This implies that if $c$ is $U$-controlled, then $f_*c$ is $f(U)$-controlled.

Let $B'$ be a bounded subset of $X'$. If $(x_0',\dots,x_n')$ is in the support of $f_*c$ and meets $B'$, then it is of the form $f_*(x_0,\dots,x_n)=(x_0',\dots,x_n')$ where $(x_0,\dots,x_n)$ meets $f^{-1}(B')$ and is in the support of $c$. There only finitely many such points. This shows that $f_*c$ is again locally finite.

We finally check that $f_*$ is compatible with the differential. Indeed,

$$\begin{aligned} df_*(x_0,\ldots,x_n) &= d(f(x_0),\ldots f(x_n)) \\ &= \sum_{i=0}^{n}(-1)^i(f(x_0),\ldots,\widehat{f(x_i)},\ldots f(x_n)) \\ &= f_*\sum_{i=0}^{n}(-1)^i(x_0,\ldots\hat{x}_i,\ldots x_n) \\ &= f_*d(x_0,\ldots,x_n) \end{aligned}$$

This finishes the construction of the functor

$$C\mathcal{X}(-):\mathbf{BornCoarse}\to\mathbf{Ch}\ .$$

We now consider a pair $(X,\mathcal{Y})$. Then we can identify the chain complex $C\mathcal{X}(\mathcal{Y}):=\mathtt{colim}_{Y\in\mathcal{Y}}C\mathcal{X}(Y)$ with the subspace of $C\mathcal{X}(X)$ consisting of chains which are supported on some member of $\mathcal{Y}$. We get a functorial exact sequence

$$0\to C\mathcal{X}(\mathcal{Y})\to C\mathcal{X}(X)\to C\mathcal{X}(X,\mathcal{Y})\to 0\ , \tag{5.1}$$

where the last complex is defined as a quotient

$$C\mathcal{X}(X,\mathcal{Y}):=\frac{C\mathcal{X}(X)}{C\mathcal{X}(\mathcal{Y})}\ .$$

In this way we define the functor

$$C\mathcal{X}:\mathbf{BC}^{\mathrm{pair}}\to\mathbf{Ch}\ ,\quad (X,\mathcal{Y})\mapsto C\mathcal{X}(X,\mathcal{Y})\ .$$

**Definition 5.4.** *We define*

$$H\mathcal{X}:=H\circ C\mathcal{X}:\mathbf{BC}^{\mathrm{pair}}\to\mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$$

*and let* $\delta:H\mathcal{X}\Rightarrow\partial H\mathcal{X}[-1]$ *be indued by the exact sequence* (5.1).

This finites the construction of the coarse $\delta$-functor $(H\mathcal{X},\delta)$.

The main theorem of the present section is the following.

**Theorem 5.5.** *The coarse* $\delta$*-functor* $(H\mathcal{X},\delta)$ *is a linear coarse homology theory.*

*Proof.* The proof of this theorem is given by the combination of the following four Lemmas 5.6, 5.7, 5.8, and 5.6. □

**Lemma 5.6.** *The functor* $H\mathcal{X}$ *is excisive.*

*Proof.* We must show that for every complementary pair $(Z, \mathcal{Y})$ on some $X$ in **BornCoarse** the map

$$H\mathcal{X}(Z, Z \cap \mathcal{Y}) \to H\mathcal{X}(X, \mathcal{Y})$$

is an isomorphism. We show the stronger statement that

$$i : C\mathcal{X}(Z, Z \cap \mathcal{Y}) \to C\mathcal{X}(X, \mathcal{Y})$$

is an isomorphism.

We show that $i$ is injective. Let $c$ be in $C\mathcal{X}(Z)$ and $[c]$ be its class in $C\mathcal{X}(Z, Z \cap \mathcal{Y})$. If $i([c]) = 0$, then $c \in C\mathcal{X}(\mathcal{Y}) \cap C\mathcal{X}(Z) = C\mathcal{X}(Z \cap \mathcal{Y})$ and hence $[c] = 0$.

We now show that $i$ is surjective. Let $d$ be in $C\mathcal{X}(X)$ and $[d]$ be its class in $C\mathcal{X}(X, \mathcal{Y})$. Let $d_{|Z}$ be its restriction to $Z$. We claim that $i([d_{|Z}]) = [d]$. Assume that $d$ is $U$-controlled. Then $d - d_{|Z}$ is supported on $U[X \setminus Z]$. Indeed, if $d(x) - d_{|Z}(x) \neq 0$, then at least one component of $x$ belongs to $X \setminus Z$. But then all components of $x$ belong to $U[X \setminus Z]$. Since $\mathcal{Y}$ is big and $X \setminus Z \in \bar{\mathcal{Y}}$, we also have $U[X \setminus Z] \in \bar{\mathcal{Y}}$. This implies that $[d] = i([d_{|Z}])$. □

**Lemma 5.7.** *The functor $H\mathcal{X}$ is coarsely invariant.*

*Proof.* We show that $H\mathcal{X}$ sends close morphisms to equal maps and appeal to Lemma 4.2. Assume that $f, g : X \to Y$ are close to each other, then we will show that $f_*$ and $g_*$ are chain homotopic by giving an explicit homotopy. To this end we define $h : C\mathcal{X}(X) \to C\mathcal{X}(X)[1]$ as the linear extension of the map determined by

$$h(x_0, \ldots, x_n) := \sum_{i=0}^{n} (-1)^i (f(x_0), \ldots, f(x_i), g(x_i), \ldots, g(x_n)) \ .$$

We now argue that $h$ indeed takes values in controlled and locally finite chains.

Assume that $f \sim_V g$ and and that $c$ is $U$-controlled. If $x, y$ are components of $(x_0, \ldots, x_n)$, then we have $(f(x), f(y)) \in f(U)$, $(g(x), g(y)) \in g(U)$ and $(f(x), g(y)) \in f(U) \circ V$ since $f(x) \sim_V g(y) \sim_U g(y)$, and similarly $(g(x), f(y)) \in V \circ f(U)$. Hence $h(x_0, \ldots, c_n)$ is controlled by $g(U) \cup f(U) \cup V \circ g(U) \cup V^{-1} \circ f(U)$.

If $B$ is a bounded subset of $Y$ and $(y_0, \ldots, y_n) \in \mathtt{supp}(h(c)) \cap B$, then $(y_0, \ldots, y_n)$, then there exists a point in $\mathtt{supp}(c)$ which meets $f^{-1}(B) \cup g^{-1}(B)$. There are only finitely may such points. Hence $h(c)$ is locally finite.

We claim that

$$dh + hd = g_* - f_* \ .$$

This is checked by a direct, but tedius calculation. It is a standard formal consequence of the fact that the map $h$ in Lemma 2.59 is a simplicial homotopy. We consider the first two degrees. In degree 0 we have

$$dh(x) + hd(x) = d(f(x), g(x)) = (g(x)) - (f(x)) = (g_* - f_*)(x) \ .$$

In degree 1 we have

$$\begin{aligned} dh(x,y)+hd(x,y) &= d[(f(x),g(x),g(y))-(f(x),f(y),g(y))]+h[(y)-(x)] \\ &= (g(x),g(y))-(f(x),g(y))+(f(x),g(x)) \\ &\quad -(f(y),g(y))+(f(x),g(y))-(f(x),f(y)) \\ &\quad +(f(y),g(y))-(f(x),g(x)) \\ &= (g_*-f_*)(x,y) \end{aligned}$$

The map $h$ is therefore a chain homotopy equivalence between $f_*$ and $g_*$. It follows that $H\mathcal{X}(f)=H\mathcal{X}(g)$. □

**Lemma 5.8.** *The functor $H\mathcal{X}$ vanishes on flasques.*

*Proof.* We show that $H\mathcal{X}(X)=0$ for every flasque $X$ in **BornCoarse**. Let $f:X\to X$ witnesses the flasqueness of $X$. We define a map $S: C\mathcal{X}(X)\to C\mathcal{X}(X)$ by

$$S:=\sum_{k=0}^{\infty} f_*^k \ .$$

In order to see that $S$ is well-defined we observe that $S$ is point-wise finite, i.e. for $c$ in $C\mathcal{X}_n(X)$ and $x$ in $X^{n+1}$ the sum $\sum_{k=0}^{\infty}(f_*^k c)(x)$ is finite. Here we use that the image of $f^k$ eventually misses every bounded set, so in particular $\bigcup_{i=0}^{n} f^{-1}(\{x_i\})$, where $x=(x_0,\dots,x_n)$. The same property and a similar reasoning implies that $S$ preserves locally finite chains. We use the property that $f$ is non-expanding in order to see that $S$ preserves controlled chains. Indeed, if $c$ is $U$-controlled, then $S(c)$ is controlled by $\bigcup_{n\geq 0} f^n(U)$ which is a coarse entourage by assumption.

We obviously have

$$\mathtt{id}_{C\mathcal{X}(X)}+f_*\circ S=S \ .$$

Applying $H$ and using that $H\mathcal{X}(f)=\mathtt{id}$ since $f\sim \mathtt{id}_X$ and $H\mathcal{X}$ is coarsely invariant we get the equality $\mathtt{id}_{H\mathcal{X}(X)}+H\mathcal{C}(S)=H\mathcal{C}(S)$. This implies $\mathtt{id}_{H\mathcal{X}(X)}=0$ and hence $H\mathcal{X}(X)=0$. □

The map $S$ in the proof above is also called an Eilenberg-swindle.

**Lemma 5.9.** *The functor $H\mathcal{X}$ is u-continuous.*

*Proof.* Every chain in $C\mathcal{X}$ is controlled by some entourage. We therefore have

$$\operatorname*{\mathtt{colim}}_{U\in\mathcal{C}_X} C\mathcal{X}(X_U)\cong C\mathcal{X}(X)$$

where the colimit amounts taking a union of subspaces in $C\mathcal{X}(X)$. Since taking homology is compatible with filtered colimits this implies

$$\operatorname*{\mathtt{colim}}_{U\in\mathcal{C}_X} H\mathcal{X}(X_U)\cong H\mathcal{X}(X) \ .$$

□

**Example 5.10.** We have $H\mathcal{X}(*) \cong \mathbb{Z}$. Indeed, $C\mathcal{X}$ is the complex

$$\dots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

where the last $\mathbb{Z}$ is in degree 0.

Hence $H\mathcal{X}(\mathbb{Z}^n) \cong \mathbb{Z}[-n]$, i.e.

$$H\mathcal{X}_k(\mathbb{Z}^n) = \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} .$$

In particular we conclude that $\mathbb{Z}^n$ (and therefore also $\mathbb{R}^n$) is not flasque. □

**Example 5.11.** We consider the action of $C_2$ on $\mathbb{Z}$ by multiplication by $-1$. Let $\epsilon$ denote the non-trivial element in $C_2$. We claim that

$$H\mathcal{X}_1(\epsilon) = -1$$

on $H\mathcal{X}_1(\mathbb{Z}) \cong \mathbb{Z}$.

In order to show this we exhibit an explicit representative of the generator of this group. We consider the chain $c : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ in $C\mathcal{X}_1(\mathbb{Z})$ given by

$$c := \sum_{n\in\mathbb{Z}} (n, n+1) .$$

This chain is $U_1$-controlled and locally finite. We have $\partial c = \sum_{n\in\mathbb{Z}}((n+1) - (n)) = 0$.

We claim that $[c] \neq 0$. To this end we calculate $\partial^{MV}([c])$. We set

$$c_+ := \sum_{n\in\mathbb{N}} ((n+1) - (n)) .$$

The class of $c_+$ in $C\mathcal{X}(\mathbb{N})/C\mathcal{X}(\{0\})$ is send to the class of $c$ in $C\mathcal{X}(\mathbb{Z})/C\mathcal{X}(-\mathbb{N})$. Consequently, by the construction of the Mayer-Vietoris boundary, $\partial^{MV}([c]) = \delta_{(\mathbb{N},\{0\})}[c_+] = [\partial c_+]$ in $H\mathcal{X}_0(\{0\})$. Note that $[\partial c_+] = (0)$. This generates $H\mathcal{X}_0(\{0\})$. Since $\partial^{MV} : H\mathcal{X}_1(\mathbb{Z}) \to H\mathcal{X}_0(\{0\})$ is an isomorphism we can conclude that $[c]$ is the generator of $H\mathcal{X}_1(\mathbb{Z})$.

We now have $\epsilon_* c = -c$. This shows the assertion.

More generally, by the functoriality of the MV-sequence the matrix $(1, \dots, 1, -1) : \mathbb{Z}^n \to \mathbb{Z}^n$ acts by $-1$ on $H\mathcal{X}_n(\mathbb{Z}^n)$. □

**Lemma 5.12.** *$H\mathcal{X}$ is strongly additive.*

*Proof.* Let $(X_i)$ be a family in **BornCoarse** and set $X := \bigsqcup_{i\in I}^{\text{free}} X_i$. For every $j$ in $I$ and $n$ in $\mathbb{N}$ we have an embedding $X_j^{\times n} \to X^{\times n}$. We call the complement $X^{\times n} \setminus \bigcup_{j\in I} X_j^{\times n}$ the mixed part of $X^{\times n}$. Note that $C\mathcal{X}_n(X)$ consists of functions $X^{\times(n+1)} \to \mathbb{Z}$. We claim that the restrictions along the embeddings $X_j^{\times n} \to X^{\times n}$ induce an isomorphism

$$C\mathcal{X}(X) \cong \prod_{i\in I} C\mathcal{X}(X_i) \ .$$

We fix $j$ in $I$. Then $c \mapsto c_{|X_j^{\times \bullet}}$ is a chain map $C\mathcal{X}(X) \to C\mathcal{X}(X_j)$. This follows from the fact $c$ in $C\mathcal{X}(X)$ is controlled and therefore vanishes on the mixed part. If $c$ is $U$-controlled for $U$ in $\mathcal{C}_X$, then $c_{X_j}$ is controlled by $U \cap (X_j \times X_j)$ in $\mathcal{C}_{X_j}$. If $B$ is bounded in $X_j$, then it is bounded in $X$. Since $\mathtt{supp}(c)$ meets only finitely many points in $B$, so does the support of $c_{|X_j}$. This finishes the justification of the chain map.

Putting all restriction maps together we get a chain map

$$C\mathcal{X}(X) \to \prod_{i\in I} C\mathcal{X}(X_i) \ .$$

This map is injective. Indeed, if $c_{|X_j} = 0$ for all $j$ in $I$, then $c = 0$, again since $c$ vanishes on the mixed part. We now show that it is surjective. Let $(c_j)_{j\in I}$ be a family in $\prod_{i\in I} C\mathcal{X}(X_i)$. Then we define $c$ in $C\mathcal{X}(X)$ such that $c_{|X_j} = c_j$ for all $j$ in $I$ and $c$ vanishes on the mixed part. Assume that $c_j$ is $U_j$-controlled for all $j$ in $I$. Then $c$ is $\bigcup_{j\in J} U_j$-controlled, and this is a coarse entourage of the free union $X$. We check locally finiteness of $c$ on generators. Let $j$ be in $I$ and $B$ be bounded in $X_j$. Then the points in $\mathtt{supp}(c)$ which meet $B$ are the points in $\mathtt{supp}(c_j)$ which meet $B$. This set is finite by the local finiteness of $c_j$. This shows that $c$ in $C\mathcal{X}(X)$ is well-defined. It is clearly a preimage of the family $(c_j)_{j\in I}$.

We finally use that for a family of chain complexes $C_i$ we have $H(\prod_{i\in I} C_i) \cong \prod_{i\in I} H(C_i)$. □

**Example 5.13.** For a set $X$ we have

$$H\mathcal{X}(X_{min,min}) \cong \prod_X \mathbb{Z} \ .$$

We have

$$H\mathcal{X}(Sq) \cong \mathbb{Z} \oplus \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z}$$

by Example 4.40.

**Example 5.14.** If $\Gamma$ is a group and $S$ be a $G$-set, then we can consider abelian group $\mathbb{Z}[S]$ with the induced action of $\Gamma$. We can consider group cohomology of $\Gamma$ with coefficients in this representation.

**Proposition 5.15.** *We have an isomorphism* $H\mathcal{X}_*(\Gamma_{can,min} \otimes S_{min,max}) \cong H_*(\Gamma, \mathbb{Z}[S])$.

This is shown by an explicit identification of the chain complex $C\mathcal{X}(\Gamma_{can,min} \otimes S_{min,max})$ with the standard chain complex calculating $H_*(\Gamma, \mathbb{Z}[S])$.

□

**Remark 5.16.** The coarse homology $H\mathcal{X}$ has another remarkable property. Let $I$ be a filtered poset and $(X_i)_{i \in I}$ be a decreasing family of subsets of $X$ in **BornCoarse**. The we consider the set-theoretic intersection $\bigcap_{i \in I} X_i$ in $X$ equipped with the induced bornological coarse structures. Then

$$C\mathcal{X}(\bigcap_{i \in I} X_i) \to \lim_{i \in I} C\mathcal{X}(X_i)$$

is an isomorphism. Indeed, we can consider $C\mathcal{X}(X_i)$ as a subcomplex of $C\mathcal{X}(X)$ of chains supported on $X_i$, and the limit on the right-hand side is realized as the intersection of these subcomplexes.

Taking homology does not commute with filtered limits. In general we have a spectral sequence with second term

$$\mathtt{lim}^r H\mathcal{X}_{q+r}(X_i) \Rightarrow H\mathcal{X}_q(\bigcap_{i \in I} X_i) \ .$$

If $I = \mathbb{N}$, then this reduces to the $\mathtt{lim}^1$-sequence

$$0 \to \mathtt{lim}^1_{i \in \mathbb{N}} H\mathcal{X}_{q+1}(X_i) \to H\mathcal{X}_q(\bigcap_{i \in \mathbb{N}} X_i) \to \lim_{i \in \mathbb{N}} H\mathcal{X}_q(X_i) \to 0 \ . \tag{5.2}$$

# 6 Coarse homotopy

We consider a bornological coarse space $X$. Let $p = (p_+, p_-)$ be a pair of functions $p_+ : X \to [0, \infty)$ and $p_- : X \to (-\infty, 0]$. We assume that the maps $p_\pm$ are bornological.

**Definition 6.1.** *The coarse cylinder associated to $p$ is the subspace $I_p$ of $\mathbb{R} \otimes X$ given by*

$$I_p X := \{(t, x) \in \mathbb{R} \times X \mid p_-(x) \le t \le p_+(x)\} \ .$$

**Lemma 6.2.** *The projection $\pi : I_p X \to X$ is a morphism of bornological coarse spaces.*

*Proof.* We must check that $\pi$ is proper. Let $B$ be bounded in $X$. Then $p_+(B)$ and $p_-(B)$ are bounded. Assume that $p_+(B) \cup p_-(B) \subseteq [-r, r]$ for $r$ in $\mathbb{R}$. Then $\pi^{-1}(B) \subseteq [-r, r] \times B$. Hence $p^{-1}(B)$ is bounded. □

**Example 6.3.** The projection $\pi : I_p X \to X$ is an equivalence of bornological coarse spaces if and only if $p_+$ and $p_-$ are bounded. In this case the map $X \to I_p X$, $x \mapsto (0, x)$ is an inverse. In this case $E(\pi) : E(I_p X) \to E(X)$ is an isomorphism for every coarsely invariant functor. For coarse homology theories we have the much stronger statement Lemma 6.4. □

Assume that $(E,\delta)$ is a linear coarse homology theory.

**Lemma 6.4.** *The projection $\pi : I_pX \to X$ induces an isomorphism $E(\pi) : E(I_pX) \to E(X)$.*

*Proof.* We consider the subspaces

$$W := (-\infty,0] \times X \cup I_pX \ , \quad [0,\infty) \times X \cap I_pX$$

of $\mathbb{R} \otimes X$. Thus $W$ is the lower half space over $X$ together with the positive part of the cyclinder, and $Z$ is the positive part of the cylinder. Note that $Z \subseteq W$.

We let furthermore let $\mathcal{Y}$ be the big family in $W$ generated by the lower half space $(-\infty,0] \times X$. We let $Y_n := W \cap (-\infty,n] \times X$. The family $\mathcal{Y}' := (Y_n)_{n\in\mathbb{N}}$ is cofinal in $\mathcal{Y}$. Then $(Z,\mathcal{Y})$ is complementary pair on $W$. The map $(t,x) \mapsto (t-1,x)$ implements flasqueness of every member of $\mathcal{Y}'$ and of $W$. Hence $E(W) = 0$ and $E(\mathcal{Y}) = 0$ since $E$ vanishes on flasques. From the long exact sequence for the pair $(W,\mathcal{Y})$ we get the isomorphism $0 = E(W,\mathcal{Y})$. We now use excision for the complementary pair $(Z,\mathcal{Y})$.

$$E(Z, Z \cap \mathcal{Y}) \cong E(W,\mathcal{Y}) \cong 0 \ .$$

From the long exact sequence of the pair $(Z, Z \cap \mathcal{Y})$ we now get the isomorphism

$$E(Z \cap \mathcal{Y}) \cong E(Z) \ .$$

We now observe that the restriction of the projection $\pi_{Z\cap Y_n} : Z \cap Y_n \to X$ is a coarse equivalence with inverse $x \mapsto (0,x)$. It follows by coarse invariance of $E$ that we have an isomorphism $E(Z \cap \mathcal{Y}) \cong E(X)$. Hence also $E(Z) \cong E(X)$.

We now consider the subsets

$$V := I_pX \cup [0,\infty) \times X \ , \quad U := [0,\infty) \times X$$

of $\mathbb{R} \otimes X$. Thus $U$ is the upper half space, and $V$ is the upper half space with the negative part of the cylinder added. Note that both spaces are flasque with flasqueness implemented by $(t,x) \mapsto (t+1,x)$. The long exact sequence for $(V,\{U\})$ yields $E(V,\{U\}) = 0$. By excision for the complementary pair $(V, I_pX \cap \{U\})$ on $I_pX$ we get

$$E(I_pX, I_pX \cap \{U\}) \cong E(V,\{U\}) \cong 0 \ .$$

From the long exact sequence of the pair $(I_pX, I_pX \cap \{U\})$ we get the isomorphism $E(I_pX \cap \{U\}) \cong E(I_pX)$. The family $(U_n)_{n\in\mathbb{N}}$ of subsets $U_n := V \cap [-n,\infty) \times X$ is cofinal in $\{U\}$. We now observe that $Z \to I_pX \cap U_n$ is a coarse equivalence for every $n$. We conclude that $E(Z) \cong E(I_pX \cap \{U\})$. In view of the commuting diagram

$$\begin{array}{ccccccc} E(Z\cap\mathcal{Y}) & \xrightarrow{\cong} & E(Z) & \xrightarrow{\cong} & E(I_p(X)\cap\{U\}) & \xrightarrow{\cong} & E(I_pX) \\ & \searrow^{\cong} & \downarrow & \swarrow & & \swarrow_{!} & \\ & & E(X) & & & & \end{array}$$

we conclude that the marked arrow is an isomorphism. All vertical maps are induced by restrictions of $\pi$. $\square$

The idea is now to define the notion of a coarse homotopy using the cylinders. To this end we must ensure that the maps

$$i_\pm : X \to I_p X \ , \quad x \mapsto (x, p_\pm(x))$$

are morphisms in **BornCoarse**. Since any bounded subset of $I_pX$ is contained in $\mathbb{R} \times B$ for a bounded subset $B$ of $X$ and $i_\pm^{-1}(\mathbb{R} \times B) = B$ it is clear that $i_\pm$ are bornological. If $p_\pm$ are controlled, then so are $i_\pm$.

Assume that that $p_\pm$ are controlled, and that $(E, \delta)$ is a linear coarse homology theory.

**Corollary 6.5.** *We have* $E(i_+) = E(i_-)$.

*Proof.* Indeed, $E(\pi)$ is an isomorphism and

$$E(\pi) \circ E(i_-) = E(\pi \circ i_-) = \mathtt{id}_{E(X)} = E(\pi \circ i_+) = E(\pi) \circ E(i_+) \ .$$

□

Let $f_\pm : X \to Y$ be two morphisms in **BornCoarse**.

**Definition 6.6.** *We say that* $f_\pm$ *are coarsely homotopic if there exists a pair* $p = (p_+, p_-)$ *of controlled and bornological maps and a morphism* $h : I_pX \to Y$ *such that* $f_\pm = h \circ i_\pm$.

**Problem 6.7.** *Show that coarse homotopy is an equivalence relation.*

**Corollary 6.8.** *If* $f_\pm$ *are coarsely homotopic, then for every coarse homology theory* $(E, \delta)$ *we have* $E(f_+) = E(f_-)$.

*Proof.* We have

$$E(f_+) = E(h \circ i_+) = E(h) \circ E(i_+) = E(h) \circ E(i_-) = E(h \circ i_-) = E(f_-) \ .$$

□

**Example 6.9.** Let $A$ be an invertible matrix in $\mathtt{Mat}(n, n; \mathbb{R})$.

**Lemma 6.10.** *Then* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is coarsely homotopic to* $\mathtt{id}_{\mathbb{R}^n}$ *iff* $\det(A) = 1$.

*Proof.* Let $\gamma : [0, 1] \to GL_n(\mathbb{R})$ be a Lischitz continuous path. We consider the coarse cylinder $I_p(\mathbb{R}^n)$ for $p_- = 0$ and $p_+(x) := 1 + \|x\|$. This map is bornological and controlled. We now define $h : I_p(\mathbb{R}^n) \to \mathbb{R}^n$ by

$$h(t, x) := \gamma((1 + \|x\|)^{-1}t)x \ .$$

We claim that this map is proper and controlled.

Let $C := \sup_{t\in[0,1]}(\|\gamma(t)\| + \|\gamma(t)^{-1}\|)$. Furthermore, let $L$ be the Lipschitz constant of $\gamma$.

For every $r$ in $(0,\infty)$ the set $h^{-1}(B_r)$ is contained in the bounded subset $([0,rC]\times B_{rC})\cap I_p(\mathbb{R}^n)$ of $I_p(\mathbb{R}^n)$. This shows that $h$ is proper.

Furthermore

$$\|h(t',x) - h(t,x)\| \le L(1+\|x\|)^{-1}|t-t'|\|x\| \le L|t-t'| \ .$$

Similarly,

$$\|h(t,x) - h(t,x')\| \le C\|x-x'\| \ .$$

These two estimates show that $h$ is controlled.

If $\det(A) > 0$, the there exists a Lipschitz path from $A$ to $\mathtt{id}_{\mathbb{R}^n}$. Consequently $A$ is coarsely homotopy to the identity.

If $\det(A) < 0$, then $A$ is coarsely homotopic to $\mathtt{diag}(1,\dots,1,-1)$. We know that $(1,\dots,1,-1)$ and hence $A$ acts on $H\mathcal{X}_n(\mathbb{R}^n) \cong \mathbb{Z}$ by multiplication with $-1$, see Example 5.11. □

□

# 7 Continuous controlled cones and homology of compact spaces

Let $Y$ by a Hausdorff topological space with a subset $A$. We call $(Y,A)$ a pair. To a pair $(Y,A)$ we associate the set $X := Y \setminus A$ with the continuously controlled structure.

A map of pairs $\tilde f : (Y,A) \to (Y',A')$ is a map $\tilde f : Y \to Y$ such that $\tilde f(A) \subseteq A'$ and $\tilde f(X) \subseteq X'$. We get a category $\mathbf{Top}^{\mathrm{pair}}_{\mathrm{Hausd}}$ of pairs and morphisms. We write $f := \tilde f_{|X}$.

**Lemma 7.1.** *If $X$ is compact, then $f : X \to X'$ is a morphism in* **BornCoarse**.

*Proof.* We first check that $f$ is proper. Let $B'$ be a bounded subset of $X'$. We must show that $f^{-1}(B')$ is bounded in $X$. We assume the contrary. Then there exists $a$ in $\overline{f^{-1}(B')}^Y \cap A$. Hence there exists a net $(x_i)_i$ in $f^{-1}(B')$ such that $\mathtt{lim}_i\, x_i = a$. But then $(f(x_i))_i$ is a net in $B'$ and $\mathtt{lim}_i\, f(x_i) = \tilde f(a) \in A'$ by continuity of $\tilde f$. This is impossible since $\overline{B}^{Y'} \cap A' = \emptyset$ by the assumption that $B$ is bounded. For this part of the argument compactness of $Y$ is irrelevant.

We now show that $f$ is controlled. Let $U$ be an entourage of $X$. We must show that $f(X)$ is controlled.

We consider a net $((y_i, z_i))_i$ in $U$ and assume that $\mathtt{lim}_i f(y_i) = a$ belongs to $A'$. We must show that then also $\mathtt{lim}_i f(z_i) = a$. Assume the contrary. By the compactness of $Y$, there exists a subnet $((y_{i'}, z_{i'}))_{i'}$ with $\mathtt{lim}_{i'} z_{i'} = z$, $\mathtt{lim}_{i'} y_i = y$ in $Y$ and such that $\tilde{f}(z) \neq a$. Note that $\tilde{f}(y) = a \in A'$, hence $y \in A$. This implies that $z = y$ and therefore $\tilde{f}(z) = a$, a contradiction. □

We let $\mathbf{Top}^{\mathrm{pair}}_{c,\mathrm{Hausd}}$ denote the full subcategory of $\mathbf{Top}^{\mathrm{pair}}_{\mathrm{Hausd}}$ of pairs $(X, A)$ such that $X$ is compact.

**Corollary 7.2.** *We have a functor* $\mathbf{Top}^{\mathrm{pair}}_{c,\mathrm{Hausd}} \to \mathbf{BornCoarse}$ *sending* $(Y, A)$ *to* $X := Y \setminus A$ *with the continuously controlled structure, and* $\tilde{f} : (Y, A) \to (Y', A')$ *to* $f := \tilde{f}_{|X} : X \to X'$.

We next construct for every $X$ in $\mathbf{Top}_{\mathrm{Hausd}}$ a bornological coarse space $\mathcal{O}_c(X)$. Restricting to compact spaces we even get a functor

$$\mathcal{O}_c : \mathbf{Top}_{c,\mathrm{Hausd}} \to \mathbf{BornCoarse} \ .$$

Let $X$ be in $\mathbf{Top}_{\mathrm{Hausd}}$.

1. The underlying set of $\mathcal{O}_c(X)$ is $[0, \infty) \times X$.

2. The coarse structure on $\mathcal{O}_c(X)$ consists of entourages $U$ such that $U$ is a coarse entourage of $[0, \infty) \otimes X_{max,max}$ and $U$ is continuously controlled with respect to the pair $([0, \infty] \times X, \{\infty\} \times X)$.

3. The bornology of $\mathcal{O}_c(X)$ is the bornology of $[0, \infty) \otimes X_{max,max}$.

One easily checks that the coarse structure is well-defined. Since it is contained in the coarse structure of $[0, \infty) \otimes X_{max,max}$ it is compatible with the bornology. □

**Remark 7.3.** In addition to continuous control we thus require that an entourage $U$ of $\mathcal{O}_c(X)$ in addition satisfies

$$\sup_{((t,x),(t',x'))\in U} |t - t'| < \infty \ . \tag{7.1}$$

If $X$ is compact, then a subset of $\mathcal{O}_c(X)$ is bounded if and only if it belongs to the continuously controlled bornology of the pair $([0, \infty] \times X, \{\infty\} \times X)$. □

If $f : X \to X'$ is a morphism in $\mathbf{Top}_{c,\mathrm{Hausd}}$ between non-empty spaces, then we define

$$\mathcal{O}_c(f) : \mathcal{O}_c(X) \to \mathcal{O}_c(X') \ , \quad (t, x) \mapsto (t, f(x)) \ .$$

This map is obviously proper and preserves continuously controlled entourages by Lemma 7.1. Since it preserves the $t$-variable it also preserves the additional condition for entourages of $\mathcal{O}_c(X)$ explained in (7.1).

**Definition 7.4.** *We call the functor* $\mathcal{O}_c : \mathbf{Top}_{c,\mathrm{Hausd}} \to \mathbf{BornCoarse}$ *the continuously controlled cone functor.*

Let $X$ be in $\mathbf{Top}_{c,\mathrm{Hausd}}$.

**Proposition 7.5.** *If $X$ is contractible, then $\mathcal{O}_c(X)$ is flasque in the generalized sense.*

*Proof.* Let $h : [0,\infty] \times X \to X$ be the contraction such that $h(0,-) = \mathtt{id}_X$ and $h(\infty,-) = \mathtt{const}_{x_0}$. We define $f : [0,\infty) \otimes \mathcal{O}_c(X) \to \mathcal{O}_c(X)$ by

$$f(s,t,x) := (s+t, h(\frac{s}{1+t}, x)) \ .$$

We check that the map $f$ is proper. Let $B$ be bounded in $\mathcal{O}_c(X)$. Then there exists $r$ in $[0,\infty)$ such that $B \subseteq [0,r] \times X$. But then $f^{-1}(B) \subseteq [0,r] \times [0,r] \times X$. This is a bounded subset of $[0,\infty) \otimes \mathcal{O}_c(X)$.

We check that the map $f$ is controlled.

Let $U$ be an entourage of $\mathcal{O}_c(X)$ and let $((s_i,t_t,x_i),(s'_i,t'_i,x'_i))_i$ be some net in $U_1 \times U$. Assume that $\mathtt{lim}_i f(s_i,t_i,x_i) = (\infty,x)$ for some $x$ in $X$. Then we must show that $\mathtt{lim}_i f(s'_i,t'_i,x'_i) = (\infty,x)$.

We assume the contrary. Then by compactness of $[0,\infty] \times [0,\infty] \times X$ there exists a subnet with the following properties:

1. $\mathtt{lim}_{i'}(s_{i'},t_{i'},x_{i'}) = (u,v,y)$ and $\mathtt{lim}_{i'}(s'_{i'},t'_{i'},x'_{i'}) = (u',v',y')$
2. $\mathtt{lim}_{i'} \frac{s_{i'}}{1+t_{i'}} = r$ and $\mathtt{lim}_{i'} \frac{s'_{i'}}{1+t'_{i'}} = r'$.
3. $\mathtt{lim}_{i'} f(s'_{i'},t'_{i'},x'_{i'})$ exists, but is not equal to $(\infty,x)$.

We know that the entries of the pairs $(u,u')$ and $(v,v')$ are either both $\infty$ or both finite. Furthermore, $u+v = u'+v' = \infty$. We have by assumption $x = h(r,y) \neq h(r',y')$. We must exclude the following cases.

1. Case: $v = \infty$: Then $y = y'$ since $U$ is continuously controlled, and also $r = r'$ since the differences $|s_{i'} - s'_{i'}|$ and $|t_{i'} - t'_{i'}|$ are uniformly bounded. In fact, the first difference is bounded by 1 since we consider the entourage $U_1 \times U$. The second difference is bounded by the condition (7.1) on $U$.

Hence $h(r,y) = h(r',y')$ which is impossible.

2. Case: $v < \infty$: Then $u' = u = \infty$ and $v' < \infty$. In this case $r = r' = \infty$, and $h(r',y') = x_0 = h(r,y)$. This is impossible.

In the upshot we have shown that $f$ is controlled. □

**Lemma 7.6.** *The functor $\mathcal{O}_c : \mathbf{Top}_{c,\mathrm{Hausd}} \to \mathbf{BornCoarse}$ sends closed subspaces to subspaces.*

*Proof.* Let $Y$ be a closed subspace of $X$. Then we must show that $\mathcal{O}_c(Y) \to \mathcal{O}_c(X)$ is a coarse embedding.

This amounts to show the equalities

$$\mathcal{B}_{\mathcal{O}_c(Y)} = \mathcal{O}_c(Y) \cap \mathcal{B}_{\mathcal{O}_c(X)} \quad \text{and} \quad \mathcal{C}_{\mathcal{O}_c(Y)} = (\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap \mathcal{C}_X \ .$$

Since the map $\mathcal{O}_c(Y) \to \mathcal{O}_c(X)$ is a morphism we clearly have

$$\mathcal{B}_{\mathcal{O}_c(Y)} \supseteq \mathcal{O}_c(Y) \cap \mathcal{B}_{\mathcal{O}_c(X)} \quad \text{and} \quad \mathcal{C}_{\mathcal{O}_c(Y)} \subseteq (\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap \mathcal{C}_{\mathcal{O}_c(X)} \ .$$

In the following we verify the reverse inclusions.

Let $B$ be in $\mathcal{B}_{\mathcal{O}_c(Y)}$. Then $B \subseteq [0,r] \times Y$ for some $r$ in $[0,\infty)$. But then $B \subseteq [0,r] \times X$ and hence $B \in \mathcal{B}_{\mathcal{O}_c(X)}$.

Let $U$ be in $\mathcal{C}_{\mathcal{O}_c(X)}$. We show that then $(\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap U \in \mathcal{C}_{\mathcal{O}_c(Y)}$. First of all

$$\sup_{((t,x),(t',x')) \in (\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap U} |t - t'| \le \sup_{((t,x),(t',x')) \in U} |t - t'| < \infty \ .$$

If $((t_i,y_i),(t'_i,y'_i))$ is a net in $(\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap U$ with $\mathtt{lim}_i(t_i,y_i) = (\infty,y)$. Then since $((t_i,y_i),(t'_i,y'_i))$ is a net in $U$ we can conclude that $\mathtt{lim}_i(t'_i,y'_i) = (\infty,y)$. □

**Remark 7.7.** In Lemma 7.6 it is important to assume that $Y$ is closed. If $Y$ is not closed then it is not clear that $\mathcal{C}_{\mathcal{O}_c(Y)} \subseteq (\mathcal{O}_c(Y) \times \mathcal{O}_c(Y)) \cap \mathcal{C}_{\mathcal{O}_c(X)}$. In order to locate the problem, let $U$ be in $\mathcal{C}_{\mathcal{O}_c(Y)}$. Let $((t_i,y_i),(t'_i,y'_i))_i$ be a net in $U$ such that $\mathtt{lim}_i(t_i,y_i) = (\infty,x)$. If $Y$ is not closed, then we do not know that $x \in Y$. If $x \notin Y$, then we do not have any information about convergence of the net $((t'_i,y'_i))_i$. □

Let $h : [0,1] \times X \to Z$ be a map in $\mathbf{Top}_{c,\mathrm{Hausd}}$. We consider the functions $p_- \equiv 0$ and $p_+(t) := 1 + t$. We consider the coarse cylinder $I_p(\mathcal{O}_c(X))$ over $\mathcal{O}_c(X)$ associated to $p = (p_-,p_+)$.

**Lemma 7.8.** *The map*

$$f : I_p(\mathcal{O}_c(X)) \to \mathcal{O}_c(Z) \ , \quad (s,t,x) \mapsto (t, h(\frac{s}{1+t}, x))$$

*is a morphism.*

*Proof.* We show that $f$ is proper. Let $B$ be a bounded subset of $\mathcal{O}_c(Z)$. Then there exists $r$ in $[0,\infty)$ such that $B \subseteq [0,r] \times Z$. Then $f^{-1}(B) \subseteq ([0,r+1] \times [0,r] \times X) \cap I_p(\mathcal{O}_c(X))$. Consequently, $f^{-1}(B)$ is bounded.

We show that $f$ is controlled. It suffices to show that the image under $f$ of any entourage of $I_p(\mathcal{O}_c(X))$ of the form $U_r \times U$ for $r$ in $(0,\infty)$ and $U$ a coarse entourage of $\mathcal{O}_c(X)$ is again a coarse entourage of $\mathcal{O}_c(X)$.

First of all we have

$$\sup_{((s,t,x),(s',t',x'))\in U_r\times U} |t-t'| = \sup_{((t,x),(t',x'))\in U} |t-t'| < \infty$$

by (7.1) since $U$ is a coarse entourage of $\mathcal{O}_c(X)$. It remains to show that $(f\times f)(U_r \times U)$ is continuously controlled.

Let $((s_i,t_i,x_i),(s'_i,t'_i,x'_i))_i$ be a net of points in $U_r \times U$ and assume that $\mathtt{lim}_i\, f(s_i,t_i,x_i) = (\infty,x)$. Then we must show that $\mathtt{lim}_i\, f(s'_i,t'_i,x'_i) = (\infty,z)$.

Assume the contrary. Then there exists a subnet such that

1. $\mathtt{lim}_{i'}(s_{i'},t_{i'},x_{i'}) = (u,v,y)$ and $\mathtt{lim}_{i'}(s'_{i'},t'_{i'},x'_{i'}) = (u',v',y')$
2. $\mathtt{lim}_{i'} \frac{s_{i'}}{1+t_{i'}} = a$ and $\mathtt{lim}_{i'} \frac{s'_{i'}}{1+t'_{i'}} = a'$
3. $\mathtt{lim}_{i'}\, f(s'_i,t'_i,x'_i)$ exists but is not equal to $(\infty,x)$.

We have $v=\infty$. Since $U$ is continuously controlled this implies that $v'=\infty$. Consequently, $\mathtt{lim}_{i'}\, f(s'_i,t'_i,x'_i) = (\infty,z')$ for some $z'$ different from $z$. Since $|s_{i'}-s'_{i'}|$ and $|t_{i'}-t'_{i'}|$ are uniformly bounded we conclude that $a=a'$. We further conclude that $y=y'$ since $U$ is continuously controlled, again. But then $z=h(a,y)=h(a',y')=z'$. This is a contradiction. □

Let $(Z,Y)$ be a closed decomposition of $X$. Then pair $(\mathcal{O}_c(Z),\{\mathcal{O}_c(Y)\})$ is a complementary pair. Note that if $Z\cap Y=\emptyset$, then every member of $Z\cap\{\mathcal{O}_c(Y)\}$ is bounded and hence coarsely equivalent to a point.

We consider the category of pairs $\mathbf{Top}^2_{c,\mathrm{Hausd}}$ of pairs $(X,A)$ of compact Hausdorff spaces and closed subspaces. A morphism $f:(X,A)\to(X',A')$ in $\mathbf{Top}^2_{c,\mathrm{Hausd}}$ is a map $f:X\to X'$ such that $f(A)\subseteq A'$.

We have a functor

$$\mathcal{O}_c : \mathbf{Top}^2_{c,\mathrm{Hausd}} \to \mathbf{BC}^{\mathrm{pair}}\ , \quad (X,A)\mapsto(\mathcal{O}_c(X),\{\mathcal{O}_c(A)\})\ .$$

Let $(E,\delta)$ be a linear coarse homology. Then we consider the $\delta$-functor $E\mathcal{O}_c(-,-) : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{M}$ obtained by precomposing with $\mathcal{O}_c(-)$.

**Proposition 7.9.**

1. *For every pair $(X,A)$ we have a long exact sequence*

$$E\mathcal{O}_c(A) \to E\mathcal{O}_c(X) \to E\mathcal{O}_c(X,A) \stackrel{\delta}{\to} E\mathcal{O}_c(A)[-1]$$

2. *If $X$ is contractible, then $E\mathcal{O}_c(X) \cong 0$.*

3. *The functor $E\mathcal{O}_c$ is homotopy invariant.*

4. *If $(A,B)$ is a non-disjoint closed decomposition of $X$, then we have a Mayer-Vietoris sequence*

$$E\mathcal{O}_c(A\cap B) \to E\mathcal{O}_c(A) \oplus E\mathcal{O}_c(B) \to E\mathcal{O}_c(X) \to E\mathcal{O}_c(A\cap B)[-1]\ .$$

5. *If $(A,B)$ is a disjoint closed decomposition of $X$, then we have a Mayer-Vietoris sequence*

$$E(*) \to E\mathcal{O}_c(A) \oplus E\mathcal{O}_c(B) \to E\mathcal{O}_c(X) \to E(*)[-1]\ .$$

6. *For a finite family of pointed compact Hausdorff spaces $((X_i,x_i))_{i\in I}$ we have $E(\bigvee_{i\in I} X_i) \cong \bigoplus_{i\in I} E(X_i)$.*

7. *We have $E\mathcal{O}_c(*\sqcup *) \simeq E(*)[-1]$*

*Proof.*

Assertion 1: The exact sequence is the exact sequence of the pair $(\mathcal{O}_c(X), \{\mathcal{O}_c(A)\})$.

Assertion 2: This follows from Proposition 7.5.

Assertion 3: Let $f,g : X \to Z$ be homotopic maps with homotopy $h : IX \to Z$. Then the map $I_p(\mathcal{O}_c(X)) \to \mathcal{O}_c(Z)$ from Lemma 7.8 is a coarse homotopy from $\mathcal{O}_c(f)$ to $\mathcal{O}_c(g)$. It follows that $E\mathcal{O}_c(f) = E\mathcal{O}_c(g)$.

Assertion 4 and Assertion 5: This follows from splicing the exact sequences for the pairs $(\mathcal{O}_c(X), \{\mathcal{O}_c(A)\})$ and $(\mathcal{O}_c(B), \mathcal{O}_c(B)\cap\{\mathcal{O}_c(A)\})$.

Assertion 6: We use the Mayer-Vietoris sequence and that $E(*) \cong 0$.

Assertion 7: We use the Mayer-Vietoris sequence (5). □

**Example 7.10.** As in ordinary topology one can calculate inductively the homology of spheres and of pairs $(D^n, S^{n-1})$. We have $E\mathcal{O}_c(S^n) \simeq E(*)[-n-1]$. We get from the long exact sequence that $E\mathcal{O}_c(D^n, S^{n-1}) \cong E\mathcal{O}_c(S^{n-1})[-1] \simeq E(*)[-n-1]$.

Assume that $f : S^n \to S^n$ is a map. Then $H\mathcal{X}\mathcal{O}_c(f)$ is multiplication by the degree of $f$. To see this for $n \geq 0$ it suffices to consider the map standard example

$$S^n \to \bigvee_{i=1}^{n} S^n \stackrel{\vee_{i=1}^n \mathtt{id}}{\to} S^n$$

of a degree $n$-map which induces

$$E\mathcal{O}_c(S^n) \stackrel{\mathtt{diag}}{\to} \bigoplus_{i=1}^{n} E\mathcal{O}_c(S^n) \stackrel{+}{\to} E\mathcal{O}_c(S^n) \ .$$

We now consider the case of $E = H\mathcal{X}$. If $X$ is a finite CW-complex, then we get the cellular complex

$$\to H\mathcal{X}\mathcal{O}_{c,3}(X_2, X_1) \to H\mathcal{X}\mathcal{O}_{c,2}(X_1, X_0) \to H\mathcal{X}\mathcal{O}_{c,1}(X_0) \to \mathbb{Z}$$

(with $\mathbb{Z}$ in degree 0) calculating $H\mathcal{X}\mathcal{O}_{c,n}(X_n, X_{n-1}) \cong \bigoplus_{Z_n(X)} \mathbb{Z}$, where $Z_n(X)$ is the set of $n$-cells of $X$. Note that this chain group lives in degree $n+1$. This allows to calculate $H\mathcal{X}\mathcal{O}_c(X)$ for finite CW-complexes. We get

$$H\mathcal{X}\mathcal{O}_{c,n}(X) \cong \begin{cases} H_{n-1}(X) & n \geq 0 \\ \mathtt{ker}(H_0(X) \to \mathbb{Z}) & n = 1 \\ 0 & else \end{cases} \ ,$$

i.e.

$$H\mathcal{X}\mathcal{O}_c(X) \cong \hat{H}(X)[-1] \ ,$$

where $\hat{H}$ denotes the reduced homology. □

**Remark 7.11.** Let $((X_i, x_i))_{i\in I}$ be an infinite family of pointed compact Hausdorff spaces. Then we can define the strong wegde $\bigvee_{i\in I}^{\mathrm{str}} X_i$ as a subspace of $\prod_{i\in I} X_i$ of tuples with at most one non-base points. It is an interesting question whether, possibly under additional assumptions on $E$, e.g. strong additivity, we have

$$E\mathcal{O}_c(\bigvee_{i\in I}^{\mathrm{str}} X_i) \cong \prod_{i\in I} E\mathcal{O}_c(X_i) \ .$$

□

Let $X$ be in $\mathbf{Top}_{c,\mathrm{Hausd}}$. Let $(X_i)_{i\in\mathbb{N}}$ be a decreasing family of closed subspaces of $X$. We consider the set-theoretic intersection $\bigcap_{i\in I} X_i$ in $X$ with the subspace topology.

**Lemma 7.12.** *We have an exact sequence*

$$0 \to \lim^{1}_{i\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(X_i)[1] \to H\mathcal{X}\mathcal{O}_c(\bigcap_{i\in\mathbb{N}} X_i) \to \lim_{i\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(X_i) \to 0 \ .$$

*Proof.* We note that the underlying set of $\mathcal{O}_c(\bigcap_{i\in\mathbb{N}} X_i)$ is the intersection of the subsets $\bigcap_{i\in I} \mathcal{O}_c(X_i)$ in $\mathcal{O}_c(X)$. The inclusion into $\mathcal{O}_c(X)$ induces a bornological coarse structure on $\bigcap_{i\in I} \mathcal{O}_c(X_i)$. We must check that this structure coincides with the intrinsic structure of $\mathcal{O}_c(\bigcap_{i\in\mathbb{N}} X_i)$. But this follows from Lemma 7.6 since $\bigcap_{i\in\mathbb{N}} X_i$ is a closed subspace of $X$.

The short exact sequence now follows from (5.2). □

**Remark 7.13.** Using the argument of [**?**, Sec.6.6. Thm. 8] one can show that a morphism between homotopy invariant excisive $\delta$-functors satisfying the conclusion of Lemma 7.12 is an isomorphism. The idea is that any pair $(X, A)$ of compact spaces can be written as the intersection of a decreasing family of compact pairs which have the homotopy type of finite polyhedral pairs.

**Lemma 7.14.** *$H\mathcal{X}\mathcal{O}_c$ satisfies the cluster axiom.*

*Proof.* (for countable families) Let $(X_i, x_i)_{i\in\mathbb{N}}$ be a family in $\mathbf{Top}_{hc}$. For every $n$ in $\mathbb{N}$ we have by Mayer-Vietoris for the decomposition of $\bigvee^{\mathrm{str}}_{i\in\mathbb{N}} X_i$ into $\bigvee_{i=0}^{n} X_i$ and $\bigvee^{\mathrm{str}}_{i\geq n+1} X_i$ with intersection $\bigvee_{i=0}^{n} X_i \cap \bigvee^{\mathrm{str}}_{i\geq n+1} X_i = *$ and $H\mathcal{X}\mathcal{O}_c(*) \cong 0$ and Proposition 7.9.6 that

$$H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq 0} X_i) \cong H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq n+1} X_i) \oplus \bigoplus_{i=0}^{n} H\mathcal{X}\mathcal{O}_c(X_i) \ .$$

Applying $\lim_{n\in\mathbb{N}}$ we get

$$H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq 0} X_i) \cong \lim_{n\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq n+1} X_i) \oplus \prod_{i\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(X_i) \ .$$

By Lemma 7.12 we have exact sequence

$$0 \to \lim^{1}_{n\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq n+1} X_i)[1] \to H\mathcal{X}\mathcal{O}_c(\bigcap_{n\in\mathbb{N}} \bigvee^{\mathrm{str}}_{i\geq n} X_i) \to \lim_{n\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq n+1} X_i) \to 0 \ .$$

Using that $\bigcap_{n\in\mathbb{N}} \bigvee^{\mathrm{str}}_{i\geq n} X_i \cong *$ we conlcude that $\lim_{n\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(\bigvee^{\mathrm{str}}_{i\geq n+1} X_i) = 0$.

For uncountable families the argument is similar. In the last step one uses the $\lim$-spectral sequence instead. □

**Example 7.15.** We consider the standard Cantor set defined by

$$C := \bigcap_{i\in\mathbb{N}} X_i \ ,$$

where the subspaces $X_i$ of $[0,1]$ are defined inductively by $X_0 := [0,1]$ and

$$X_{i+1} := 1/3X_i \cup (2/3 + 1/3X_i)$$

for $i \geq 1$. The subset $X_i$ consists of $2^i$ disjoint intervals and is therefore homotopy equivalent to a set of $2^i$ points. We have

$$H\mathcal{X}\mathcal{O}_c(X_i) \cong \mathtt{ker}(\sum : \bigoplus_{k=0}^{2^i-1} \mathbb{Z} \to \mathbb{Z})[-1] \ .$$

The connecting map is

$$H\mathcal{X}\mathcal{O}_c(X_{i+1}) \to H\mathcal{X}\mathcal{O}_c(X_i)$$

is given by

$$(n_i)_{i=0}^{2^{i+1}-1} \mapsto (n_{2i} + n_{2i+1})_{i=0}^{2^i-1} \ .$$

One can calculate that

$$H\mathcal{X}(\mathcal{O}_c(C)) \cong \mathtt{lim}_{i\in\mathbb{N}} H\mathcal{X}\mathcal{O}_c(X_i) \cong \prod_{\mathbb{N}} \mathbb{Z} \ .$$

To this end use the coordinate change

$$\bigoplus_{i=0}^{2^i} \mathbb{Z} \xrightarrow{\cong} \bigoplus_{i=0}^{2^i} \mathbb{Z} \ , \quad (n_0, \ldots, 2^i - 1) \mapsto (\sum_{i=0}^{2^i-1} n_i, \sum_{i=1}^{2^i-1} n_i, \ldots, \ldots, n_{2^i-1}) \ .$$

Under this identification $H\mathcal{X}\mathcal{O}_c(X_i)$ is the subgroup characterized by the equation $n_0 = 0$, and the transition maps $H\mathcal{X}\mathcal{O}_c(X_{i+1}) \to H\mathcal{X}\mathcal{O}_c(X_i)$ are given by

$$(0, n_1, \ldots, n_{2^{i+1}-1}) \mapsto (0, n_2, n_4, \ldots, n_{2^i-2}) \ .$$

There is no $\mathtt{lim}^1$-contribution since $H\mathcal{X}\mathcal{O}_{c,1}(X_i) = 0$ for all $i$. □

**Example 7.16.** Let $G$ be the Sierpinski gasket. It is defined as $G := \bigcap_{i=0}^{\infty} X_i$ where $X_i$ is the subset of $[0,1] \times [0,1]$ inductively defined by $X_0 := \{t_0 + t_1 \leq 1\}$ and

$$X_{i+1} := 1/2X_0 \cup ((1/2, 0) + 1/2X_0) \cup ((0, 1/2) + 1/2X_0) \ .$$

One can calculate using Mayer-Vietoris of the CW-structure and Example 7.10 that

$$H\mathcal{X}\mathcal{O}_c(X_i) \cong (\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \cdots \oplus \mathbb{Z}^{3^i})[-2] \ .$$

The connecting map

$$H\mathcal{X}\mathcal{O}_c(X_{i+1}) \to H\mathcal{X}\mathcal{O}_c(X_i)$$

is the obvious projection

$$(\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \cdots \oplus \mathbb{Z}^{3^{i+1}})[-2] \to (\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \cdots \oplus \mathbb{Z}^{3^i})[-2] \ .$$

It is surjective. The Mittag-Leffler condition is satisfied and excludes $\mathtt{lim}^1$-terms. We conclude that

$$H\mathcal{X}\mathcal{O}_c(G) \cong \prod_{\mathbb{N}} \mathbb{Z}[-2] \ .$$

**Example 7.17.** We consider the Hawaian earring space $R$. It is defined as the compact subset

$$R := \bigcup_{i=0}^{\infty} [(0, 2^{-i}) + 2^{-i} S^1]$$

of $\mathbb{R}^2$.

We let

$$X_n := R \cup [(0, 2^{-n}) + 2^{-n} D^2] \ ,$$

i.e., we fill the $n+1$th circle by a disc. Then we have an equality

$$R = \bigcap_{n \in \mathbb{N}} X_n \ .$$

Using Mayer-Vietors we calculate that have

$$H\mathcal{X}\mathcal{O}_c(X_n) \cong \mathbb{Z}^n[-2] \ .$$

The connecting map $H\mathcal{X}\mathcal{O}_c(X_{n+1}) \to H\mathcal{X}\mathcal{O}_c(X_n)$ is given by the projection onto the first $n$-components $\mathbb{Z}^{n+1}[-2] \to \mathbb{Z}^n[-2]$. It is surjective. The Mittag-Leffler condition is satisfied and excludes $\mathtt{lim}^1$-terms. We conclude that

$$H\mathcal{X}\mathcal{O}_c(R) \cong \prod_{\mathbb{N}} \mathbb{Z}[-2] \ .$$

**Example 7.18.** Let $Z$ be the mapping cylinder of the map $f : D^2 \to D^2$ given by $z \mapsto 1/2z^2$. We have $Z = [0,1] \times D^2 \cup_f D^2$. We call $\{0\} \times D^2$ the base and the second copy of $D^2$ the top. We form the infinite telescope by glueing the top of the $i+1$th copy of $Z$ with the base of the $i$th copy. We let $X$ be the one-point compactification of this telescope. This telescope is contractible. The subset $[0,1] \times S^1 \cup_f D^2 \setminus \mathrm{int}(1/2D^2)$ is called the boundary of $Z$. For $n$ in $\mathbb{N}$ we let $X_n$ be the subspace of $X$ obtained by replacing the first $n$-copies of $Z$ by their boundaries. Let $A := \bigcap_{n \in \mathbb{N}} X_n$. We are interested in calculating $H\mathcal{X}\mathcal{O}_c(A)$.

Note that $X_n$ is homotopy equivalent to the mapping cylinder of the map $S^1 \to D^2$ given by $z \mapsto z^{2^n}$.

We calculate using the Mayer-Vietoris sequence

$$H\mathcal{X}\mathcal{O}_c(X_n) \cong (\mathbb{Z}/2^n\mathbb{Z})[-2] \ .$$

The inclusion $H\mathcal{X}\mathcal{O}_c(X_{n+1}) \to H\mathcal{X}\mathcal{O}(X_n)$ induces the projection $(\mathbb{Z}/2^{n+1}\mathbb{Z})[-2] \to (\mathbb{Z}/2^n\mathbb{Z})[-2]$.

We have

$$\lim_{n \in \mathbb{N}} (\mathbb{Z}/2^n\mathbb{Z})[-2] \cong \mathbb{Z}_2[-2] \ , \qquad \lim_{n \in \mathbb{N}}{}^1 (\mathbb{Z}/2^n\mathbb{Z})[-2] \cong 0 \ ,$$

where $\mathbb{Z}_2$ denotes the $p$-adic integers. Consquently we get

$$H\mathcal{X}\mathcal{O}_{c,i}(A) \cong \left\{ \begin{array}{cc} \mathbb{Z}_2 & i = 2 \\ 0 & else \end{array} \right.$$

# 8 Controlled objects in additive categories

We start with the notion of an additive category. The typical example is the category of abelian groups **Ab**. The main goal of the present section the construction of a functor

$$\mathbf{A}(-) : \mathbf{BornCoarse} \to \mathbf{Add}$$

which associates to a bornological coarse space $X$ the additive category of $X$-controlled objects in an additive category $\mathbf{A}$. We then study the basic properties of this functor.

Let $\mathbf{A}$ be a category. It is called pointed if it has a zero-object, i.e. an object 0 which is both final and initial.

We assume that $\mathbf{A}$ is pointed and admits finite coproducts and products. Then for every two objects $A, A'$ we have a canonical map

$$A \sqcup A' \to A \times A'$$

given by $a \mapsto (a, 0)$ and $a' \mapsto (0, a')$.

Let $\mathbf{A}$ be a category.

**Definition 8.1.** $\mathbf{A}$ *is called semiadditive if it is pointed, admits finite coproducts and products, and if the canonical map* $A \sqcup A' \to A \times A'$ *is an isomorphism for all objects* $A, A'$ *in* $\mathbf{A}$.

In a semiadditive category we call the coproduct, or equivalently the product, of objects the sum.

Given two maps $f, f' : A \to A'$ we can define their sum as the composition

$$f + f' : A \stackrel{\mathtt{diag}}{\to} A \times A \stackrel{(f,f')}{\to} A' \times A' \cong A' \sqcup A' \stackrel{\text{fold}}{\to} A' \ .$$

One can check that this defines the structure of commutative monoids on the sets $\mathtt{Hom}_{\mathbf{A}}(A, A')$ such that the composition is bilinear.

**Definition 8.2.** *A functor between semiaddive categores is called additive if it preserves finite coproducts.*

An additive functor then preserves zero objects (empty coproducts) and induces homomorphisms between the morphism sets (is enriched in abelian monoids).

Let $\mathbf{A}$ be semiadditive.

**Definition 8.3.** $\mathbf{A}$ *is called additive if* $\mathtt{Hom}_{\mathbf{A}}(A, A')$ *is a group for all* $A, A'$ *in* $\mathbf{A}$.

We get the category **Add** of additive categories and additive functors.

**Example 8.4.** If $\hat{\mathbf{A}}$ is an additive category and $\mathbf{A}$ is a full subcategory which is closed under isomorphisms and forming finite sums, then $\mathbf{A}$ is also additive. $\square$

**Example 8.5.** The typical example of an additive category is the category $\mathrm{Mod}(R)$ of left $R$-modules for a ring $R$. The category of abelian groups $\mathbf{Ab} = \mathrm{Mod}(\mathbb{Z})$ is a special case.

For modules $A, A'$ the direct sum $A \oplus A'$ with the canonical inclusions represents the coproduct, and the same object $A \oplus A'$ with the canonical projections represents the product. The canonical map turns out to be the identity, and the monoid structure on the morphism sets is the usual sum of morphisms.

Subexamples are the categories $\mathrm{Mod}^{\mathrm{fg}}(R)$ and $\mathrm{Mod}^{\mathrm{fg,proj}}(R)$ of finitely generated projective modules. $\square$

Let $\mathbf{A}$ be an additive category and $A$ be an object of $\mathbf{A}$.

**Definition 8.6.**

1. *An idempotent on $A$ is an element $p$ in $\mathtt{End}(A)$ such that $p^2 = p$.*

2. *It is called split of we have an isomorphism $A \cong A' \oplus A''$ under which $p$ corresponds to $\mathtt{id}_{A'} \oplus 0$.*

In Point 2 the object $A'$ is unique up to isomorphism. The datum of $A'$ (or better $A' \to A \to A'$) is called an image of $p$. It is unique up to (unique) isomorphism.

We let $\mathrm{Idem}(A)$ denote the set of idempotents on an object $A$. We say that $\mathbf{A}$ is idempotent complete if every idempotent in $\mathbf{A}$ is split.

**Example 8.7.** Let $R$ be a ring. Then the additive categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}^{\mathrm{fg,proj}}$ are idempotent complete. $\square$

Let $\mathbf{A}$ be an additive functor. Then the sum can be conidered as a functor $\oplus : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$. Let $S, T : \mathbf{A} \to \mathbf{A}'$ be two functors between additive categories, then we can define their sum by

$$S \oplus T : \mathbf{A} \stackrel{\mathtt{diag}}{\to} \mathbf{A} \times \mathbf{A} \stackrel{(S,T)}{\to} \mathbf{A}' \times \mathbf{A}' \stackrel{\oplus}{\to} \mathbf{A}' .$$

Let $\mathbf{A}$ be an additive category.

**Definition 8.8.** $\mathbf{A}$ *is called flasque, if it admits an additive endofunctor $S : \mathbf{A} \to \mathbf{A}$ such that there exists an equivalence $S \oplus \mathtt{id} \simeq S$.*

**Example 8.9.** The category $\mathrm{Mod}(R)$ is flasque. We define the functor $S : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ such that $S(A) := \bigoplus_{\mathbb{N}} A$.

In general, the category $\mathrm{Mod}^{\mathrm{fg,proj}}$ is not flasque. If we have non-trivial a rank function

$$\mathtt{rk} : \mathrm{Mod}^{\mathrm{fg,proj}} \to \mathbb{N}$$

which is additive under sums, then the equivalence $S \oplus \mathtt{id} \simeq S$ would imply

$$\mathtt{rk}(S(A)) + \mathtt{rk}(A) = \mathtt{rk}(S(A))$$

and hence $\mathtt{rk}(A) = 0$ for all objects $A$. This is a contradiction. □

We fix an idempotent complete category $\hat{\mathbf{A}}$ which admits all sums together with a full additive subcategory $\mathbf{A}$. The typical example is

$$\hat{\mathbf{A}} = \mathrm{Mod}(R) \ , \quad \mathbf{A} = \mathrm{Mod}^{\mathrm{fg,proj}} \ .$$

Let $X$ be a set.

**Definition 8.10.**

1. *An $X$-controlled object in $\hat{\mathbf{A}}$ is a pair $(M, \mu)$, where*
   a) *$M$ is an object of $\hat{\mathbf{A}}$.*
   b) *$\mu$ is a finitely additive projection-valued measure on $X$, i.e. a map $\mu : \mathcal{P}_X \to \mathrm{Idem}(M)$ such that*
      i. *For all subsets $Y, Z$ in $\mathcal{P}_X$ with $Y \subseteq Z$ we have $\mu(Y) + \mu(Z \setminus Y) = \mu(Z)$.*
      ii. *$\mu(X) = \mathtt{id}_M$.*
2. *A morphism $a : (M, \mu) \to (M', \mu')$ is simply a morphism $a : M \to M'$ in $\hat{\mathbf{A}}$.*

We obtain the category $\hat{\mathbf{A}}(X)$ of $X$-controlled objects in $\hat{\mathbf{A}}$. It is again additive. The sum of two objects is represented by

$$(M, \mu) \oplus (M', \mu') \cong (M \oplus M', \mu \oplus \mu') \ ,$$

and the sum of morphisms $a, a' : (M, \mu) \to (M', \mu')$ is given by the morphism $a + a' : M \to M'$.

Let $X \to X'$ be a map of sets. Then we define functor

$$f_* : \hat{\mathbf{A}}(X) \to \hat{\mathbf{A}}(X')$$

by $f_*(M,\mu) := (M, f_*\mu)$, where $(f_*\mu)(Y') := \mu(f^{-1}(Y'))$ for all $Y'$ in $\mathcal{P}_{X'}$. On morphisms we set $f_*(a) := a$. This functor is obviously additive. If $g : X' \to X''$ is a second functor, then we have an equality $(g \circ f)_* = g_* \circ f_*$ of functors $\hat{\mathbf{A}}(X) \to \hat{\mathbf{A}}(X'')$.

We get a functor

$$\hat{\mathbf{A}}(-) : \mathbf{Set} \to \mathbf{Add}$$

which associates to a set $X$ the additive category of $X$-controlled objects in $\hat{\mathbf{A}}$.

Recall that we have fixed the additive subcategory $\mathbf{A}$ which we assume to be closed under isomorphisms. Note that $\hat{\mathbf{A}}$ is assumed to be idempotent complete. We now assume that $X$ is in **Born**. Let $(M,\mu)$ be in $\hat{\mathbf{A}}(X)$. We define the support of $(M,\mu)$ by

$$\mathtt{supp}(M,\mu) := \{x \in X \mid \mu(\{x\}) \neq 0\} \ .$$

We will write $M_x$ or $M(Y)$ for the choice of an image of the idempotents $\mu(\{x\})$ or $\mu(Y)$.

**Definition 8.11.** *An object $(M,\mu)$ is called locally finite if:*

1. *For every $x$ in $X$ we have $M_x \in \mathbf{A}$.*
2. *$\mathtt{supp}(M,\mu)$ is locally finite.*
3. *For every subset $Y$ of $X$ the canonical map $\bigoplus_{x\in Y} M_x \to M(Y)$ is an isomorphism.*

Let $f : X \to X'$ be a morphism in **Born**. Then $f_*$ preserves locally finite objects. Indeed let $(M,\mu)$ be locally finite in $\hat{\mathbf{A}}(X)$.

1. If $B'$ is bounded in $X$', then $(f_*\mu)(B') = \mu(f^{-1}(B))$ implies $\mathrm{Im}((f_*\mu)(B')) \cong \mathrm{Im}(\mu(f^{-1}(B')))$. Since $f$ is proper the set $f^{-1}(B')$ is bounded in $X$. Hence $\mathrm{Im}(\mu(f^{-1}(B'))) \in \mathbf{A}$, and therefore also $\mathrm{Im}((f_*\mu)(B'))$.
2. If $Y'$ is a subset of $X'$, then

$$\bigoplus_{x'\in Y'} \mathrm{Im}(f_*\mu(\{x'\})) \cong \bigoplus_{x'\in Y'} \mathrm{Im}(\mu(f^{-1}(\{x'\})) \cong \bigoplus_{x\in f^{-1}(\{x'\})} \bigoplus_{x'\in Y} \mathrm{Im}(\mu(\{x\})) \cong \bigoplus_{x\in f^{-1}(Y')} \mathrm{Im}(\mu(\{x\})) \cong \mathrm{Im}(f_*\mu)(Y') \ .$$

The sum of locally finite objects in $\hat{\mathbf{A}}(X)$ is again locally finite. Consequently, the full subcategory $\hat{\mathbf{A}}_{\mathrm{lf}}(X)$ of $\mathbf{A}(X)$ of locally finite objects is again additive. We get a subfunctor

$$\hat{\mathbf{A}}_{\mathrm{lf}}(-) : \mathbf{Born} \to \mathbf{Add} \ .$$

We now assume that $X$ is in **BornCoarse**. If $U$ is an entourage of $X$ and $Y, Y'$ are subsets, then we say that $Y'$ is $U$-separated from $Y$ if $Y' \cap U[Y] = \emptyset$.

Let $(M,\mu)$ and $(M',\mu')$ be in $\hat{\mathbf{A}}_{\mathrm{lf}}(X)$ and $a : (M,\mu) \to (M',\mu')$ be a morphism.

**Definition 8.12.** *$a$ is controlled if there exists a coarse entourage $U$ of $X$ such that $\mu(Y')a\mu(Y) = 0$ for all subsets $Y', Y$ such that $Y'$ is $U$-separated from $Y$.*

If we want to stress the choice of $U$, then say that $a$ is $U$-controlled.

**Lemma 8.13.** *The composition and the sum of controlled morphisms is again controlled.*

*Proof.* If $a, a' : (M, \mu) \to (M', \mu')$ are $U$, respectively $U'$-controlled, then $a + a'$ is $U \cup U'$-controlled.

If $a : (M, \mu) \to (M', \mu')$ is $U$-controlled and $a' : (M', \mu') \to (M'', \mu'')$ is $U'$-controlled, then $a' \circ a$ is $U' \circ U$-controlled. □

Let $f : X \to X'$ be a morphism between bornological coarse spaces. Let $a : (M, \mu) \to (M', \mu')$ be controlled.

**Lemma 8.14.** *$f_*(a) : f_*(M, \mu) \to f_*(M', \mu')$ is controlled.*

*Proof.* If $a$ is $U$-contrrolled, then $f_*$ is $f(U)$-controlled. Indeed, if $Y'$ is $f(U)$-separated from $Y$, then $f^{-1}(Y')$ is $U$-separated from $f^{-1}(Y)$. □

We let $\mathbf{A}(X)$ denote the wide subcategory of $\hat{\mathbf{A}}_{\mathrm{lf}}(X)$ of controlled morphisms. This category is again additive. To this end we observe that the structure maps of sums in $\hat{\mathbf{A}}_{\mathrm{lf}}(X)$ are $\mathtt{diag}(X)$-controlled.

We have thus defined a functor

$$\mathbf{A}(-) : \mathbf{BornCoarse} \to \mathbf{Add}$$

which sends $X$ to the additive category of locally small $X$-controlled objects in $\hat{\mathbf{A}}$ and controlled morphisms.

In the following we study the basic properties of the functor $\mathbf{A}(-)$.

**Lemma 8.15.** *The functor $\mathbf{A}(-)$ is u-continuous.*

*Proof.* Let $X$ be in **BornCoarse**. We note that $\mathbf{A}(X_U)$ is a wide subcategory of $\mathbf{A}(X)$ consisting of morphisms which are controlled by entourages in $\mathcal{C}_{X_U}$. We can calculate $\mathtt{colim}_{U \in \mathcal{C}_X} \mathbf{A}(X_U)$ as the union of the subcategories $\mathbf{A}(X_U)$ inside $\mathbf{A}(X)$. Since every morphism in $\mathbf{A}(X)$ is controlled by some entourage we actually have an equality $\mathtt{colim}_{U \in \mathcal{C}_X} \mathbf{A}(X_U) = \mathbf{A}(X)$. □

**Lemma 8.16.** *The functor $\mathbf{A}$ sends close maps to equivalent functors.*

*Proof.* Let $f, g : X \to Y$ be morphisms such that $g \sim_V f$ for some coarse entourage $V$ of $Y$. We define a natural transformation $u : f_* \to g_*$ by the family $u = (u_{(M,\mu)})_{(M,\mu)\in \mathbf{A}(X)}$ with $u_M : f_*(M, \mu) \to g_*(M, \mu)$ given by $\mathtt{id}_M : M \to M$ for all $(M, \mu)$ in $\mathbf{A}(X)$. We must show that $u_M : f_*(M, \mu) \to g_*(M, \mu)$ is a morphism in $\mathbf{A}(Y)$. We claim that it is $V$-controlled. Assume that $Y'$ is $V$-separated from $Y$. Then

$$(g_*\mu)(Y')u_M\mu_M(f_*\mu)(Y) = \mu(g^{-1}(Y'))\mu(f^{-1}(Y)) = \mu(g^{-1}(Y') \cap f^{-1}(Y)) = 0 \ .$$

since $g^{-1}(Y') \cap f^{-1}(Y) = \emptyset$. Ideed, if there exists a point $x$ in this intersection, then $g(x) \in Y' \cap V[Y]$. But $Y' \cap V[Y] = \emptyset$ by assumption. □

**Lemma 8.17.** *If $X$ is flasque, then $\mathbf{A}(X)$ is flasque.*

*Proof.* Assume that $f : X \to X$ implements flasqueness. We define $S : \mathbf{A}(X) \to \mathbf{A}(X)$ by

$$S(M, \mu) := (\bigoplus_{n\in\mathbb{N}} M, \oplus_{n\in\mathbb{N}} f_*\mu)$$

and

$$S(a) := \oplus_{n\in\mathbb{N}} f_*(a) = \oplus_{n\in\mathbb{N}} a \ .$$

We must show that $S(M) \in \hat{\mathbf{A}}_{\mathrm{lf}}(X)$. Let $B$ be a bounded subset of $B$. Then there exists $n_0$ in $\mathbb{N}$ such that $f^n)^{-1}(B) = \emptyset$ for all $n$ in $\mathbb{N}$ with $n \geq n_0$. Hence for $n \geq n_0$ we have $(f^n_*\mu)(B) = 0$.

If $B = \{x\}$, then we conclude that $S(M, \mu)_x \cong \bigoplus_{n=0}^{n_0-1} M_x \in \mathbf{A}$. Furthermore, if $a$ is $U$-controlled, then $S(a)$ is $\bigcup_{n\in\mathbb{N}} f^n(U)$-controlled.

Finally we have an isomorphism of functors $f_* \circ S \oplus \mathtt{id} \simeq S$. Since $f$ is close to $\mathtt{id}_X$ we have $f_* \simeq \mathtt{id}_{\mathbf{A}(X)}$ by Lemma 8.16. Consequently $S \oplus \mathtt{id} \simeq S$. □

Let $\mathbf{C} \to \mathbf{D}$ be the inclusion of a full additive subcategory. Then we can define the quotient category $\mathbf{D}/\mathbf{C}$ as follows. We first define an equivalence relation on the morphism sets $\mathtt{Hom}_{\mathbf{D}}(D, D')$ of $\mathbf{D}$ by declearing $f \sim f'$ provided there exists a factorization

$$\begin{array}{ccc} D & \xrightarrow{\ f-f'\ } & D' \\ & \searrow \quad \nearrow & \\ & C & \end{array} \ .$$

for some object $C$ of $\mathbf{C}$. It is clear that this relation is compatible with the composition. We can therefore define the quotient category $\mathbf{D}/\mathbf{C}$ with the same objects as $\mathbf{D}$, and with the morphism sets

$$\mathtt{Hom}_{\mathbf{D}/\mathbf{C}}(D, D') := \mathtt{Hom}_{\mathbf{D}}(D, D')/\sim \ .$$

The equivalence relation is compatible with the group structure on the morphism sets. Indeed assume that $f \sim f'$ and $g \sim g'$. Then we must show that $f + g \simeq f' + g'$. By assumption we have factorizations

$$\begin{array}{ccc} D & \xrightarrow{f-f'} & D' \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle i} & \\ & C & \end{array} \ , \qquad \begin{array}{ccc} D & \xrightarrow{g-g'} & D' \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle j} & \\ & C' & \end{array} \ .$$

Then get the factorization

$$\begin{array}{ccc} D & \xrightarrow{f+g-f'-g'} & D' \\ & {\scriptstyle p\oplus q}\searrow \quad \nearrow{\scriptstyle i+j} & \\ & C \oplus C' & \end{array} \ .$$

This implies that the enrichment of $\mathbf{D}$ in abelian groups descends to $\mathbf{D}/\mathbf{C}$. The existence of finite sums in $\mathbf{D}/\mathbf{C}$ is inherited from $\mathbf{D}$.

If the bold

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C}' \\ \downarrow & & \downarrow \\ \mathbf{D} & \longrightarrow & \mathbf{D}' \\ \downarrow & & \downarrow \\ \mathbf{D}/\mathbf{C} & \dashrightarrow & \mathbf{D}'/\mathbf{C}' \end{array}$$

diagram of additive functors is given, then we get the extension indicated by the dotted arrow in the natural way.

Let $X$ be in **BornCoarse** and let $\mathcal{Y}$ be a big family on $X$. Then we define

$$\mathbf{A}(\mathcal{Y}) \subseteq \operatorname{\mathtt{colim}}_{Y \in \mathcal{Y}} \mathbf{A}(Y) \ .$$

This colimit can be realized as the full subcategory of $\mathbf{A}(X)$ of objects which are supported on some member of $\mathcal{Y}$. We can therefore form $\mathbf{A}(X)/\mathbf{A}(\mathcal{Y})$. We actually have a functor

$$\mathbf{BC}^{\mathrm{pair}} \to \mathbf{Add} \ , \quad (X, \mathcal{Y}) \mapsto \mathbf{A}(X)/\mathbf{A}(\mathcal{Y}) \ .$$

**Lemma 8.18.** *The functor* $\mathbf{A}$ *is excisive in the sense that for every* $X$ *in* **BornCoarse** *and complementary pair* $(Z, \mathcal{Y})$ *the canonical functor*

$$i : \mathbf{A}(Z)/\mathbf{A}(Z \cap \mathcal{Y}) \to \mathbf{A}(X)/\mathbf{A}(\mathcal{Y})$$

*is an equivalence.*

*Proof.* We define an inverse equivalence $p : \mathbf{A}(X)/\mathbf{A}(\mathcal{Y}) \to \mathbf{A}(Z)/\mathbf{A}(Z \cap \mathcal{Y})$ as follows. For every object $(M, \mu)$ in $\mathbf{A}(X)/\mathbf{A}(\mathcal{Y})$ we choose $p(M, \mu) := (M(Z), \mu_Z)$ in $\mathbf{A}(Z)/\mathbf{A}(Z \cap \mathcal{Y})$. For every morphism $[a] : (M, \mu) \to (M', \mu')$ we set $p([a]) := [a_Z]$.

One must check that $(M(Z), \mu_Z)$ is locally finite. Furthermore, one must check that $a_Z$ is controlled.

Furthermore this construction is compatible with the composition because of the rule

$$(a' \circ a)_Z - a'_Z \circ a_Z = \mu''(Z)a''\mu'(X \setminus Z)a\mu(Z) \ .$$

If $a'$ is $U$-controlled, then

$$\mu''(Z)a'\mu'(X \setminus Z)a\mu(Z) = \mu''(Z \cap U[X \setminus Z])a'\mu'(X \setminus Z)a\mu(Z) \ .$$

Consequently the composition has a factorization over the object

$$(M(Z \cap U[X \setminus Z]), \mu_{Z \cap U[X \setminus Z]})$$

which belongs to $\mathbf{A}(Z \cap \mathcal{Y})$. By construction $p \circ i \cong \mathtt{id}$. On the other hand $i \circ p \cong \mathtt{id}$ since

$$a - a_Z = a - \mu(Z)a\mu(Z) = \mu'(X \setminus Z)a\mu(Z) + \mu'(Z)a\mu(X \setminus Z) \ .$$

Both summands have obvious factorizations over objects of $\mathbf{A}(\mathcal{Y})$. Hence $[a] = i(p([a]))$. $\square$

**Definition 8.19.** *The inclusion $\mathbf{C} \to \mathbf{D}$ is called a Karoubi filtration if for every diagram $C \to D \to C'$ in $\mathbf{D}$ with $C, C'$ in $\mathbf{C}$ there exists an extension to a diagram*

$$\begin{array}{ccccc} C & \longrightarrow & D & \longrightarrow & C' \\ \downarrow & & \downarrow\cong & & \uparrow \\ C'' & \xrightarrow{\text{incl}} & C'' \oplus D' & \xrightarrow{\mathtt{pr}} & C'' \end{array}$$

*for some $C''$ in $\mathbf{C}$.*

**Lemma 8.20.** *The inclusion $\mathbf{A}(\mathcal{Y}) \to \mathbf{A}(X)$ is a Karoubi filtration.*

*Proof.* We consider a diagram

$$(M, \mu) \xrightarrow{a} (N, \nu) \xrightarrow{b} (M', \mu')$$

in $\mathbf{A}(\mathbf{X})$ where $(M, \mu)$ and $(M', \mu')$ belong to $\mathbf{A}(\mathcal{Y})$.

Then $Y := \mathtt{supp}(M, \mu)$ belongs to $\bar{\mathcal{Y}}$. We assume that $a$ is $U$-controlled and $b$ is $V$-controlled. We define $Z := (V^{-1} \cup U)[Y]$. Then we consider the decomposition

$$(N, \nu) \cong (N(Z), \nu_Z) \oplus (N(X \setminus Z), \nu_{X \setminus Z}) \ .$$

We let $i : N(Z) \to N$ and $p : N \to N(Z)$, $q : N \to N(X \setminus Z)$ be the inclusion and projections. Since $\mathcal{Y}$ is big we have $Z \in \bar{\mathcal{Y}}$. We consider the diagram

$$\begin{array}{ccccc} (M, \mu) & \xrightarrow{\quad a \quad} & (N, \nu) & \xrightarrow{\quad b \quad} & (M', \mu') \\ \downarrow u & & p\oplus q \downarrow \cong & & v \uparrow \\ (N(Z), \nu_Z) & \xrightarrow{\text{incl}} & (N(Z), \nu_Z) \oplus (N(X \setminus Z) & \xrightarrow{\mathtt{pr}} & (N(Z), \nu_Z) \end{array} \ .$$

with $u = p \circ a$ and $v := b \circ i$. In order to show that it commutes we must check that $(p \oplus q) \circ a = (u \oplus 0)$ and $b = v \circ p$. The first equality is equivalent to $q \circ a = 0$ and the second equality is equvalent to $b \circ j = 0$. By construction $X \setminus Z$ is $U$-separated from $Y$. Since $a$ is $U$-controlled we get $q \circ a = 0$. Furthermore $Y$ is $V$-separated from $X \setminus Z$, and this implies that $b \circ j = 0$. $\square$

Let $(Y, Z)$ be a coarsely disjoint partition of $X$ in **BornCoarse**.

Then we have functors $i_* : \mathbf{A}(Y) \to \mathbf{A}(X)$ and $j_* : \mathbf{A}(Z) \to \mathbf{A}(X)$ induced by the inclusions $i : Y \to X$ and $j : Z \to X$ and can define

$$i \oplus j : \mathbf{A}(Y) \times \mathbf{A}(Z) \to \mathbf{A}(X) .$$

This is an isomorphism. Indeed we can define an inverse functor $\mathbf{A}(X) \to \mathbf{A}(Y) \times \mathbf{A}(Z)$ by

$$(M, \mu) \mapsto (M(Y), \mu_Y), (M(Z), \mu_Z) , \quad a \mapsto (a_{|M(Y)}, a_{|M(Z)}) .$$

The point here is that the mixed terms $\mu'(Z)a\mu(Y) = 0$ and $\mu'(Y)a\mu(Z) = 0$ vanish for all morphisms $a : (M, \mu) \to (M', \mu')$ in $\mathbf{A}(X)$.

Let $(X_i)_{i\in I}$ be a family in **BornCoarse** and form the free union $X := \bigsqcup_{i\in I}^{\mathrm{free}} X_i$. Then we have projections $\mathbf{A}(X) \to \mathbf{A}(X_i)$ for all $i$ in $I$.

**Lemma 8.21.** *The induced map*

$$\mathbf{A}(X) \to \prod_{i\in I} \mathbf{A}(X_i) , \quad (M, \mu) \mapsto ((M(X_i), \mu_{X_i})_{i\in I}$$

*is an equivalence of categories.*

*Proof.* The inverse functor sends $((M_i, \mu_i)_{i\in I}$ to $(\bigoplus_{i\in I} M_i, \bigoplus_{i\in I} \mu_i)$ and $(a_i)_{i\in I}$ to $\bigoplus_{i\in I} a_i$. It is straightforward to check that this functor is well-defined and an inverse equivalence. $\square$

# 9 Coarse algebraic $K$-homology

We fix a pair $\hat{\mathbf{A}}$ of an idempotent complete additive category and an additive subcategory $\mathbf{A}$.

**Example 9.1.** The typical example is $\hat{A} = \mathrm{Mod}(R)$ and $A = \mathrm{Mod}^{\mathrm{fg,proj}}$ for some ring $R$. $\square$

In the present section we construct the (linear ) coarse algebraic $K$-homology theory with coefficients in $\mathbf{A}$, a delta functor $(K\mathbf{A}\mathcal{X}, \delta)$ with

$$K\mathbf{A}\mathcal{X} : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}} .$$

It will be obtained from $\mathbf{A}(-) : \mathbf{BornCoarse} \to \mathbf{Add}$ by post-composeing with an algebraic $K$-theory functor.

We let $\mathbf{Add}^{\mathrm{pair}}$ denote the category of pairs $\mathbf{C} \to \mathbf{D}$ of full inclusions of subcategories which are Karoubi filtrations.

**Definition 9.2.** *An algebraic $K$-theory functor is a pair $(K, \delta)$ of a functor*

$$K : \mathbf{Add} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}}$$

*together with a map*

$$\delta : K(\mathbf{D}/\mathbf{C}) \to K(\mathbf{C})[-1]$$

*for every Karoubi filtration $\mathbf{C} \to \mathbf{D}$ such that*

1. *the sequence*
$$K(\mathbf{C}) \to K(\mathbf{D}) \to K(\mathbf{D}/\mathbf{C}) \xrightarrow{\delta} K(\mathbf{C})[-1]$$
*is natural in $(\mathbf{C} \to \mathbf{D})$ in $\mathbf{Add}^{\mathrm{pair}}$ and exact for every Karoubi filtration $\mathbf{C} \to \mathbf{D}$.*

2. *$K$ sends isomorphic functors to equal maps.*

3. *$K$ preserves filtered colimits.*

4. *$K$ is additive on morphisms.*

We will use the following results as a black-box.

**Theorem 9.3** (M. Schlichting)**.** *There exists a $K$-theory functor with the property that $K_0(\mathbf{A})$ is the Grothendieck group of the monoid of isomorphism classes of objects of $A$ provided $\mathbf{A}$ is idempotent complete.*

**Lemma 9.4.** *An algebraic $K$-theory functor annihilates flasque additive categories.*

*Proof.* Assume that $\mathbf{C}$ is a flasque additive category. Let $S : \mathbf{C} \to \mathbf{C}$ implement flasqueness, i.e. $S \oplus \mathtt{id}_{\mathbf{C}} \cong S$. Then using the additivity of the $K$-theory functor on functors and its invariance under isomorphisms of functors we get

$$K(S) + K(\mathtt{id}_{\mathbf{C}}) = K(S \oplus \mathtt{id}_{\mathbf{C}}) = K(S) \ .$$

This implies $K(\mathtt{id}_{\mathbf{C}}) = 0$ and hence $K(\mathbf{C}) = 0$. □

**Example 9.5.** The last condition implies non-triviality. For example $K_0(\mathrm{Mod}(k)^{\mathrm{fg}}) \cong \mathbb{Z}$ for any field $k$.

For a ring $R$ we have

$$K_i(\mathrm{Mod}^{\mathrm{fg,proj}}(R)) \cong K_i(R)$$

for all $i \geq 0$ where $K_i(R) := \pi_i(K_0(R) \times BGL(R)^+)$ is Quillen's K-theory for rings. If $R$ is regular, then $K_i(\mathrm{Mod}^{\mathrm{fg,proj}}(R)) \cong 0$ provided $i < 0$. □

**Theorem 9.6** (G. Carlsson)**.** *The $K$-theory functor in addition preserves products.*

**Definition 9.7.** *We define the coarse algebraic $K$-homology $(K\mathbf{A}\mathcal{X}, \delta)$ by*

$$K\mathbf{A}\mathcal{X} : \mathbf{BC}^{\mathrm{pair}} \to \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}} , \quad K\mathbf{A}\mathcal{X}(X, \mathcal{Y}) := K(\mathbf{A}(X)/\mathbf{A}(\mathcal{Y}))$$

*and*

$$\delta : K(\mathbf{A}(X)/\mathbf{A}(\mathcal{Y})) \to K(\mathbf{A}(\mathcal{Y}))[-1]$$

*(induced by the Karoubi filtration $\mathbf{A}(\mathcal{Y}) \to \mathbf{A}(X)$).*

**Theorem 9.8.** *$(K\mathbf{A}\mathcal{X}, \delta)$ is a strongly additive coarse homology theory.*

*Proof.* For every complementary pair $(X, \mathcal{Y})$ we have functially the Karoubi filtration $\mathbf{A}(\mathcal{Y}) \subseteq \mathbf{A}(X)$ and hence the functorial exact sequence

$$K\mathbf{A}\mathcal{X}(\mathcal{Y}) \to K\mathbf{A}\mathcal{X}(X) \to K\mathbf{A}\mathcal{X}(X, \mathcal{Y}) \stackrel{\delta}{\to} K\mathbf{A}\mathcal{X}(\mathcal{Y})[-1] .$$

In view of Lemma 8.16 and since $K$ sends isomorphic functors to equal maps the functor $K\mathbf{A}\mathcal{X}$ is coarsely invariant. In view of Lemma 8.15 and since $K$ preserves filtered colimit the functor $K\mathbf{A}\mathcal{X}$ is $u$-continuous. If $X$ is flasque, then by Lemma 8.17 $\mathbf{A}(X)$ is flasque, and hence $K\mathcal{A}\mathcal{X}(X) = 0$ by Lemma 9.4. □

**Proposition 9.9.** *$K\mathbf{A}\mathcal{X}$ is strongly additive.*

*Proof.* This follows from Theorem 9.6 and 8.21. □

**Example 9.10.** One can use coarse geometry in order to give a geometric model for the negative $K$-groups of an additive category. Note that $A(*) \simeq \mathbf{A}$ and therefore $K\mathbf{A}\mathcal{X}(*) \cong K(\mathbf{A})$. We have

$$K\mathbf{A}\mathcal{X}_k(\mathbb{Z}^n) \cong K\mathbf{A}\mathcal{X}_{k-n}(*) \cong K_{k-n}(\mathbf{A}) .$$

In particular

$$K_{-n}(\mathbf{A}) \cong K\mathbf{A}\mathcal{X}_0(\mathbb{Z}^n) .$$

Thus $K_{-n}(\mathbf{A})$ is the Grothendieck group of isomorphism class of objects in the additive category $\mathbf{A}(\mathbb{Z}^n)$.

# 10 The equivariant case

Let $G$ be a group. Then we can consider the category $\mathbf{Fun}(BG, \mathbf{BornCoarse})$ of bornological coarse spaces with an action of $G$ by automorphisms.

**Example 10.1.**

We consider $G = \mathbb{Z}$. It acts on $\mathbb{R}$ by $(n, x) \mapsto x + n$ (shift).

It can also act by $(n, x) \mapsto 2^n x$ (scaling).

Both actions are fundamentally different □

**Definition 10.2.** *A $G$-bornological coarse space is a bornological coarse space $X$ with an action by automorphisms such that for every entourage $U$ of $X$ also $\bigcup_{g \in G}(g \times g)(U) \in \mathcal{C}_X$.*

Equivalently one can require that the set of $G$-invariant entourages $\mathcal{C}_X^G$ is cofinal in the set of all entourages $\mathcal{C}_X$.

**Example 10.3.**

$\mathbb{R}$ with the shift action is a $G$-bornological coarse space.

$\mathbb{R}$ with the scaling action is not.

**Example 10.4.** If $G$ acts on a metric space $(X, d)$ by quasi-isometries, then it acts on the underlying $G$-bornological coarse space $X_d$. Recall that this means that for every $g$ there are constants $C, C'$ such that

$$C^{-1} d(gx, gx') - C' \leq d(x, x') \leq C d(gx, gx') + C'$$

for all $x, x'$ in $X$.

But $X_d$ is a $G$-bornological coarse space if one can choose the constants independently of $g$ in $G$. This is the case in particular if $G$ acts by isometries.

**Example 10.5.** The group $G$ gives rise to the $G$-coarse space $G_{can}$. The coarse structure is generated by the subsets $G\{(g, e)\}$ for all $g$ in $G$. Note that $\pi_0^{coarse}(G_{can,min}) = *$.

The coarse structure on $\mathbb{Z}_{can,min}$ is the metric coarse coarse structure of $\mathbb{Z}$.

If $G$ is finitely generated, then $G_{can}$ is the coarse structure associated to any choice of word metricl

Since thickenings of points are finite it is compatible with the miminal bornology.

The coarse structure on $\mathbb{R}^{can}$ is much smaller than the metric coarse structure. But $\mathbb{R}$ (with the translation action) is also a $\mathbb{R}$-bornological coarse spaces,

**Definition 10.6.** *We let $G$***BornCoarse** *be the full subcategory of* **Fun**$(BG,$ **BornCoarse**$)$ *of $G$-bornological coarse spaces and equivariant morphisms.*

**Example 10.7.** We have a functor

$$G\mathbf{Set} \to G\mathbf{BornCoarse}\ , \quad S \mapsto S_{min,max}\ .$$

The definition of an equivariant coarse homology theory we must adopt the following modifications of definitions from the non-equivariant case.

1. flasqueness: The morphism implementing flasqueness must be equivariant.
2. $u$-continuity: We must replace $\mathcal{C}_X$ by $\mathcal{C}_X^G$ as the index category of the index colimit. The equivariant condition is $\mathtt{colim}_{\mathcal{C}_X^G} E(X_U) \to E(X)$ is an isomorphism. Indeed, if $U$ is in $\mathcal{C}_X^G$, then $X_U \in G\mathbf{BornCoarse}$.
3. equivariant big family: We consider families of $G$-invariant subsets.
4. Excision: We consider excisive pairs $(Z, \mathcal{Y})$ of invariant subsets and equivariant big families.

In the definition of coarse cylinders it is natural to require the components pf $p = (p_+, p_-)$ to be $G$-invariant.

We now modify the construction of $H\mathcal{X}$ in order to define the equivariant coarse homology $(H\mathcal{X}^G, \delta)$. To this end we replace the complex $C\mathcal{X}(X)$ by the subcomplex $C\mathcal{X}^G(X)$ of $G$-invariant chains. Recall that $c$ in $C\mathcal{X}_n(X)$ is a function $c : X^{n+1} \to \mathbb{Z}$ which is controlled and locally finite. It belongs to the subgroup $C\mathcal{X}^G(X)$ of $C\mathcal{X}_n(X)$ if and only if $c(gx_0, \ldots, gx_n) = c(x_0, \ldots, gx_n)$ for all $g$ in $G$ and $(x_0, \ldots, x_n)$ in $X^{n+1}$. We get a chain complex

$$(C\mathcal{X}^G(X, \partial)\ .$$

By repeating the construction of $H\mathcal{X}$ with this replcaement we get the equivariant coarse homology theory $(H\mathcal{X}^G, \delta)$. The verification of the axioms is by the same arguments.

**Theorem 10.8.** *$(H\mathcal{X}^G, \delta)$ is an equivariant coarse homology theory.*

**Example 10.9.** We have a canonical isomorphism

$$H\mathcal{X}^G(G_{min,min} \otimes X) \cong H\mathcal{X}(\mathrm{Res}^G X)\ .$$

The follows from the explicit isomorphism

$$C\mathcal{X}^G(G_{min,min} \otimes X) \to C\mathcal{X}(X)$$

given by

$$C\mathcal{X}_n^G(G_{min,min} \times X) \ni c \mapsto (c((1, x_0), \ldots, (1, x_n)) \in C\mathcal{X}_n(X)$$

The inverse is given by

$$\tilde{c} \mapsto ((g_0, x_0), \ldots, (g_n, x_n)) \mapsto c(x_0, \ldots, x_n)\ .$$

□

There are situations where one starts with an equivariant coarse homology theory $E^G$. One can define an associated non-equivariant one by $E(X) := E^G(G_{min,min} \otimes X)$. □

**Remark 10.10.** The following is a reminder on group homology. It starts with the observation that abelian groups with a $G$-action are the same as left $\mathbb{Z}[G]$-modules. Expressed in categorical terms, we have an equivalence

$$\mathbf{Fun}(BG, \mathbf{Ab}) \simeq \mathrm{Mod}^l(\mathbb{Z}[G]) .$$

The classical home for group homology is the derived category $D(\mathrm{Mod}(\mathbb{Z}[G]))$ of the abelian category $\mathrm{Mod}(\mathbb{Z}[G])$. If $M$ is in $\mathrm{Mod}^r(\mathbb{Z}[G])$, then we can define the functor

$$- \otimes_{\mathbb{Z}[G]} M : \mathrm{Mod}^l(\mathbb{Z}[G]) \to \mathbf{Ab} , \quad N \mapsto N \otimes_{\mathbb{Z}[G]} M .$$

This functor is right-exact and preserves projectives and is therefore left derivable. The derived version of this functor is usually denoted by

$$M \otimes^L_{\mathbb{Z}[G]} - : D(\mathrm{Mod}(\mathbb{Z}[G])) \to D(\mathbf{Ab}) .$$

We consider $\mathbb{Z}$ as an object of $\mathrm{Mod}^l(\mathbb{Z}[G])$. By definition, the group homology of $G$ with coefficients in $M$ is defined by

$$H_*(G, M) := H_*(M \otimes^L_{\mathbb{Z}[G]} \mathbb{Z}) .$$

We let $G^\bullet$ be the simplicial $G$-set induced by the $G$-map $G \to *$. The $G$-set of$n$-simplices of $G^\bullet$ is the $G$-set $G^{n+1}$. We get a simplicial $G$-module $\mathbb{Z}[G^\bullet]$ and let $(C_*(G), \partial)$ be the associated chain complex. Since $G^{n+1}$ is a free $G$-set the $\mathbb{Z}[G]$-module $C_n(G)$ is free. Indeed, if $X$ is a free $G$-set, then fixing a base point in each orbit we get an equivariant bijection $X \cong \bigsqcup_{G\backslash X} G$. Hence $\mathbb{Z}[X] \cong \bigoplus_{G\backslash X} \mathbb{Z}[G]$. The map $G \to *$ induces a map $C_0(G) \to \mathbb{Z}$. One can check that the induced map $C(G) \to \mathbb{Z}[0]$ is a quasi-isomorphism. Hence it is a free resolution of $\mathbb{Z}$. Consequently, we can calculate group homology using the so called standard complex

$$H_*(G, M) := H_*(M \otimes_{\mathbb{Z}[G]} C(G) .$$

If $H$ is a subgroup of $G$, then the inclusion $H \to G$ induces a homomorphism $\mathbb{Z}[H] \to \mathbb{Z}[G]$. We consider the induction functor

$$\mathtt{Ind}_H^G : \mathrm{Mod}^r(\mathbb{Z}[H]) \to \mathrm{Mod}^r(\mathbb{Z}[G]) , \quad N \mapsto N \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] .$$

Here we consider $\mathbb{Z}[G]$ as a left-module over $\mathbb{Z}[H]$ using the left multiplication and as a right $\mathbb{Z}[G]$-module. Note that $G$ is a free $H$-set and therefore $\mathbb{Z}[G]$ is free as a right $\mathbb{Z}[H]$-module. Hence $\mathtt{Ind}_H^G$ is exact and descends to the derived category.

We have the following induction isomorphism. Let $M$ be in $\mathrm{Mod}^r(\mathbb{Z}[H])$

**Lemma 10.11.** *We have a canonical isomorphism*

$$H_*(H,M) \cong H_*(G, \mathtt{Ind}_H^G(M)) \ .$$

*Proof.* We can consider the complex $C(H)$ as a free resolution of $\mathbb{Z}$ in $\mathrm{Mod}^l(\mathbb{Z}[H])$ since $G^{n+1}$ is a free $H$-set for every $n$. Then

$$H_*(H,M) \cong H_*(M \otimes_{\mathbb{Z}[H]} C(G)) \cong H_*((M \otimes_{\mathbb{Z}(H)} \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C(G)) \cong H_*(G, \mathtt{Ind}_H^G(M)) \ .$$

□

□

**Proposition 10.12.** *We have natural (in $S$) isomorphism*

$$H\mathcal{X}_*^G(S_{min,max} \otimes G_{can,min}) \cong H_*(G, \mathbb{Z}[S]) \ .$$

*In particular, for a subgroup $K$ of $G$ we have*

$$H\mathcal{X}_*^G((G/K)_{min,max} \otimes G_{can,min}) \cong H_*(K, \mathbb{Z}) \ .$$

*Proof.* We claim that there is a natural isomorphism between $C\mathcal{X}^G(G_{can,min} \otimes S_{min,max})$ and the standard complex $C(G, \mathbb{Z}[S])$. We first note that $S$ is a right $G$-set by the action $(g,s) := g^{-1}s$. To do so, we identify

$$\mathbb{Z}[G^{n+1}] \cong \mathbb{Z}[G^{n+1} \times S] \ ,$$

where $G^{n+1} \times S$ carries the diagonal $G$-action. Then we define the homomorphism

$$\phi_n \colon C_n(G, \mathbb{Z}[S]) \cong \mathbb{Z}[S] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G^{n+1}] \to C\mathcal{X}^G(G_{can,min} \otimes S_{min,max}) \tag{10.1}$$

as the linear extension of

$$[s] \otimes [g_0, g_1, \dots, g_n] \mapsto \sum_{g \in G} ((gg_0, gs), \dots, (gg_n, gs)) \ . \tag{10.2}$$

Note that all summands are different points on $(G \times S)^{n+1}$ so that the infinite sum makes sense, and it is $G$-invariant by construction. Furthermore this map sends $[h^{-1}s] \otimes [g_0, g_1, \dots, g_n]$ and $[s] \otimes [hg_0, hg_1, \dots, hg_n]$ to the same element. Hence it factorizes over $\otimes_{\mathbb{Z}[G]}$.

Every $((gg_0, gs), \dots, (gg_n, gs))$ is controlled by the entourage $G\{(g_i, g_j) \mid 0 \le i,j \le n\} \times \mathtt{diag}_S$ of the $G$-bornological coarse space $G_{can,min} \otimes S_{min,max}$. To show that this chain is also locally finite, it suffices to check that there are only finitely many points in the support of the chain (10.2) which meet bounded sets of the form $B \times S$, where $B$ is some finite subset of $G$. This is clear since $G$ acts freely on $G^{n+1}$. This finishes the argument for the assertion that (10.1) is well-defined.

It is straightforward to check that the collection $\{\phi_n\}_n$ is a chain map. The boundary is defined by the same formula on both sides.

We now argue that the map (10.1) is an isomorphism. To this end we define an inverse

$$\psi\colon C\mathcal{X}_n(G_{can,min}\otimes S_{min,max})\to \mathbb{Z}[S]\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G^{n+1}]\cong C_n(G,\mathbb{Z}[S])\ .$$

Let

$$c=\sum_{x\in (G\times S)^{n+1}} n_x x$$

be an invariant, controlled and locally finite $n$-chain on $G_{can,min}\otimes S_{min,max}$. We then define

$$\psi(c):=\sum_{(g_1,\dots,g_n,s)\in G^n\times S} n_{((1,s),(g_1,s)\dots,(g_n,s))}[s]\otimes[1,g_1,\dots,g_n]\ .$$

Assume that $c$ is $U$-controlled. Then only summands with $\{g_1,\dots,g_n\}\subseteq U[\{1\}]$ contribute to the sum. Since $U[\{1\}]$ is bounded and $c$ is locally finite we see that the number of non-trivial summands is finite. This implies that $\psi(c)$ is well-defined.

It is straightforward to check that $\phi$ and $\psi$ are inverse to each other: To see that $\psi\circ\phi=\mathtt{id}$, use that

$$[s]\otimes[g_0,g_1,\dots,g_n]=[g_0^{-1}s]\otimes(1,g_0^{-1}g_1,\dots,g_0^{-1}g_n)$$

in $\otimes_{\mathbb{Z}[G]}$. The equality $\phi\circ\psi=\mathtt{id}$ follows from the $G$-invariance of a chain $c=\sum_{x\in(G\times S)^{n+1}} n_x x$ together with the observation that $n_{((g_0,s_0),\dots,(g_n,s_n))}=0$ unless $s_0=\dots=s_n$. The latter fact is due to $S$ carrying the minimal coarse structure.

One easily checks that $\phi$ is natural for maps between $G$-sets.

The second assertion follows from the induction isomorphism Lemma 10.11. To this end we note that $\mathbb{Z}[G/K]=\mathtt{Ind}_K^G(\mathbb{Z})$. □

**Remark 10.13.** It turns out that in applications often the evaluation of the coarse homology theory at the objects of the form $X\otimes G_{can,min}$ is the most relevant one. □

For the following assume that $G$ is finite.

For the following example we will use a better cone functor

$$\mathcal{O}^\infty_c:\mathbf{Top}_{c,\mathrm{Hausd}}\to\mathbf{BornCoarse}\ .$$

The underlying set of $\mathcal{O}^\infty(X)$ is $\mathbb{R}\times X$. An entourage $U$ is controlled if it is continuously controlled for $\mathbb{R}\times X\to(-\infty,\infty]\times X$ and $U$ is contained in $U_d\times(X\times X)$ for some coarse entourage $U_d$ of $\mathbb{R}$.

**Lemma 10.14.** *$\mathcal{O}^\infty(X)$ is a $G$-bornological coarse space*

*Proof.* It is clear that $\mathcal{O}^\infty(X) \in \mathbf{Fun}(BG, \mathbf{BornCoarse})$. Since $G$ is finite we have $G\mathbf{BornCoarse} = \mathbf{Fun}(BG, \mathbf{BornCoarse})$. □

**Remark 10.15.** For infinite groups $G$ we need another definition of cones using uniform structures. We will discuss this later.

One then shows for an equivariant coarse homology theory $E : G\mathbf{BornCoarse} \to \mathbf{M}$ (by the same proofs as non-equivariant and for $\mathcal{O}$):

1. $E\mathcal{O}^\infty_c(-)$ is homotopy invariant

2. $E\mathcal{O}^\infty_c(-)$ is satisfies closed excision

**Lemma 10.16.** *$E\mathcal{O}^\infty_c(Z_{disc}) \cong E(Z_{min,min})[-1]$ for finite $G$-set $Z$.*

*Proof.* By the long exact sequence for $(\mathcal{O}^\infty(Z_{disc}), ((-\infty, n] \times Z)_{n\in\mathbb{N}})$ and flasqueness of the members of the family we get an isomorphism

$$E\mathcal{O}^\infty_c(Z_{disc}) \cong E(\mathcal{O}^\infty_c(Z_{disc}), ((-\infty, n] \times Z)_{n\in\mathbb{N}}) \ .$$

We now use $u$-continuity and get

$$E(\mathcal{O}^\infty_c(Z_{disc}), ((-\infty, n] \times Z)_{n\in\mathbb{N}})) \cong \operatorname*{\mathtt{colim}}_U E((\mathbb{R} \times Z)_U, ((-\infty, n] \times Z)_{n\in\mathbb{N}}) \ .$$

For every sufficiently large $U$ we have $U_{d'} \times \mathtt{diag}(Z) \subseteq U$. Then the decomposition $((-\infty, m] \times Z, [m, \infty) \times Z)$ is coarsely excisive. We further can fix $m$ in $\mathbb{N}$ so large $U_{|[m,\infty)\times Z} \subseteq U_d \times \mathtt{diag}(Z)$ since $U$ is contiuously controlled and $Z$ is finite. We use excision and get

$$E(\mathcal{O}^\infty_c(Z_{disc}), ((-\infty, n] \times Z)_{n\in\mathbb{N}}) \cong E(([m, \infty) \times Z)_U, ([m, m + n] \times Z)_{n\in\mathbb{N}}) \ .$$

Now $([m, \infty) \times Z)_U \cong [0, \infty) \otimes Z_{min,min}$. Do the same steps backwards with the coarse structure of $\mathbb{R} \otimes Z_{min,min}$ we get

$$E(([m, \infty) \times Z)_U, ([m, m + n] \times Z)_{n\in\mathbb{N}}) \cong E(\mathbb{R} \times Z_{min,min}) \simeq \Sigma E(Z_{min,min}) \ .$$

□

**Example 10.17.** A $G$-$CW$-complex a topological $G$-space $X$ with a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots X$$

such that

1. $X \cong \mathtt{colim}_{n\in\mathbb{N}} X_n$

2. For every $n$ in $\mathbb{N}$ there is a push-out

$$\begin{array}{ccc} Z_n \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ Z_n \times D^n & \longrightarrow & X_n \end{array}$$

where $Z_n$ is some $G$-set.

Since $Z_n$ is a union of $G$-orbits we can build $X$ iteratively by attaching $G$-cells of the form $G/K \times D^n$ for subgroups $K$ of $G$.

If $X$ is a finite $G$-$CW$-complex, then $X$ belongs to $G\mathbf{Top}_{c,\mathrm{Hausd}}$.

We want to calculate $H\mathcal{X}^G_*(\mathcal{O}^\infty_c(X))$. This goes by induction on cells.

Assume that $X$ is obtained from $Y$ by attaching the cell $G/K \times D^n$. Then we have a push-out

$$\begin{array}{ccc} G/K \times S^{n-1} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ G/K \times D^n & \longrightarrow & X \end{array} \quad .$$

This gives a Mayer-Vietoris sequence

$$\begin{aligned} &H\mathcal{X}^G(\mathcal{O}^\infty_c(G/K \times S^{n-1}) \otimes G_{can,min}) \\ &\to H\mathcal{X}^G(\mathcal{O}^\infty_c(G/K \times D^n) \otimes G_{can,min}) \oplus H\mathcal{X}^G(\mathcal{O}^\infty_c(Y) \otimes G_{can,min}) \\ &\to H\mathcal{X}^G(\mathcal{O}^\infty_c(X) \otimes G_{can,min}) \to H\mathcal{X}^G(\mathcal{O}^\infty_c(G/K \times S^{n-1}) \otimes G_{can,min})[-1] \end{aligned}$$

The usual calculations for spheres show

$$H\mathcal{X}^G(\mathcal{O}^\infty_c(G/K \times S^{n-1}) \otimes G_{can,min}) \cong H(K,\mathbb{Z})[-1] \oplus H(K,\mathbb{Z})[-n]$$

and

$$H\mathcal{X}^G(\mathcal{O}^\infty_c(G/K \times D^n)) \otimes G_{can,min}) \cong H(K,\mathbb{Z})[-1] \ .$$

The Mayer-Vietoris sequence therefore gives

$$H(K,\mathbb{Z})[-n] \to H\mathcal{X}^G(\mathcal{O}^\infty_c(Y) \otimes G_{can,min}) \to H\mathcal{X}^G(\mathcal{O}^\infty_c(X) \otimes G_{can,min}) \to H(K,\mathbb{Z})[-n-1] \, .$$

$\square$

**Example 10.18.** We consider the group $C_2$ and $S^n$ with the action by the antipodal map. We obtain $S^n$ from $S^{n-1}$ (equator) by attaching a cell $C_2 \times D^n$ (pair of upper and lower hemisphere). We get

$$\mathbb{Z}[-n] \to H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^{n-1}) \otimes C_{2,can,min}) \to H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^n) \otimes C_{2,can,min}) \to \mathbb{Z}[-n-1]$$

One must calculate boundary operators. Or by general groups, since $C_2$ acts freely,

$$H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^n) \cong H\mathcal{X}(\mathcal{O}_c^\infty(\mathbb{RP}^n)) \cong H(\mathbb{RP}^n;\mathbb{Z})[-1]$$

Example:

| | $H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^3))$ | $H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^4))$ | $\mathbb{Z}[5]$ |
|---|---|---|---|
| 5 | 0 | 0 | $\mathbb{Z}$ |
| 4 | $\mathbb{Z}$ | $C_2$ | 0 |
| 3 | 0 | 0 | 0 |
| 2 | $C_2$ | $C_2$ | 0 |
| 1 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |
| 0 | 0 | 0 | 0 |

The only non-trivial map is multiplication by 2

| | $H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^4))$ | $H\mathcal{X}^{C_2}(\mathcal{O}_c^\infty(S^5))$ | $\mathbb{Z}[6]$ |
|---|---|---|---|
| 6 | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 5 | 0 | 0 | 0 |
| 4 | $C_2$ | $C_2$ | 0 |
| 3 | 0 | 0 | 0 |
| 2 | $C_2$ | $C_2$ | 0 |
| 1 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |
| 0 | 0 | 0 | 0 |

□

**Example 10.19.** We consider $X$ with trivial action. If $H_*(X;\mathbb{Z})$ is free, then

$$H\mathcal{X}^G(\mathcal{O}_c^\infty(X)) \cong H_*(X;\mathbb{Z}) \otimes H_*(G,\mathbb{Z})[-1]$$

Otherwise we have a Künneth formula with `Tor`-terms. □

**Remark 10.20.** The following is good to know. If one works with the $\infty$-category $D^\infty(\mathbf{Ab})$, then one can consider $H\mathcal{X}^G$ as a functor $G\mathbf{BornCoarse} \to D^\infty(\mathbf{Ab})$ (by not taking the homology of the complex $C\mathcal{X}^G$ but viewing it as an object of $D^\infty(\mathbf{Ab})$). It gives rise to the functor

$$G\mathbf{Orb} \ni S \mapsto H\mathcal{X}^G(\mathcal{O}_c^\infty(S_{min,max}) \otimes G_{can,min}) \in D^\infty(\mathbf{Ab}) \ .$$

The theorem of Elmendorf says that any functor $G\mathbf{Orb} \to D^\infty(\mathbf{Ab})$ determines an up to equivalence unique equivariant homology theory $G\mathbf{Top} \to D^\infty(\mathbf{Ab})$. Above we calculated the value of this functor on $X$. □

Next we extend the construction of $X$-controlled objects in an additive category to the equivariant case. Let $\hat{\mathbf{A}}$ be an additive category with a strict $G$-action, i.e., an object of $\mathbf{Fun}(BG, \mathbf{Add})$. We assume that the underlying category of $\hat{\mathbf{A}}$ is idempotent complete and admits all sums.

**Example 10.21.** We can consider categories with the trivial $G$-action.

Let $R$ be a ring with $G$-action. Then we can consider $\mathrm{Mod}^G(R)$.

The objects are $R$-modules with $G$-action $(M, \kappa)$ such that $\kappa(g)(mr) = \kappa(g)(m)g(r)$. Morphisms are equivariant maps. The group $G$ acts on this category by sending $(M, \kappa)$ to $g(M, \kappa) = (M, (h \mapsto \kappa(g^{-1}hg))$ and the new $R$-module structure is given $m \cdot r := mg^{-1}r$. This action fixes morphisms. □

Let $X$ be in $G\mathbf{BornCoarse}$.

**Definition 10.22.** *An equivariant $X$-controlled $\mathbf{A}$-module is a triple $(M, \rho, \mu)$ where $(M, \mu)$ is in $\mathbf{A}(X)$ and $\rho = (\rho_g)_{g \in G}$ is a family of morphisms $\rho_g : M \to g(M)$ such that:*

1. *cocycle condition: $g(\rho_h)\rho_h = \rho_{gh}$ for all $g, h$ in $G$.*

2. *measure is invariant: $\mu(g(Y)) = \rho_g^{-1} \circ g(\mu(Y)) \circ \rho_g$.*

If $(M, \rho, \mu)$ and $(M', \rho', \mu')$ are two equivariant $X$-controlled $\mathbf{A}$-objects, then for a morphism $a : M \to M'$ we set $g \cdot a := g(\rho_g^{-1}) \circ g(a) \circ \rho_g : M \to M'$.

**Definition 10.23.** *A morphism $a : (M, \rho, \mu) \to (M', \rho', \mu')$ between equivariant $X$-controlled $\mathbf{A}$-objects is a morphism $a : M \to M'$ such that $g \cdot a = a$ for all $g$ in $G$.*

We get the additive category $\mathbf{A}^G(X)$ of equivariant $X$-controlled $\mathbf{A}$-objects and morphisms. If $f : X \to X'$ is a morphism in $G\mathbf{BornCoarse}$, then we have the additive functor

$$f_* : \mathbf{A}^G(X) \to \mathbf{A}^G(X') , \quad f_*(M, \rho, \mu) = (M, \rho, f_*\mu) .$$

**Proposition 10.24.** *The functor $\mathbf{A}^G(-)$ is*

1. *coarsely invariant*

2. *u-continuous*

3. *annihilates flasques*

4. *excisive.*

5. *It sends $\mathcal{Y} \to X$ to a Karoubi filtration $\mathbf{A}^G(\mathcal{Y}) \to \mathbf{A}^G(X)$.*

*Proof.* The arguments are the same as in the non-equivariant case. □

We define the equivariant coarse algebraic $K$-homology with coefficients in $\mathbf{A}$ by $(K\mathbf{A}\mathcal{X}^G, \delta)$ such that
$$K\mathbf{A}\mathcal{X}^G(X) \cong K(\mathbf{A}^G(X)) \ .$$

It is again interesting to calculate some values of $K\mathbf{A}\mathcal{X}^G$. In order to state a nice result we restrict to the case $\mathbf{A} = \mathrm{Mod}^{\mathrm{fg,proj}}(R)$ with the trivial $G$-action.

For a group $K$ we consider the group ring $R[K]$. Furthermore, for $\mathbf{Fun}(BK, \mathrm{Mod}^{\mathrm{fg,proj}})$ is the category of finitely generated projective $R$-modules with an action of $G$.

**Theorem 10.25.** *We have*
$$K\mathbf{A}\mathcal{X}^G((G/K)_{min,max} \otimes G_{can,min}) \cong K(\mathrm{Mod}^{\mathrm{fg,proj}}(R[K]))$$
*and*
$$K\mathbf{A}\mathcal{X}^G((G/K)_{min,min}) \cong K(\mathbf{Fun}(BK, \mathrm{Mod}(R)^{\mathrm{fg,proj}}) \ .$$

*Proof.* [BEKW] □

# 11 Uniform bornological coarse spaces

Our first goal is to introduce the category of uniform bornological coarse spaces. The new notion is that of a uniform structure, which we introduce in the following. A uniform structure is a structure which is more special than a topology but more general than a metric.

Let $X$ be a set.

**Definition 11.1.** *A uniform structure on $X$ is a subset $\mathcal{U}$ of $X$ with the following properties.*

1. *If $U$ is in $\mathcal{U}$, then $\mathtt{diag}(X) \subseteq U$.*
2. *$\mathcal{U}$ is closed under forming supersets.*
3. *$\mathcal{U}$ is closed under forming inverses.*
4. *$\mathcal{U}$ is closed under forming finite intersections.*
5. *For every $U$ in $\mathcal{U}$ there exists $V$ in $\mathcal{U}$ with $V^2 \subseteq U$.*

**Example 11.2.** If $X$ is a set, then we have the discrete uniform structure $\mathcal{U}_{disc}$ which is characterized by $\mathtt{diag}(X) \in \mathcal{U}_{disc}$.

We also have the chaotic uniform structure $\mathcal{U}_{\text{chaot}} := \{X \times X\}$. □

**Example 11.3.** Let $(X, d)$ be a quasi metric space. Then we define the metric uniform structure

$$\mathcal{U}_d := \{U \in \mathcal{P}_{X\times X} \mid (\exists r \in (0,\infty) \mid U_r \subseteq U)\} .$$

**Lemma 11.4.** *$\mathcal{U}$ is a uniform structure.*

*Proof.* We check that axioms.

1. If $U$ is in $\mathcal{U}_d$, then $\mathtt{diag}(X) \subseteq X$ since there exists $r$ in $(0,\infty)$ such that $U_r \subseteq U$ and $\mathtt{diag}(X) \subseteq U_r$.

2. It is immediate from the definition that $\mathcal{U}_d$ is closed under forming supersets

3. $\mathcal{U}_d$ is closed under forming inverses since $U_r^{-1} = U_r$ for all $r$ in $(0,\infty)$.

4. $\mathcal{U}_d$ is closed under forming finite intersections. Indeed, if $(U_i)_{i\in I}$ is a finite family in $\mathcal{U}_d$, then we choose a family $(r_i)_{i\in I}$ in $(0,\infty)$ such that $U_{r_i} \subseteq U_i$ for all $i$ in $I$. Let $s := \min_{i\in I} r_i$. Then $s \in (0,\infty)$ and $U_s \subseteq \bigcap_{i\in I} U_i$. Hence $\bigcap_{i\in I} U_i \in \mathcal{U}_d$ as required.

5. Let $U$ be in $\mathcal{U}_d$ and choose $r$ in $(0,\infty)$ such that $U_r \subseteq U$. Set $s := r/2$ and $V := U_s$. Then $V^2 \subseteq U_r \subseteq U$ and $V \in \mathcal{U}_d$.

□

The topology $\mathcal{T}_{\mathcal{U}}$ is Hausdorff exactly if

$$\bigcap_{U\in\mathcal{U}} U = \mathtt{diag}(X) .$$

□

A uniform space is a pair $(X,\mathcal{U})$ of a set and a uniform structure.

Let $(X,\mathcal{U})$ be a uniform space.

**Definition 11.5.** *The underlying topology $\mathcal{T}_{\mathcal{U}}$ of $X$ is determined as follows: A subset $W$ of $X$ belongs to $\mathcal{T}_{\mathcal{U}}$ if for every $w$ in $W$ there exists $U$ in $\mathcal{U}$ such that $U[\{w\}] \subseteq W$.*

**Lemma 11.6.** *The topology associated to a uniform structure is well-defined.*

*Proof.* The set $\mathcal{T}_{\mathcal{U}}$ contains $\emptyset$ and $X$. It is closed under arbitrary unions. If $(W_i)$ is a finite family in $\mathcal{T}_{\mathcal{U}}$ and $w$ is in $W := \bigcap_{i\in I} W_i$, then we can choose a family $(U_i)_{i\in I}$ such that $U_i[\{w\}] \subseteq W_i$ for all $i$ in $I$. Then $U := \bigcap_{i\in I} U_i \in \mathcal{U}$ and $U[\{w\}] \subseteq W$. □

**Example 11.7.** On a compact Hausdorff space there exists exactly one uniform structure determining the topology. □

**Remark 11.8.** Quasi metric spaces are always Hausdorff. Furthermore, the uniform structure of a metric space has a countable cofinal set e.g. $(U_{n^{-1}})_{n\in\mathbb{N}}$.

General uniform structures may not have these properties.

Let $(X,\mathcal{U})$ and $(X',\mathcal{U}')$ be uniform spaces and $f : X \to X'$ be a map between the underlying sets.

**Definition 11.9.** *$f$ is uniformly continuous if $f^{-1}(\mathcal{U}') \subseteq \mathcal{U}$.*

We obtain a category **Uniform** of uniform spaces and uniformly continuous maps.

Let $f : (X,\mathcal{U}) \to (X',\mathcal{U}')$ be uniformly continuous.

**Lemma 11.10.** *$f$ is continuous as a map between the underlying topological spaces.*

*Proof.* Let $W'$ be open in $X'$. We consider $w$ in $f^{-1}(W')$. We choose $U'$ in $\mathcal{U}'$ such that $U'[\{f(w)\}] \subseteq W'$. Then $f^{-1}(U')[w] \subseteq f^{-1}(W')$ and $f^{-1}(U') \in \mathcal{U}$. □

**Example 11.11.** For maps between metric spaces the notion of uniform continuity reduces to the classical one. Let $(X, d)$ and $(X', d')$ be metric spaces. A map $f : X \to X'$ is uniformly continuous if and only if for every $\epsilon$ in $(0,\infty)$ there exists $\delta$ in $(0,\infty)$ such that $d(x, y) \le \delta$ implies $d'(f(x), f(y)) \le \epsilon$. □

**Lemma 11.12.** *We have adjunctions*

$$(-)_{disc} : \mathbf{Set} \leftrightarrows \mathbf{Uniform} : forget$$

*and*

$$forget : \mathbf{Uniform} \leftrightarrows: \mathbf{Set} : (-)_{\mathrm{chaot}} \ .$$

*Proof.* We observe the obvious equalities

$$\mathtt{Hom}_{\mathbf{Uniform}}(X_{disc}, Y) = \mathtt{Hom}_{\mathbf{Set}}(X, Y)$$

and

$$\mathtt{Hom}_{\mathbf{Set}}(Y, X) = \mathtt{Hom}_{\mathbf{Uniform}}(Y, X_{\mathrm{chaot}})$$

for all sets $X$ and all uniform spaces $Y$. □

Let $\mathcal{A}$ be a subset of $\mathcal{P}_{X\times X}$ which has the property that for every $A$ in $\mathcal{A}$ there exists $B$ in $\mathcal{A}$ such that $B^2 \subseteq A$. Then we can form a uniform structure $\mathcal{U}\langle\mathcal{A}\rangle$ which consists of supersets of finite intersections of elements of $\mathcal{A}$ and their inverses. Note that if $(A_i)_{i\in I}$ is a finite family in $\mathcal{A}$ and $(B_i)_{i\in I}$ is a family in $\mathcal{A}$ such that $B_i^2 \subseteq A_i$, then $(\bigcap_{i\in I} B_i)^2 \subseteq \bigcap_{i\in I} A_i$.

**Example 11.13.** Let $(X,\mathcal{U})$ be a uniform space and $f : Y \to X$ be a map of sets. Then $f^*\mathcal{U} := \langle f^{-1}\mathcal{U}\rangle$ is the uniform on $Y$ induced via $f$. Note that $f^{-1}\mathcal{U}$ has the required conditions since $f^{-1}(V) \circ f^{-1}(V) \subseteq f^{-1}(V \circ V)$. The map $f : Y \to X$ becomes a uniformly continuous.

**Proposition 11.14.** *The category* **Uniform** *is complete and cocomplete.*

*Proof.* In view of the adjunctions in Lemma 11.12 the underlying sets of potential limits or colimits are calculated in **Set**.

We describe (co)products and (co)equalizers.

Let $(X_i)_{i\in I}$ be a family in **Uniform**. Then we set $X := \prod_{i\in I} X_i$ and let $p_i : X \to X_i$ be the projection. We form the uniform structure $\mathcal{U}_X$ generated by the set $\bigcup_{i\in I} p_i^{-1}(\mathcal{U}_{X_i})$. The uniform space $(X,\mathcal{U}_X)$ together with $(p_i)_{i\in I}$ has the required properties of the product.

For the coproduct we consider $Y := \bigsqcup_{i\in I} X_i$ and the inclusions $e_i : X_i \to Y$. We define the uniform structure on $Y$ consistsing of the subsets $\bigsqcup_{i\in I} U_i$ for families $(U_i)_{i\in I}$ with $U_i \in \mathcal{U}_{X_i}$. Then $(Y,\mathcal{U}_Y)$ together with the family $(e_i)_{i\in I}$ has the required properties of a coproduct.

If

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is an equalizer diagram, then the subset

$$Z := \{x \in X \mid f(x) = g(x)\} \to X$$

with the induced uniform structure has the required universal property.

If we consider the diagram as a coequalizer diagram, then we equip the coequalizer $Y \to Q$ in **Set** with the uniform structure characterized as follows. For every uniform map $f : Y \to Z$ equalizing $f$ and $g$ we get a set map $\bar{f} : Q \to Z$. We then let $\mathcal{U}_Q$ be the uniform structure generated by the set $\bar{f}^{-1}(\mathcal{U}_Z)$ for all such maps. One checks that this has the required properties of the coequalizer. □

A homotopy between uniformly continuous maps $f, g : X \to Y$ is given by a map $h : [0,1] \times X \to Y$ such that $h_{|\{0\}\times X} = f$ and $h_{|\{1\}\times X} = g$. Note that this notion is more restrictive as being homotopic in the sense of continuous maps between the underlying topological spaces.

**Example 11.15.** The identity of $\mathbb{R}$ and the constant map with value zero are homotopic as continuous maps. But they are not homotopic as uniformly continuous maps.

In fact, if $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a map, then by uniform continuity there exists a $\delta$ in $(0,\infty)$ such that $|s-t| < \delta$ implies $|h(s,x) - h(t,x)| \le 1$ for all $x$ in $X$ and $s,t$ in $[0,1]$. But that $|h(0,x) - h(1,x)| \le \delta^{-1}$ for all $x$. This is incompatible with $h(0,x) = x$ and $h(1,x) = 0$ for all $x$ in $\mathbb{R}$ if $h$ would be a potential homotopy between the identity and the zero map. $\square$

We now combine coarse and uniform structures. Let $X$ be a set with a uniform structure $\mathcal{U}$ and a coarse structure $\mathcal{C}$.

**Definition 11.16.** *We say that $\mathcal{U}$ and $\mathcal{C}$ are compatible if $\mathcal{C} \cap \mathcal{U} \neq \emptyset$.*

**Example 11.17.** The metric uniform and coarse structures of a metric space are compatible. $\square$

**Definition 11.18.** *A uniform bornological coarse space is a quadrupel $(X, \mathcal{C}, \mathcal{B}, \mathcal{U})$ such that $(X, \mathcal{C}, \mathcal{B})$ is a bornological coarse space and $\mathcal{U}$ is a uniform structure which is compatible with $\mathcal{C}$.*

*A morphism of uniform bornological coarse spaces is a morphism of bornological coarse spaces which is in addition uniformly continuous.*

We let **UBC** denote category of uniform bornological coarse spaces and morphisms. We have a forgetful functors **UBC** $\to$ **BornCoarse** and **UBC** $\to$ **Top**.

**Example 11.19.** We consider $\mathbb{R}^n$ and its subsets as uniform bornological coarse spaces with the structures induced by the standard metric. $\square$

In **UBC** we have a symmetric monoidal structure. We define $X \otimes Y$ such that the underlying bornological coarse space is given by the tensor product of bornological coarse spaces, and the underlying uniform space is the cartesian product. One checks easily that the compatibility condition is preserved under taking products.

We want to consider homology theories on **UBC**. The appropriate notion is that of a local homology theory. In order to formulate the excision axiom we need the notion of a uniformly excisive descomposition.

Let $X$ be in **UBC**. Let $(A, B)$ be a decomposition of $X$ into two subsets. Recall that it is coarsely excisive if for every entourage $W$ in $\mathcal{C}_X$ there exists $U$ in $\mathcal{C}_X$ such that

$$W[A] \cap W[B] \subseteq U[A \cap B] \ .$$

Coarse homology theories have Mayer-Vietoris sequences for coarsely excisive decompositions. The notion of a uniformly excisive decomposition is similar.

**Definition 11.20.** *$(A, B)$ is uniformly excisive if there exists a uniform entourage $U$ and a function $\kappa : \mathcal{P}^{\subseteq U}_{X\times X} \to \mathcal{P}_{X\times X}$ which is*

1. *monotone: if $W, W' \in \mathcal{P}^{\subseteq U}_{X\times X}$ and $W \subseteq W'$, then $\kappa(W) \subseteq \kappa(W')$.*

2. *cofinal: for every $V$ in $\mathcal{U}_X$ here exists $W$ in $\mathcal{P}^{\subseteq U}_{X\times X}$ such that for all $W' \in \mathcal{P}^{\subseteq W}_{X\times X}$ we have $\kappa(W') \subseteq U$.*

3. *excisive: for every $W$ in $\mathcal{P}^{\subseteq U}_{X\times X}$ we have $W[A] \cap W[B] \subseteq \kappa(W)[A \cap B]$.*

**Remark 11.21.** Note that in contrast to the notion of being coarsely excisive the notion of being uniformly excisive involves the choice of a function $\kappa$. This seems to be stronger than the condition that for every $W$ in $\mathcal{P}^{\subseteq U}_{X\times X}$ there exists $W'$ such that $W[A] \cap W[B] \subseteq \kappa(W)[A \cap B]$. The point is that want that $W'$ becomes small if $W$ is small. □

**Example 11.22.** Assume that $(X, d)$ is a path quasi-metric space.

**Lemma 11.23.** *Then every closed decomposition is uniformly excisive.*

*Proof.* We set $U = U_1$. For every $W$ in $\mathcal{P}^{\subseteq U}_{X\times X}$ we define $k(W) := \inf\{r \in [0, 1] \mid W \subseteq U_r\}$ and $\kappa(W) := U_{k(W)}$. This function is monotonous by construction. Furthermore, since $\kappa(U_r) = U_r$ it is cofinal. Assume that $x$ is a point in $W[A] \cap W[B]$. Then there exist paths $\gamma$ and $\sigma$ of length $\le k(W)$ from some $a$ in $A$ to $x$, and from $x$ to some $b$ in $B$, respectively. The concatenation $\sigma\sharp\gamma$ is a path from $a$ to $b$ of length $\le 2k(W)$. Then there exists a point on this path in $A \cap B$ at distance at most $k(W)$ from $x$. Hence $x \in \kappa(W)[A \cap B]$. □

We now let $\mathbf{UBC}^2$ be the categories of pairs $(X, A)$ in $\mathbf{UBC}$ where $A$ is a closed subset.

**Definition 11.24.** *A $\delta$-functor on $\mathbf{UBC}$ is a pair $(E, \delta)$ of a functor $E : \mathbf{UBC}^2 \to \mathbf{Ab}^{\mathbb{Z}}$ and a natural transformation $\delta : E(X, A) \to E(A)[-1]$ such that*

$$E(A) \to E(X) \to E(X, A) \stackrel{\delta}{\to} E(A)[-1]$$

*is exact for every pair $(X, A)$ in $\mathbf{UBC}^2$.*

Here as usual we write $E(X) := E(X, \emptyset)$.

In the following we consider a $\delta$-functor $(E, \delta)$ on $\mathbf{UBC}^2$.

**Definition 11.25.** *We say that $E$ is homotopy invariant if the projection $[0, 1] \times X \to X$ induces an isomorphism $E([0, 1] \otimes X) \to E(X)$.*

Let $X$ be in $\mathbf{UBC}$.

**Definition 11.26.** *$X$ is called flasque if it admits a selfmap $f : X \to X$ such that $f$ implements flasqueness of the underlying bornological coarse spaces and $f$ is homotopic to the identity.*

**Definition 11.27.** *We say that $E$ vanishes on flasques $E(X) \cong 0$ for all flasque $X$ in* **UBC**.

Let $X$ be in **UBC**. Then there exists $V$ in $\mathcal{C}_X \cap \mathcal{U}_X$. If $U$ is in $\mathcal{C}_X$ such that $V \subseteq U$, then $\mathcal{C}\langle U\rangle$ is compatible with $\mathcal{U}$. We write $X_U$ for the objects of **UBC** obtained from $X$ by replacing the coarse structure by $\mathcal{C}\langle U\rangle$. Therefore the following definition makes sense since we can form the colimit over all sufficiently large coarse entourages.

**Definition 11.28.** *We say that $E$ is u-continuous if for all $X$ in* **UBC** *we have*

$$\operatorname{colim}_{U \in \mathcal{C}_X} E(X_U) \cong E(X) .$$

**Definition 11.29.** *We say that $E$ satisfies closed excision if for every $X$ in* **UBC** *and closed coarsely and uniformly excisive decomposition $(A, B)$ we have an isomorphism*

$$E(B, B \cap A) \stackrel{\cong}{\to} E(X, A)$$

*induced by the map $(B, B \cap A) \to (X, A)$ in* $\mathbf{UBC}^2$.

**Definition 11.30** ([BEa])**.** *A local homology theory is a $\delta$-functor $(E, \delta)$ on* **UBC** *which is:*

1. *homotopy invariant*
2. *u-continuous*
3. *vanishing on flasques*
4. *excisive.*

**Remark 11.31.** The name *local* comes from the stronger condition of being locally finite which we actually wanted to axiomatize. The condition of vanishing on flasques is weaker, but much easier to handle.

In [BEb] we axiomatize the notion of a locally finite homology theory on the category **TopBorn** of topological bornological spaces. There is a forgetful functor

$$\mathcal{F}_{\mathcal{U}/2,\mathcal{C}} : \mathbf{UBC} \to \mathbf{TopBorn} .$$

The pull-back of a locally finite homology theory on **TopBorn** (with closed excision) yields a local homology theory in the sense above. □

**Example 11.32.** We have a forgetful functor $\mathcal{F}_{\mathcal{U}} : \mathbf{UBC}^2 \to \mathbf{BornCoarse}^2$. It sends $(X, A)$ to $(X, \{A\})$. Note that in the case of **BornCoarse** the second entry always was a big family.

Let $(E, \delta)$ be a coarse homology theory. Then we can consider the functor

$$E\mathcal{F} := E \circ \mathcal{F}_{\mathcal{U}} : \mathbf{UBC}^2 \to \mathbf{Ab}^{\mathbb{Z}} .$$

**Lemma 11.33.** *$(E\mathcal{F}, \delta)$ is a local homology theory.*

*Proof.* We check the axioms.

1. Homotopy invariance is implied by coarse invariance.
2. $u$-continuity is the same condition.
3. Vanishing on flasques is implied by the corresponding condition for $E$. When $f : X \to X$ is homotopic to the identity, then it is close to the identity.
4. Excision follows from the excisiveness of $E$ since we require that $(A, B)$ is coarsely excisive. We further use that $E(A) \cong E(\{A\})$.

□

**Example 11.34.** Here are some basic calculations for any local homology theory $E$.

1. We have $E([0, \infty) \otimes X) \cong 0$. Indeed $[0, \infty) \otimes X$ is flasque with flasqueness implemented by $(t, x) \mapsto (t + 1, x)$.
2. We have $E(\mathbb{R}^n \otimes X) \cong E(X)[-n]$. We argue by induction. We use excision for the decomposition $((-\infty, 0] \times X, [0, \infty) \times X)$ of $\mathbb{R} \otimes X$ and Case 1 in order to show that $E(\mathbb{R} \otimes X) \cong E(X)[-1]$.

We now construct the cone functor

$$\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{BornCoarse} .$$

Let $X$ be a set. The underlying bornological space of $\mathcal{O}^\infty(X)$ will be the one of $\mathbb{R} \otimes X$. The coarse structure is the so-call hybrid structure introduced by N. Wright, see [BEb] for references.

**Definition 11.35.** *A scale for $X$ is a pair $(\kappa, \phi)$, where*

1. *$\kappa : \mathbb{R} \to [0, 1]$ is monotonously decreasing and satisfies $\mathtt{lim}_{t \to \infty} \kappa(t) = 0$.*

2. *$\phi : \mathbb{R} \to \mathcal{P}_{X\times X}$ is monotonously decreasing such that for every $U$ in $\mathcal{U}$ there exists $t$ in $\mathbb{R}$ such that $\phi(t) \subseteq U$.*

**Remark 11.36.** If the uniform structure $\mathcal{U}$ does not admit a cofinal sequence, and if $(\kappa, \phi)$ is a scale, then $\phi(t) = \mathtt{diag}(X)$ for large $X$.

Metric uniform structure spaces always admit countable cofinal sets. □

For a scale $(\kappa, \phi)$ we define the entourage

$$U_{(\kappa,\phi)} := \{((s,x),(t,y)) \in (\mathbb{R}\times X)^2 \mid |s-t| \le \kappa(s\vee t) \ \& \ (x,y) \in \phi(s \vee t)\}$$

The coarse structure of $\mathcal{O}^\infty(X)$ is generated by the entourages $U \cap U_{(\kappa,\phi)}$ for all $U$ in $\mathcal{C}_{\mathbb{R}\times X}$ and scales $(\kappa, \phi)$.

This finitshes the construction of the cone.

**Lemma 11.37.** *If $f : X \to X'$ is a morphism in* **UBC***, then*

$$\mathcal{O}^\infty(f) : \mathcal{O}^\infty(X) \to \mathcal{O}^\infty(X') \ , \quad (t,x) \mapsto (t, f(x))$$

*is a morphism in* **BornCoarse**.

*Proof.* The only non-trivial point is show that for every scale $(\kappa, \phi)$ of $X$ there is a scale $(\kappa', \phi')$ such that $f(U \cap U(\kappa, \phi)) \subseteq U_{(\kappa',\phi')}$. It suffices to set $\kappa' = \kappa$ and $\phi'(t) := f(\phi(t'))$. This function is decreasing. If $W'$ is in $\mathcal{U}'$, then by the uniform continuity of $f$ we have $f^{-1}(W') \in \mathcal{U}$. Hence there exists $t$ in $\mathbb{R}$ such that $\phi(t) \subseteq f^{-1}(W')$. Then $f(\phi(t)) \subseteq W'$. □

We let $\mathcal{O}(X)$ denote the subspace $[0,\infty) \times X$ of $\mathcal{O}^\infty(X)$ with the induced structures.

**Definition 11.38.** *We define the cone-at-$\infty$ functor*

$$\mathcal{O}^\infty(-) : \mathbf{UBC} \to \mathbf{BornCoarse} \ .$$

*We further define the functor*

$$\mathcal{O} : \mathbf{UBC} \to \mathbf{BornCoarse} \ .$$

*The cone sequence is the sequence of natural transformations*

$$\mathcal{F}_{\mathcal{U}}(-) \to \mathcal{O}(-) \to \mathcal{O}^\infty(-) \stackrel{\partial^{\mathrm{cone}}}{\to} \mathbb{R} \otimes \mathcal{F}_{\mathcal{U}}(-)$$

*of functors* **UBC** $\to$ **BornCoarse***, where $\partial^{\mathrm{cone}}$ is the identity on underlying sets. It is called the cone boundary.*

Let $E$ be a coarse homology theory. We define the functors

$$E\mathcal{O} := E \circ \mathcal{O} \ , E\mathcal{O}^\infty := E \circ \mathcal{O}^\infty : \mathbf{UBC} : \mathbf{UBC} \to \mathbf{Ab}^{\mathbb{Z}} \ .$$

Let $X$ $X$ be in **UBC** and note that $E\mathcal{F}(\mathbb{R} \otimes X) \cong E\mathcal{F}(X)[-1]$. The isomorphism is given by the Mayer-Vietoris boundary for the decomposition $((-\infty, 0] \otimes X, [0, \infty) \otimes X)$ of $\mathbb{R} \otimes X$. This identification will be used implicitly in the last term of the sequence in the following statement.

**Proposition 11.39.** *The cone sequence induces a long exact sequence*

$$E\mathcal{F}(X) \to E\mathcal{O}(X) \to E\mathcal{O}^\infty(X) \stackrel{\partial^{\text{cone}}}{\to} E\mathcal{F}(X)[-1] \ .$$

*Proof.* We consider the coarsely excisive decomposition $((-\infty, 0] \times X, [0, \infty) \times X)$ of $\mathcal{O}^\infty(X)$. The subspace $(-\infty, 0] \times X$ with the induced structures is flasque with flasqueness implemented by $(t, x) \mapsto (t-1, x)$. Furthermore $\{0\} \times X \cong X$. The Mayer-Vietoris sequence gives

$$E\mathcal{F}(X) \to E\mathcal{O}(X) \to E\mathcal{O}^\infty(X) \stackrel{\partial^{MV}}{\to} E\mathcal{F}(X)[-1] \ .$$

From the naturality of Mayer-Vietoris sequences we have a commutative diagram

$$\begin{array}{ccc} E\mathcal{O}^\infty(X) & \xrightarrow{\partial^{MV}} & E\mathcal{F}(X)[-1] \\ \downarrow{\scriptstyle \partial^{\text{cone}}} & & \| \\ E\mathcal{F}(\mathbb{R} \otimes X) & \xrightarrow{\partial^{MV}} & E\mathcal{F}(X)[-1] \end{array}$$

which shows that the boundary operator is the correct one. □

Note that $(-\infty, 0] \times X$ is a flasque subset of $\mathcal{O}^\infty$. Using the exact sequence of the pair $(\mathcal{O}^\infty(X), ((-\infty, n] \times X)_{n \in \mathbb{N}})$ and excision we get an isomorphism

$$E\mathcal{O}^\infty(X) \stackrel{\cong}{\to} E(\mathcal{O}^\infty(X), ((-\infty, n] \times X)_{n \in \mathbb{N}}) \stackrel{\cong}{\leftarrow} E(\mathcal{O}(X), ([0, n] \times X)_{n \in \mathbb{N}}) \ .$$

The cone sequence is isomorphic to the pair sequence for $E$ and $(\mathcal{O}(X), ([0, n] \times X)_{n \in \mathbb{N}})$.

The proof of the following theorem is quite technical. This in particular applies to the homotopy invariance. We therefore refrain from giving a proof and use it as a black box.

Let $E$ be a coarse homology theory.

**Theorem 11.40** ([BEKW, Sec. 9.4 & 9.5])**.** *The functors $E\mathcal{O}$ and $E\mathcal{O}^\infty$ are homotopy invariant and excisive.*

In order to ensure that $E\mathcal{O}$ and $E\mathcal{O}^\infty$ vanish on flasques we need an additional assumption on $E$.

**Remark 11.41.** If $f : X \to X$ implements flasqueness of $X$, then $\mathcal{O}^\infty(f) : \mathcal{O}^\infty(X) \to \mathcal{O}^\infty(X)$ does not implement flasqueness of $\mathcal{O}^\infty(X)$. The problem is that $f$ is in general not close to the identity. So we can not simply conclude that $E\mathcal{O}^\infty(X) \cong 0$.

**Definition 11.42.** *We say that $X$ in* **BornCoarse** *is weakly flasque if it admits a selfmap $f : X \to X$ such that*

1. *For every coarse homology theory $E$ we have $E(f) = E(\mathtt{id}_X)$.*
2. *For every $U$ in $\mathcal{C}_X$ we have $\bigcup_{n\in\mathbb{N}} f^n(U) \in \mathcal{C}_X$.*
3. *For every $B$ in $\mathcal{B}_X$ there exists $n$ in $\mathbb{N}$ such that $f^n(X) \cap B = \emptyset$.*

So we changed the first condition. Surprisingly we can check this condition for cones.

**Proposition 11.43.** *If $X$ in* **UBC** *is flasque, then $\mathcal{O}(X)$ and $\mathcal{O}^\infty(X)$ are weakly flasque.*

*Proof.* Assume that $f$ implements flasqueness of $X$. Let $E$ be any coarse homology theory. Since $f$ is homotopic to the identity $E\mathcal{O}(f) = E\mathcal{O}(\mathtt{id}_X)$ and $E\mathcal{O}^\infty(f) = E\mathcal{O}^\infty(\mathtt{id}_X)$. □

**Definition 11.44.** *A coarse homology theory is called strong if it annihilates weakly flasques.*

**Corollary 11.45.** *If $E$ is a stong coarse homology theory, then $E\mathcal{O}$ and $E\mathcal{O}^\infty$ vanish on flasques.*

**Lemma 11.46.** *The coarse chomology theories $H\mathcal{X}$ and $K\mathbf{A}\mathcal{X}$ are strong.*

*Proof.* Let $E$ be one of these. In the verification of the fact that $E$ vanishes on flasques (see Lemma 5.8 and Lemma 9.4) we have constructed an endomorphism $S : E(X) \to E(X)$ such that $\mathtt{id}_{E(X)} + E(f) \circ S = S$. Since $E$ is strong we know that $E(f) = \mathtt{id}_{E(X)}$. The resulting equation $\mathtt{id}_{E(X)} + S = S$ implies $\mathtt{id}_{E(X)} = 0$. □

In order to discuss $u$-continuity of $E\mathcal{O}^\infty$ and $E\mathcal{O}$ we introduce the notion of a coarsening.

**Definition 11.47.** *A map $X \to X'$ in* **UBC** *is called a coarsening if it is an isomorphism of the underlying uniform and bornological spaces.*

**Proposition 11.48.** *If $X \to X'$ is a coarsening, then we have an isomorphism $E\mathcal{O}^\infty(X) \xrightarrow{\cong} E\mathcal{O}^\infty(X')$*

*Proof.* We let $\mathcal{X}'_n$ be the big family in $[n, \infty) \times X'$ generated by $\{n\} \times X'$. We start with the isomorphism

$$E\mathcal{O}^\infty(X') \cong E(\mathcal{O}(X'), \mathcal{X}'_0) .$$

We use $u$-continuity in order to get

$$E(\mathcal{O}(X'),\mathcal{X}'_0)\cong \mathop{\mathtt{colim}}_{U\in\mathcal{C}_{\mathcal{O}(X')}} E([0,\infty)\otimes X')_U,\mathcal{X}'_0)\ .$$

Then we can write by excision

$$\mathop{\mathtt{colim}}_{U\in\mathcal{C}_{\mathcal{O}(X')}} E([0,\infty)\otimes X)_U,\mathcal{X}'_0)\cong \mathop{\mathtt{colim}}_{m}\mathop{\mathtt{colim}}_{U\in\mathcal{C}_{\mathcal{O}(X')}} E([m,\infty)\otimes X)_U,\mathcal{X}'_m)\ \ .$$

We now interchange the order of colimis:

$$\mathop{\mathtt{colim}}_{m}\mathop{\mathtt{colim}}_{U\in\mathcal{C}_{\mathcal{O}(X')}} E([m,\infty)\otimes X)_U,\mathcal{X}'_m)\cong \mathop{\mathtt{colim}}_{U\in\mathcal{C}_{\mathcal{O}(X')}}\mathop{\mathtt{colim}}_{m} E([m,\infty)\otimes X)_U,\mathcal{X}'_m)\ .$$

Note that the generating entourages of $\mathcal{C}_{\mathcal{O}(X')}$ are of the form $V'\cap V_{(\kappa,\phi)}$ with $V'$ in $\mathcal{C}_{\mathbb{R}\otimes X'}$. There exists $W$ in $\mathcal{U}_X\cap\mathcal{C}_X$. There exists $m$ in $\mathbb{N}$ such that $\phi(m)\subseteq W$. But then $(V'\cap V_{(\kappa,\phi)})_{|[m,\infty)\otimes X}$ is the restriction of $\phi(m)\cap V'\cap V_{(\kappa,\phi)}$ which belongs to $\mathcal{C}_{\mathcal{O}(X)}$. Hence we can replace the index poset $U\in\mathcal{C}_{\mathcal{O}(X')}$ of the colimit by the subset $U\in\mathcal{C}_{\mathcal{O}(X)}$. Doing all steps backwards we get the desired isomorphism. □

**Corollary 11.49.** *The functors $E\mathcal{O}$ and $E\mathcal{O}^\infty$ are u-continuous.*

*Proof.* We use that a filtered colimit of exact sequences is exact. We consider the map of exact sequence sequence

$$\begin{array}{ccccccccc}
\mathtt{colim}_{U\in\mathcal{X}}\,E\mathcal{O}^\infty(X_U)[1] & \longrightarrow & \mathtt{colim}_{U\in\mathcal{X}}\,E\mathcal{F}(X_U) & \longrightarrow & \mathtt{colim}_{U\in\mathcal{X}}\,E\mathcal{O}(X_U) & \longrightarrow & \mathtt{colim}_{U\in\mathcal{X}}\,E\mathcal{O}^\infty(X_U)[-1] & \longrightarrow & \mathtt{colim}_{U\in\mathcal{X}}\,E\mathcal{F}(X_U)[-1]\\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong\\
E\mathcal{O}^\infty(X)[1] & \longrightarrow & E\mathcal{F}(X) & \longrightarrow & E\mathcal{O}(X) & \longrightarrow & E\mathcal{O}^\infty(X) & \longrightarrow & E\mathcal{F}(X)[-1]
\end{array}$$

The vertical arrows for $E\mathcal{F}$ is an isomorphism by Lemma 11.33. The vertical morphisms at $E\mathcal{O}^\infty$ are isomorphisms since $X_U\to X$ is a coarsening provided $U$ is large enough. We now conclude by the Five Lemma that the vertical arrow for $E\mathcal{O}$ is also an isomorphism. □

We now have finished the proof of the following theorem.

**Theorem 11.50.** *If $E$ is a strong coarse homology theory, then $E\mathcal{O}$ and $E\mathcal{O}^\infty$ are local homology theories.*

Let $X$ be in **UBC**. We let $X_{\mathcal{C}_{min}}$ be the space obtained from $X$ by replacing the coarse structure by the minimal one. It is only compatible with the uniform structure if the latter is discrete.

**Corollary 11.51.** *If $X$ is discrete as a uniform space, then we have an isomorphism*

$$E\mathcal{O}^\infty(X)\cong E\mathcal{F}(X_{\mathcal{C}_{min}})[-1]\ .$$

*Proof.* If $X$ is discrete as a uniform space, then the discrete coarse structure on $X$ is compatible with the uniform structure. Hence $X_{\mathcal{C}_{min}} \to X$ is a coarsening. Since furthermore $\mathcal{O}(X_{\mathcal{C}_{min}})$ is flasque we have from the cone sequence that

$$\partial^{\mathrm{cone}} : E\mathcal{O}^{\infty}(X_{min}) \stackrel{\cong}{\to} E\mathcal{F}(X_{\mathcal{C}_{min}})[-1] .$$

□

Our main examples of objects in **UBC** are given by simplicial complexes with their spherical path metric.

Let $X$ be a set. By $\mathcal{P}_X^{\mathrm{fin}}$ we denote the set of finite subsets of $X$.

**Definition 11.52.** *An abstract simplicial complex is pair $(X,\mathcal{S})$ of a set $X$ and a subset $\mathcal{S}$ of $\mathcal{P}_X^{\mathrm{fin}}$ which is closed under taking subsets and contains all singletons.*

*A map between abstract simplicial complexes $f : (X,\mathcal{S}) \to (X',\mathcal{S}')$ is a map of sets $f : X \to X'$ such that $f(\mathcal{S}) \subseteq \mathcal{S}'$.*

We get a category $^{\mathbf{a}}$**Simpl** of abstract simplicial complexes and maps.

Let $(X,\mathcal{S}_X)$ be in $^{\mathbf{a}}$**Simpl**. Usually we only write $X$. The set $\mathcal{S}$ is the set of simplices of $X$. We let $X_n := \{B \in \mathcal{S}_X \mid |B| = n+1\}$ be the set of $n$-simplices. Note that $X = X_0$.

**Definition 11.53.** *We define the dimension of $X$ by*

$$\dim(X) := \max\{|B| \mid B \in \mathcal{S}\} - 1 .$$

**Example 11.54.** If $X$ is finite, then we can take $\mathcal{S} = \mathcal{P}_X$.

In general, we can take $\mathcal{S}$ to be the set of all finite subsets. □

We let $|X|$ be the set of probability measures $\mu$ on $X$ such that $\mathtt{supp}(\mu) \in S$. Any such measure is given by

$$\sum_{x\in X} \mu(\{x\})\delta_x ,$$

where $\delta_x$ is the Dirac measure at $x$.

**Example 11.55.** Any point $x$ of $X$ gives rise to a point $\delta_x$ in $|X|$. In this way we get a canonical map $\delta : X \to |X|$.

If $B$ is in $S$, then $\mu_B := \frac{1}{|B|}\sum_{b\in B}\delta_b$ in $|X|$ is the center of mass (barycenter) of $B$. □

We want to equip the set $|X|$ a metric which will then induce a topology. The metric we are going to construct is called the spherical path metric.

We first recall the definition of the length of a path $\gamma : [0,1] \to Y$ in a general metric space $(Y,d)$. It is defined by

$$\ell(\gamma) := \sup_{n\in\mathbb{N}, 0=t_0\le t_1\le\dots\le t_n=1} \sum_{i=1}^{n} d(\gamma(t_i),\gamma(t_{i-1})) \ .$$

Note that the length can be infinite.

Let $X$ be in ${}^{\mathbf{a}}\mathbf{Simpl}$. Consider a finite subset $F$ of $X$. We consider $F$ as the basis of $\mathbb{R}^F$. We equip $\mathbb{R}^F$ with the euclidean metric such that this basis is orthonormal.

We let $|F|$ be the subset of $|X|$ of measures supported on $F$. Then we define a map

$$|F| \to S(\mathbb{R}^F) \ , \quad \mu \mapsto \sum_{f\in F} \sqrt{\mu(\{f\})} f \ .$$

Here $S(\mathbb{R}^F)$ denotes the unit sphere.

Let $\gamma : [0,1] \to |X|$ be a map. We say that $\gamma$ is continuous if there exists a finite subset of $X$ such that $\gamma([0,1]) \subseteq |F|$ and the composition

$$[0,1] \xrightarrow{\gamma} |F| \to S(\mathbb{R}^F)$$

is continuous. Continuity does not depend on the choice of $F$.

The length $\ell(\gamma)$ of $\gamma$ is the length of this composition. This length does not depend on the choice of the finite subset $F$.

**Definition 11.56.** *We define the spherical path quasi-metric on $|X|$ such that*

$$d(\mu,\mu') = \inf_{\gamma,\gamma(0)=\mu,\gamma(1)=\mu'} \ell(\gamma) \ .$$

It is clear $d$ is symmetric. Since we concatenate path's $\gamma,\gamma'$ with $\gamma(0)=\gamma'(1)$ (written as $\gamma\sharp\gamma'$ and $\ell(\gamma\sharp\gamma') = \ell(\gamma)+\ell(\gamma')$ we have the triangle identity. Since $d(\mu,\mu')$ is bounded from below by the distance of the images of $\mu$ and $\mu'$ in $S(\mathbb{R}^F)$ we conclude that $d(\mu,\mu')=0$ if and only if $\mu=\mu'$.

Note that the diameter of a simplex in this metric is $\pi/2$ independent of the dimension. We now define the uniform bornological coarse structure on $|X|$ as the structure induced by $d$. Later we will also consider different bornologies.

If $f : X \to X'$ is a map of abstract simplicial complexes, then $|f| : |X| \to |X'|$ is given by the push-forward of measures. We have

$$f_*(\mu) = \sum_{x'\in X'} \sum_{x\in f^{-1}(x')} \mu(\{x\})\delta_{x'} \ .$$

One can check that

$$d(f_*\mu, f_*\mu') \leq d(\mu, \mu') ,$$

i.e., that this map is 1-Lipschitz. In general it is not proper.

**Example 11.57.** Let $X$ be finite and $\mathcal{S} = \mathcal{P}_X$. Then we have $|X| \cong \Delta^{|X|-1}$ with the metric induced by its identification with the positive quadrant of $S(\mathbb{R}^X)$. We have $\dim(X) = |X| - 1$

□

**Example 11.58.** Let $X := \mathbb{Z}$ and $\mathcal{S} := \mathbb{Z} \cup \{(n, n+1) \mid n \in \mathbb{Z}\}$. Then $|X| \cong \mathbb{R}_{\pi/2}$ as metric spaces such that the unit interval has length $\pi/2$. We have $\dim(X) = 1$. □

In the following we explain how one can calculate the local homology of $|X|$ for $X$ in $^{\mathbf{a}}$**Simpl**. In order to avoid spectral sequences we stick to the case where $E(*)$ is supported in a single degree. So the basic example is $E(-) = H\mathcal{X}\mathcal{O}^\infty(-)[1]$. We shift by one in order to get the usual formulas without shift later.

We now calculate $H\mathcal{X}\mathcal{O}^\infty(|X|)[1]$ using the cellular complex. We start with finite simplicial complexes. We use that $H\mathcal{X}\mathcal{O}^\infty$ is homotopy invariant and excisive for cell attachements and satisfies $H\mathcal{X}\mathcal{O}^\infty[1](*) \cong \mathbb{Z}$. This suffices to construct the cellular chain complex in the standard way

$$C(X): \quad \dots \to C_n(X) \to C_{n-1}(X) \to \dots \to C_0(X)$$

where $C_n := \mathbb{Z}[X_n]$. In this case, since $X_n$ is finite, we have isomorphisms

$$\mathbb{Z}[X_n] \cong \bigoplus_{X_n} \mathbb{Z} \cong \prod_{X_n} \mathbb{Z} .$$

We have

$$H\mathcal{X}\mathcal{O}^\infty(|X|)[1] \cong H(C(X)) .$$

This allows to import all calculations of the homology of finite CW-complexes from topology.

For us the interesting case are infinite complexes. The construction of the cellular chain complex above used the Mayer-Vietoris sequence for cell attachements. If we attach infnitely many cells in a dimension, then in usual algebraic topology we use the wedge axiom saying that $H_*(T_{disc}) = \bigoplus_T \mathbb{Z}$.

Note that $H\mathcal{X}\mathcal{O}^\infty$ does not satisfy the wedge axioms. In order to control what happes if we attach ininitely many cells in a dimension we assume that any bounded subset meets only finitely many cells.

**Definition 11.59.** *We call the simplicial set $X$ proper, if every bounded subset of $|X|$ only contains finitely may points of $X$.*

**Remark 11.60.** Properness is equivalent to the requirement that for every $x$ in $X$ the set $\{y \in X \mid \{x, y\} \in X_1\}$ is finite. It implies that $X$ is locally finite-dimensional.

Furthermore it implies that $|X|$ is a proper metric space, i.e., that balls are compact.

We can consider the disjoint union of $X = \bigsqcup_{n\in\mathbb{N}}([n], \mathcal{P}_{[n]})$. Then is is proper, but not globally finite-dimensional.

If $X$ is proper, then the set of barycenters of the $n$-simplices has the minimal induced bornology. Consequently we get the group of $n$-chains

$$\hat{C}_n(X) := \prod_{X_n} \mathbb{Z} \ .$$

Since the (geometric) boundary of every simplex meets only finitely many other simplices the boundary map of the cellular chain complex is still well-defined. We get a chain complex

$$\hat{C}(X): \quad \dots \to \hat{C}_n(X) \to \hat{C}_{n-1}(X) \to \dots \to \hat{C}_0(X)$$

The standard argument from algebraic topology yields:

**Proposition 11.61.** *If $X$ is a finite-dimension simplicial complex, then*

$$H\mathcal{X}\mathcal{O}^\infty(|X|)[1] = H_*(\hat{C}(X)) \ .$$

If $X$ is finite, then of course $\hat{C}(X) = C(X)$.

**Example 11.62.** We consider the simplicial complex from Example 11.58. It is proper.

We have $\hat{C}_0(X) = \prod_{\mathbb{Z}} \mathbb{Z}$ and $\hat{C}_1(X) = \prod_{\mathbb{Z}} \mathbb{Z}$, where the component with index $n$ corresponds to the interval $[n, n+1]$. The boundary map is given by

$$\prod_{\mathbb{Z}} \mathbb{Z} \to \prod_{\mathbb{Z}} \mathbb{Z} \ , \quad (a_n) \mapsto (a_{n-1} - a_n)_n \ .$$

We calculate the homology:

$$H_i(\hat{C}(X)) \cong \begin{cases} \mathbb{Z} & i = 1 \\ 0 & else \end{cases}$$

Indeed we have $H_1(\hat{C}(X)) \cong \mathbb{Z}$ realized as constant sequences. In order to see that $H_0(\hat{C}(X)) \cong 0$ we note that given a sequence $(b_n)_n$ we can solve the recursion $a_{n-1} - a_n = b_n$ for $a_n$ inductively in both directions starting with $a_0 := 0$.

Of course, we expected this result since $H\mathcal{X}\mathcal{O}^\infty(\mathbb{R}) \cong H\mathcal{X}\mathcal{O}^\infty(*)[-1] \cong \mathbb{Z}[-2]$ and hence $H\mathcal{X}\mathcal{O}^\infty(\mathbb{R})[1] = \mathbb{Z}[-1]$.

Note that in contrast

$$H_i(C(X)) \cong \left\{ \begin{array}{cc} \mathbb{Z} & i=0 \\ 0 & else \end{array} \right. .$$

We modify the example and consider the complex $X = \mathbb{N}$ with $S = \mathbb{N} \cup \{(n, n+1) \,|\, n \in \mathbb{N}\}$. Then $|X| \cong [0, \infty)$. The chain complex is given by

$$\prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \ , \quad (a_n) \mapsto \left( \begin{array}{cc} a_{n-1} - a_n & n \geq 1 \\ -a_0 & n = 0 \end{array} \right)_n .$$

We get $H_*(\hat{C}(X)) \cong 0$. The argument for $H_0(\hat{C}(X)) \cong 0$ is the same as above. If $(a_n)$ in $\hat{C}_1(X)$ is a cycle, then $a_0 = 0$ and then inductively $a_n = 0$ for all $n \in \mathbb{N}$.

We again expected this since $H\mathcal{X}\mathcal{O}^\infty([0, \infty)) \cong 0$. □

**Remark 11.63.** If $E$ is a general additive strong coarse homology theory, then we have an Atiyah-Hirzebruch spectral sequence with first term

$$E^1_{p,q} \cong \prod_{X_p} E_{q-1}(*) \ .$$

and boundary map

$$E^1_{p,q} \to E^1_{p-1,q}$$

calculating an associated graded of $E\mathcal{O}^\infty(|X|)$. The difference to the usual spectral sequence is again the appearance of the product.

Note that we do not claim that $H(\hat{C}(X))$ has anything to do with $H\mathcal{X}\mathcal{O}^\infty(|X|)$ if $X$ is not finite-dimensional. □

We now construct abstract simplicial complexes from coarse spaces. Let $X$ be a set and $U$ be an entourage such that $\mathtt{diag}(X) \subseteq U$. Then we can define a simplicial complex $\mathcal{X}_U := (X, \mathcal{S}_U)$, where $\mathcal{S}_U$ is the set of all $U$-bounded finite subsets. We let

$$P_U(X) := |\mathcal{X}_U|$$

denote its realization. Note that we have a canonical map of sets $i : X \to P_U(X)$ which sends $x$ in $X$ to $\delta_x$.

**Lemma 11.64.** *The canonical inclusion $X_U \to P_U(X)$ is a coarse equivalence.*

*Proof.* If $(x, y)$ is in $U$, then $d(i(x), i(y)) = \pi/2$. This shows that $i(U) \subseteq U_{\pi/2}$.

Since the diameters of the simplices are bounded by $\pi/2$ we furthermore have $U_{\pi/2}[i(X)] = P_U(X)$. On the other hand, if $d(i(x), i(y)) \leq k\pi/2$, then $(x, y) \in U^{k}$[1]. Hence $i$ is a coarse equivalence.

□

[1] Exercise!

We now assume that $X$ is in **BornCoarse**. Then we equip $X$ with the bornology such that $X_U \to P_U(X)$ is an equivalence. This bornology is generated by the subsets $B$ considered as subsets of $P_U(X)$ for all bounded subsets of $X$. In this way we get an object $P_U(X)$ in **UBC** together with a coarse equivalence map

$$X \to \mathcal{F}_{\mathcal{U}}(P_U(X)) .$$

If $f : X \to X'$ is a map such that $f(U) \subseteq U'$, then we get a map $\mathcal{X}_U \to \mathcal{X}'_{U'}$ by $x \mapsto f(x)$. It induces a map $P_U(f) : P_U(X) \to P_{U'}(X')$.

If $X$ is in **BornCoarse**, then we get the ind-system $P(X) := (P_U(X))_{U\in\mathcal{C}_X}$ in **UBC**. A morphism $f : X \to X'$ induces a map of ind-objects $P(X) \to P(X')$ given by $P_U(X) \to P_{f(U)}(X')$ for every $U$ in $\mathcal{C}_X$. Similarly, if $\mathcal{Y}$ is a big family in $X$, then we can form the system $P(\mathcal{Y}) := (P_U(Y))_{U\in\mathcal{C}_X, Y\in\mathcal{Y}}$. We get the familiy of pairs

$$(P(X), P(\mathcal{Y})) := (P_U(X), P_U(Y))_{U\in\mathcal{C}_X, Y\in\mathcal{Y}} .$$

We get a functor

$$P : \mathbf{BornCoarse} \to \mathtt{Ind}(\mathbf{UBC}^2) .$$

**Definition 11.65.** *If $E : \mathbf{UBC}^2 \to \mathbf{Ab}^{\mathbb{Z}}$ is a functor, then we define*

$$E\mathbf{P} : \mathbf{BornCoarse}^2 \to \mathbf{Ab}^{\mathbb{Z}} , \quad E\mathbf{P}(X, \mathcal{Y}) := \mathtt{colim}\, E(P(X), P(\mathcal{Y})) .$$

**Proposition 11.66.** *If $E$ is a local homology theory, then $E\mathbf{P}$ is a coarse homology theory.*

*Proof.* $u$-continuity essentially holds true by definition:

$$\begin{aligned} E\mathbf{P}(X) &\cong \mathop{\mathtt{colim}}_{U\in\mathcal{C}_X} E(P_U(X)) \\ &\cong \mathop{\mathtt{colim}}_{V\in\mathcal{C}_X} \mathop{\mathtt{colim}}_{U\in\mathcal{C}_{X_V}} E(P_U(X)) \\ &\cong \mathop{\mathtt{colim}}_{V\in\mathcal{C}_X} E\mathbf{P}(X_V) \end{aligned}$$

We now show coarse invariance. Let $f, g : X \to Y$ be $U$-close maps. If $U^{-1}VU$, then then $f_*, g_* : P_V(X) \to P_{U^{-1}VU}(X)$ are are linearly homotopic by $h(t, \mu) := (1-t)f_*\mu + rg_*\mu$.

If $f : X \to X$ implements flasqueness, that $f \sim^U \mathtt{id}_X$. Set $V := \bigcup_n f^n(U)$. Then $f(V) \subseteq V$. Then $f_* : P_V(X) \to P_V(X)$ is defined. One observes that $f_*$ implements flasqueness of $P_V(X)$. Indeed $f_*$ is homotopic to $\mathtt{id}_{P_V(X)}$ by the linear homotopy. This implies $E\mathbf{P}(X) \cong 0$.

Finally we show excision. Let $(Z, \mathcal{Y})$ be a complementary pair on $X$. Fix $U$ in $\mathcal{C}_X$. If $Y$ is so large that every $U$-bounded subset of $X$ is in $Y$ or $Z$, then we know that $(P_U(Z), P_U(Y))$ is a closed coarsely and uniformly excisive decomposition fo $P_U(X)$. Using that $P_U(Z) \cap P_U(Y) = P_U(Z \cap Y)$ we get and isomorphism

$$E(P_U(Z)), P_U(Z \cap Y)) \stackrel{\cong}{\to} E(P_U(X), P_U(Y)) .$$

Taking the colimit over $Y$ and then over $U$ we get

$$E\mathbf{P}(Z, Z\cap\mathcal{Y}) \stackrel{\cong}{\to} E\mathbf{P}(X,\mathcal{Y}) \ .$$

□

**Definition 11.67.** *The coarse homology theory $E\mathbf{P}$ is called the coarsification of the local homology theory $E$.*

**Example 11.68.** Let $E$ be a coarse homology theory. Then we have a natural transformation $E \to E\mathcal{F}\mathbf{P}$. On $X$ in **BornCoarse** it is given by the map

$$E(i_X) : E(X) \to E\mathcal{F}(P_U(X))$$

induced by the coarse equivalence $i_X : X \to P_U(X)$ for any $U$ in $\mathcal{U}$. This transformation is an isomorphism.

**Definition 11.69.** *We call the natural exact sequence exact sequence*

$$E(X) \to E\mathcal{O}\mathbf{P}(X) \to E\mathcal{O}^\infty\mathbf{P}(X) \stackrel{\mu_{E,X}}{\to} E(X)[-1]$$

*the fundamental sequence and the map $\mu_{E,X}$ the coarse Baum-Connes assembly map for $X$ and $E$.*

**Remark 11.70.** Every strong coarse homology theory $E$ gives rise to a cone sequence

$$E\mathcal{F} \to E\mathcal{O} \to E\mathcal{O}^\infty \to E\mathcal{F}[-1] \ .$$

1. $E\mathcal{F}$ factorizes over the forgetful functor $\mathcal{F}_\mathcal{U}$ which fits into an adjunction

   $$(-)_{disc} : \mathbf{BornCoarse} \leftrightarrows \mathbf{UBC} : \mathcal{F}_\mathcal{U} \ .$$

   The counit of this adjunction is the canonical map

   $$\mathcal{F}_\mathcal{U}(X_{disc}) \to X \ .$$

   The value $E\mathcal{F}(X)$ does not depend on the uniform structure of $X$. In particular, the counit of the adjunction induces an isomorphism

   $$E\mathcal{F}(X_{disc}) \cong E\mathcal{F}(X) \ .$$

2. $E\mathcal{O}$ is a local homology theory. As a special p̈roperty, it vanishes on a point.

3. $E\mathcal{O}^\infty$ is a local homology which is almost independent of the coarse structure in the sense that it is coarsening invariant. For discrete $X$ we have $E\mathcal{O}^\infty(X) \cong E\mathcal{F}(X_{min})[-1]$.

**Remark 11.71.** In this remark we discuss the question whether the cone sequence can be characterized by a universal property. We take $H = E\mathcal{O}$ as the starting point and ask how one can reconstruct the analogs of $E\mathcal{F}$ and $E\mathcal{O}^\infty$.

Let $H$ be a local homology theory. One could ask whether there is coarse homology theory $H\mathcal{X}$ and a natural transformation $H\mathcal{X}\mathcal{F} \to H$ which is the best approximation of $H$ by a local homology of this form. One could then construct a long exact sequence

$$H\mathcal{X}\mathcal{F} \to H \to H^\infty \to H\mathcal{X}\mathcal{F}[-1]$$

and ask whether $H$ is coarsening invariant or what else properties it has.

We have pull-back

$$\begin{array}{ccc} H(X_{disc}) & \longrightarrow & H^\infty(X_{disc}) \\ \downarrow & & \downarrow \\ H(X) & \longrightarrow & H^\infty(X) \end{array} .$$

One could also try the other way and construct a best approximation

$$H \to HO^\infty$$

by a coarsening invariant local homology and a fibre sequence

$$HF \to H \to HO^\infty \to HF[-1] .$$

Then one can ask whether $HF$ comes from a coarse homology theory. Using homotopy theoretic techniques (Bousfield localization) one can construct best approximations by functors with these properties and exact sequences. The problem is that one does not stay in homology theories. $\square$

**Remark 11.72.** Here we discuss the properties of $\mathbf{P}$.

Given a local homology theory $H$ we can construct a coarse homology theory $H\mathbf{P}$. If $H = E\mathcal{F}$, then $H\mathbf{P} \cong E\mathcal{F}\mathbf{P} \cong E$, so the construction reproduces $E$.

One could ask what the universal property of this construction has. The local homology theory $H\mathbf{P}\mathcal{F}$ is a version of $H$ which factorizes $\mathcal{F}$. But it is not clear how construct a comparison morphism between $H\mathbf{P}\mathcal{F}$ and $H$.

If $X$ is nice (e.g. finite-dimensional simplicial complex) then for some $U$ we have map $X \to P_U(\mathcal{F}_\mathcal{U}(X))$ which induces $H(X) \to H\mathbf{P}\mathcal{F}(X)$.

**Definition 11.73.** *We say that $X$ is coarsifying if there exists such a map an $H(X) \to H\mathbf{P}\mathcal{F}(X)$ is an equivalence for any local homology theory.*

If $G$ is a group and $EG$ is a $G$-finite simplicial model for the classifying space, then $EG$ is coarsifying by a result of Gromov. Note that in this case the coarse Baum-Connes assembly map is isomorphic to the map

$$E\mathcal{O}^\infty(X) \to E(X)[-1] \ .$$

□

**Remark 11.74.** One can ask for a universal property of the Baum-Connes assembly map. Its domain is given by

$$E\mathcal{O}^\infty\mathbf{P}(X) = \operatorname{colim}_U E\mathcal{O}^\infty(P_U(X))$$

and $E\mathcal{O}^\infty(P_U(X))$ is as a local homology theory applied to a simplicial complex computable by means of algebraic topology.

The values $E(X)$ are mystery. □

The Baum-Connes conjecture for the coarse homology theory $E$ and the space $X$ is the question whether it is an isomorphism. If it holds true, then one can calculate the mysterious $E(X)$ in terms of algebra topology via $E\mathcal{O}^\infty\mathbf{P}(X)$.

In view of the exact cone sequence the coarse Baum-Connes conjecture holds true for $X$ and $E$ exactly the case if

$$E\mathcal{O}\mathbf{P}(X) \cong 0 \ .$$

**Remark 11.75.** This is a disclaimer for the moment. What we have defined above is a homotopy theoretic version of the classical Baum-Connes assembly map as a natural transformation defined for all spaces and strong coarse homology theories.

The classical Baum-Connes assembly map is only considered for topological coarse $K$-theory and constructed using analytic methods. For nice spaces (essentially all of classical interest) on can identify our homotopy theoretic version version with the classical one. But this is non-trivial will be worked out in future paper by B-Engel-Land. □

**Remark 11.76.** The classical coarse Baum-Connes conjecture for $E = K\mathcal{X}$ is of particular importance since it implies the Novikov conjecture (by a rather complicated argument).

Similarly, for $E = KA\mathcal{X}$ it implies the injectivity of Farell-Jones assembly map for algebraic $K$-theory stating e.g. that for a discrete torsion-free group the assembly map

$$KR_*(BG) \to K_*(R[G]) \ .$$

is injective. □

We now now discuss the question whether $\mu_{E,X}$ is an isomorphism in greater detail.

Note that

$$\mu_X : E\mathcal{O}^\infty\mathbf{P} \to E\mathbf{P}$$

is a natural transformation of coarse homology theories. This makes it possible to apply uniqueness results about coarse homology theories a la Eilenberg-Steenrod.

**Definition 11.77.** *We say that $X$ satisfies the coarse Baum-Connes conjecture motivically if $\mu_{E,X}$ is an isomorphism for any strong coarse homology theory.*

**Proposition 11.78.** *If $X$ is discrete, then it satisfies the coarse Baum-Connes conjecture motivically.*

*Proof.* If $X$ is discrete, then $\mathtt{diag}(X)$ is the maximal coarse entourage of $X$. Then $P_{\mathtt{diag}(X)}(X) \cong X_{min,\mathcal{B},disc}$ as a uniform bornological coarse space. Since $\mathcal{O}(X_{min,\mathcal{B},disc})$ is flasque we conclude

$$E\mathcal{O}\mathbf{P}(X) \cong \operatorname*{colim}_{U\in\mathcal{C}_X} E\mathcal{O}(P_U(X)) \cong E\mathcal{O}(X_{min,\mathcal{B},disc}) \cong 0\ .$$

□

**Definition 11.79.** *We say that $E$ satisfies the coarse Baum-Connes conjecture if $\mu_{E,X}$ is an isomorphism for all $X$.*

**Theorem 11.80.** *$H\mathcal{X}$ satisfies the coarse Baum-Connes conjecture.*

*Proof.* This is an extended exercise which could lead to a master thesis. □

In the following we will describe larger class of spaces which satisfy the coarse Baum-Connes conjecture motivically.

Let $\mathcal{V}$ be a covering of $X$ by subsets and $U$ be some entourage of $X$. Recall that a subset $Y$ of $X$ is $U$-bounded if $Y \times Y \subseteq U$.

**Definition 11.81.**

1. *$U$ is a Lebesgue entourage of $\mathcal{V}$ if every $U$-bounded subset of $X$ is contained in some member of $\mathcal{V}$.*

2. *$U$ is a bound of $\mathcal{V}$ if all members of $\mathcal{V}$ are $U$-bounded.*

**Example 11.82.** If $\mathcal{V}$ is the collection of all $U$-bounded subsets, then $U$ is a Lebesgue entourage of $\mathcal{V}$ and a bound at the same time. □

Let $\mathcal{V} := (\mathcal{V}_i)_{i\in i}$ be a family of coverings indexed by a partially ordered set $I$.

**Definition 11.83.** *$\mathcal{V}$ is called an anti-Čech system if the following are true:*

1. *For every $i, j$ in $I$ with $i < j$ there exists a bound of $\mathcal{V}_i$ which is Lebesgue for $\mathcal{V}_j$.*

2. *Every $U$ in $\mathcal{C}_X$ is Lebesgue entourage for some member of $\mathcal{V}$.*

Note that a covering $\mathcal{V}$ of $X$ gives rise to a simplicial set $\mathtt{N}(\mathcal{V})$ whose underlying set is $I$ and whose simplices are the finite subsets $F$ of $I$ such that $\bigcap_{V\in\mathcal{V}} V \neq \emptyset$. In particular $\dim \mathtt{N}((V_j)_{j\in J}) = \{|F| \in \mathcal{P}_J^{\mathrm{fin}} \mid \bigcap_{j\in F} V_j \neq \emptyset\} - 1$.

**Definition 11.84.** *$X$ has finite asymptotic dimension if it admits an anti-Čech system $(\mathcal{V}_{i\in I})_i$ such that $\sup_{i\in I} \dim \mathtt{N}(\mathcal{V}_i) < \infty$.*

**Example 11.85.** $\mathbb{R}^n$ has finite asymptotic dimension. Consider the covering of $\mathbb{R}^n$ given by $\mathcal{V}_1 := (B_r(n))_{n\in\mathbb{Z}^n}$ for $r$ sufficiently large $\geq \sqrt{n}$. Then take $\mathcal{V}_k := (B_{kr}(rk))_{n\in\mathbb{Z}^n}$. Then $\dim \mathtt{N}(\mathcal{V}_k) = \dim \mathtt{N}(\mathcal{V}_1)$. Then $\hat{\mathcal{V}} := (\mathcal{V}_k)_{k\in\mathbb{N}}$ is an anti-Čech system. □

**Remark 11.86.** We define the asymptotic dimension $a - \dim(X)$ of $X$ as the minimum of the the numbers $\sup_{i\in I} \dim \mathtt{N}(\mathcal{V}_i) < \infty$ for all anti Čech systems $(\mathcal{V}_i)_{i\in I}$. For example $a - \dim(\mathbb{Z}^n) = n$.

**Remark 11.87.** The asymptotic dimension of $X$ is a coarse invariant. So also $a - \dim(\mathbb{R}^n) = n$.

Let $X$ be in **UBC**.

**Definition 11.88.** *We say that $X$ has weakly finite asymptotic dimension if there exists a cofinal set of $U$ in $\mathcal{C}_X$ such that $X_U$ has finite asymptotic dimension.*

**Theorem 11.89** (B-Engel)**.** *If $X$ has weakly finite asymptotic dimension, then $X$ satisfies the coarse Baum-Connes conjecture motivially.*

**Remark 11.90.** The condition can be weakend to finite decomposition complexity, but this notion is more difficult to define and check.

**Problem 11.91** (Open)**.** *Does $E\mathcal{O}^\infty\mathbf{P}$ satisfy the coarse Baum-Connes conjecture? Is $E\mathcal{O}^\infty\mathbf{P}$ the best approximation of $E$ which satisfies the coarse Baum-Connes conjecture?*

# 12 Topological $K$-theory

We explain the case of equivariant coarse $K$-homology with coefficients in the $C^*$-category $\mathbf{Hilb}(\mathbb{C})$. Everything easily generalizes to a general $C^*$-category which is idempotent complete, countably additive and may have a non-trivial $G$-action.

Let $X$ be in $G$**Born**. For a Hilbert space $H$ we let $\mathbf{Proj}(H)$ denote the set of orthogonal projections on $H$.

**Definition 12.1.** *An equivariant $X$-controlled Hilbert space is a pair $(H, \rho, \mu)$, where:*

1. *$H$ is a Hilbert space*
2. *$\rho : G \to U(H)$ is a homomorphism of groups*
3. *$\mu : \mathcal{P}(X) \to \mathbf{Proj}(H)$ is a function satisfying:*
   a) *$\mu(Y) = \mu(Z) + \mu(Y \setminus Z)$ for all subsets $Z, Y$ of $X$ such that $Z \subseteq Y$.*
   b) *$\mu(gY) = \rho(g)^{-1}\mu(Y)\rho(g)$*

We let $H(Y) := \mathtt{im}(\mu(Y))$ for any subset $Y$ of $X$. We let

$$\mathtt{supp}(H, \rho, \mu) := \{x \in X \mid H(x) \neq 0\} \ .$$

**Definition 12.2.** *We call $(H, \rho, \mu)$ locally finite if*

1. *$\mathtt{supp}(\mu)$ is a locally finite subset of $X$.*
2. *$\dim(H(x)) < \infty$ for every $x$ in $X$.*
3. *$H \cong \bigoplus_{x \in X} H(x)$.*

**Example 12.3.** Let $X$ be a $G$-set. Then $(L^2(X), \rho, \mu)$ with $\rho(g) = g^*$ and $\mu$ the counting measure is a locally finite $X$-controlled Hilbert space on $X_{min}$.

The $X$-controlled Hilbert space $(L^2(X) \otimes \ell^2, \rho \otimes 1, \mu \otimes 1)$ is not locally finite on $X_{min}$.

On $G_{max}$ there is a non-trivial locally finite equivariant controlled Hilbert space only if $G$ is finite. □

We now assume that $X$ is in $G\mathbf{BornCoarse}$. Let $(H, \rho, \mu)$ and $(H', \rho', \mu')$ are equivariant $X$-controlled Hilbert spaces, $A : H \to H'$ be a bounded operator, and $U$ be an entourage.

**Definition 12.4.**

1. *$A$ is equivariant if $\rho'(g)A = A\rho(g)$ for all $g$ in $A$.*
2. *$A$ is $U$-controlled, if $\mu'(Z)A\mu(Z) = 0$ for all $Z, Z'$ in $\mathcal{P}_X$ such that $Z'$ is $U$-separated from $Z$. We say that $A$ is controlled if it is $U$-controlled for some $U$ in $\mathcal{C}_X$.*
3. *$A$ is locally compact if $\mu(B)A, A\mu(B) \in K(X)$ for every $B$ in $\mathcal{B}_X$.*

We construct the Roe category $\mathbf{V}^G(X)$ as follows:

1. objects: equivariant $X$-controlled Hilbert spaces

2. morphisms: $\mathtt{Hom}_{\mathbf{V}(X)}((H,\rho,\mu),(H',\rho',\mu'))$ is the closure w.r.t the norm induced from $B(H,H')$ of the set of bounded, controlled and equivariant operators.

3. involution: adjoint

The following is clear by definition:

**Proposition 12.5.** *$\mathbf{V}^G(X)$ is a $C^*$-category.*

We let $\mathbf{V}^G_{lc}(X)$ be the ideal generated by the locally compact controlled operators.

If $f : X \to X'$ is a morphism in $G$**BornCoarse** we get an induced functor $f_* : \mathbf{V}(X) \to \mathbf{V}(X')$ given by

$$f_*((H,\rho,\mu)) := (H,\rho,f_*\mu)\ , \quad f_*(A) = A\ .$$

One easily checks that $f_*$ is well-defined. Since is controlled and proper one checks that $f_*$ preserves controlled and locally compact operators.

This construction yields a functor

$$\mathbf{V}^G : G\mathbf{BornCoarse} \to \mathbf{C^*Cat}^{\mathrm{nu}}\ .$$

**Definition 12.6.** *The algebra $C^*(X,(H,\rho,\mu)) := \mathtt{End}_{\mathbf{V}^G(X)}((H,\rho,\mu))$ is the Roe algebra associated to $X$.*

**Definition 12.7.** *We let $\mathbf{V}^G_{\mathrm{lf}}(X)$ be the full subcategory of locally finite objects.*

Morphisms between locally finite objects are automatically locally compact. In particular, $\mathbf{V}^G_{\mathrm{lf}}(X)$ is unital. We get a subfunctor

$$\mathbf{V}^G_{\mathrm{lf}} : G\mathbf{BornCoarse} \to \mathbf{C^*Cat}\ .$$

**Example 12.8.** We calculate $\mathbf{V}^G_{\mathrm{lf}}(G_{min,min})$ and $\mathbf{V}^G_{\mathrm{lf}}(G_{man,min})$. Let $(H,\rho,\mu)$ be in $\mathbf{V}^G_{\mathrm{lf}}(G_{min,min})$. Then we have $H = \bigoplus_{g\in G} H(g)$. We have isomorphism $\rho(g) : H(e) \to H(g)$. Let $(H,\rho,\mu)$ and $(H',\rho',\mu')$ be in $\mathbf{V}^G_{\mathrm{lf}}(G_{min,min})$. Then a morphism is given by a matrix $(\rho'(h)A_{h,g}\rho(g^{-1}))_{h,g\in G}$ with $A_{h,g} : H(e) \to H(e)$. Thereby $A_{h,g} = 0$ if $h \neq g$ abd $A_{\ell h,\ell g} = A_{h,g}$ for all $\ell$ in $G$ by equivariance.

Hence we have a functor $\mathbf{V}^G_{\mathrm{lf}}(G_{min,min}) \to \mathbf{Hilb}(\mathbb{C})^{fin}$ given by $(H,\rho,\mu) \mapsto H(e)$ and $A \mapsto A_{e,e}$. It is an equivalence of categories.

**Lemma 12.9.** *If $f,g : X \to X'$ are close, then $f_*, g_* : \mathbf{V}_{\mathrm{lf}}(X) \to \mathbf{V}_{\mathrm{lf}}(X')$ are unitarily isomorphic.*

*Proof.* The unitary equivalence is given on $(H, \rho, \mu)$ by

$$\mathtt{id}_H : (H, \rho, f_*\mu) \to (H, \rho, g_*\mu) \ .$$

The fact that $f$ is close to $g$ translates into the statement that $\mathtt{id}_H$ is controlled. □

**Lemma 12.10.** $\mathbf{V}^G_{\mathrm{lf}}$ *is u-continuous.*

*Proof.* We must show that

$$\operatorname*{\mathtt{colim}}_{U \in \mathcal{C}^G_X} \mathbf{V}^G_{\mathrm{lf}}(X_U) \cong \mathbf{V}^G_{\mathrm{lf}}(X) \ .$$

The whole system and the r.h.s. have the same sets of objects. Moreover, all morphism spaces of the system are subspaces of the morphism spaces of the r.h.s. So the colimit is simply given by the closure of the union of the morphism spaces. The assertion follows since every generator of the morphisms of $\mathbf{V}^G_{\mathrm{lf}}(X)$ is controlled. □

A $C^*$-category $\mathbf{C}$ is called flasque if it is additive and admits an endofunctor $S$ such that $\mathtt{id} \oplus S \cong S$.

**Example 12.11.** If $\mathbf{C}$ admits countable sums, the $\mathbf{C}$ is flasque. Indeed, let $S : \mathbf{C} \to \mathbf{C}$ be given by $S(C) := \bigoplus_{\mathbb{N}} C$ and $S(f) := \oplus_{\mathbb{N}} f$. Then $S \oplus \mathtt{id} \cong S$,

**Lemma 12.12.** *If $X$ is flasque, then $\mathbf{V}_{\mathrm{lf}}(X)$ is flasque.*

*Proof.* One checks that $S : \mathbf{V}^G_{\mathrm{lf}}(X) \to \mathbf{V}^G_{\mathrm{lf}}(X)$ given by

$$S(H, \rho, \mu) := (\bigoplus_{\mathbb{N}} H, \oplus_{\mathbb{N}} \rho, \oplus_{n \in \mathbb{N}} f^n_* \mu)$$

and

$$f_*(A) := \bigoplus_{\mathbb{N}} A$$

is well-defined. Furthermore, $\mathtt{id} \oplus f_* \circ S \cong S$ (unitary isomorphism). Finally $f_* \cong \mathtt{id}$ so that $\mathtt{id} \oplus S \cong S$. So $S$ implements flasqueness of $\mathbf{V}^G_{\mathrm{lf}}(X)$. □

A $C^*$-subcategory $\mathbf{I}$ of $\mathbf{C}$ is called an ideal if it is wide and it is closed under left- and right composition with morphisms from $\mathbf{C}$.

An exact sequence of $C^*$-categories is a sequence

$$\mathbf{A} \to \mathbf{B} \to \mathbf{C}$$

where the functors induce bijections on objects and short exact sequences on morphism spaces. Then $\mathbf{A}$ is an ideal in $\mathbf{B}$.

**Example 12.13.** Let $\mathbf{Hilb}(\mathbb{C})_c$ be the subcategory of $\mathbf{Hilb}(\mathbb{C})$ with the same objects, but with morphisms $H \to H'$ the compact operators. Then $\mathbf{Hilb}(\mathbb{C})_c$ is an ideal in $\mathbf{Hilb}(\mathbb{C})$ and we have an exact sequence

$$\mathbf{Hilb}(\mathbb{C})_c \to \mathbf{Hilb}(\mathbb{C}) \to \mathbf{Q}(\mathbb{C}) \ .$$

The $C^*$-category $\mathbf{Q}(\mathbb{C})$ is the Calkin category. The $C^*$-algebra $Q(\ell^2) := \mathtt{End}_{\mathbf{Q}(\mathbb{C})}(\ell^2)$ is called the Calkin algebra. It fits into the exact seqence

$$0 \to K(\ell^2) \to B(\ell^2) \to Q(\ell^2) \to 0 \ .$$

Let $Y$ be an invariant subset of $X$. Then we define $\mathbf{V}^G_{\mathrm{lf}}(Y \subseteq X)$ as the wilde subcategory of $\mathbf{V}^G_{\mathrm{lf}}(X)$ with morphisms $(H, \rho, \mu) \to (H', \rho', \mu')$ of the form $\mu'(Y)A\mu(Y)$ for $A : (H, \rho, \mu) \to (H', \rho', \mu')$ in $\mathbf{V}^G_{\mathrm{lf}}(X)$.

Note that $\mathbf{V}^G_{\mathrm{lf}}(Y \subseteq X)$ is unital.

**Lemma 12.14.** *$i : \mathbf{V}^G_{\mathrm{lf}}(Y) \to \mathbf{V}^G_{\mathrm{lf}}(Y \subseteq X)$ is a unitary equivalence.*

*Proof.* An inverse functor $p$ sends $(H, \rho, \mu)$ to $(H(Y), \rho_{|H(Y)}, \mu_{|Y})$ and $A$ to $A$. Then $p \circ i = \mathtt{id}$ and $i \circ p \to \mathtt{id}$ is given by $H(Y) \to H$ on $(H, \rho, \mu)$.

□

Let $\mathcal{Y}$ be an invariant big family in $X$. We define the subcategory $\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X)$ of $\mathbf{V}^G_{\mathrm{lf}}(X)$ generated by the morphisms $\mu(Y)A\mu(Y)$ for all $Y$ in $\mathcal{Y}$. Note that this involves taking closures. A general element of $\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X)$ is not of the form $\mu(Y)A\mu(Y)$ for some $Y$, but can be approximated by those. Note that

$$\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X) \cong \operatorname*{\mathtt{colim}}_{Y \in \mathcal{Y}} \mathbf{V}^G_{\mathrm{lf}}(Y \subseteq X) \ .$$

**Lemma 12.15.** *$\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X)$ is an ideal in $\mathbf{V}^G_{\mathrm{lf}}(X)$.*

*Proof.* Assume that $B$ is $U$-controlled and $\mathtt{diag}(X) \subseteq U$. Then $\mu(U[Y])B\mu(Y) = B\mu(Y)$. Hence

$$B\mu(Y)A\mu(Y) = \mu(U[Y])B\mu(Y)A\mu(Y)\mu(U[Y]) = \mu(Y')BA\mu(Y')$$

for some $Y'$ in $\mathcal{Y}$ with $U[Y] \subseteq Y'$. Therefore multiplication by $B$ sends generators to generators. □

Thus a pair $(X, \mathcal{Y})$ gives rise to an exact sequence

$$0 \to \mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X) \to \mathbf{V}^G_{\mathrm{lf}}(X) \to \mathbf{V}^G_{\mathrm{lf}}(X, \mathcal{Y}) \to 0 \ . \tag{12.1}$$

Assume that $(Z, \mathcal{Y})$ is an equivariant complementary pair in $X$.

**Lemma 12.16.** *The canonical functor*

$$i : \mathbf{V}^G_{\mathrm{lf}}(Z, Z \cap \mathcal{Y}) \to \mathbf{V}^G_{\mathrm{lf}}(X, \mathcal{Y})$$

*is a unitary isomorphism.*

*Proof.* An inverse functor is given as follows. It sends $(H, \rho, \mu)$ in $\mathbf{V}^G_{\mathrm{lf}}(X, \mathcal{Y})$ to $(H(Z), \rho_{|H(Z)}, \mu_{|Z})$ and $[A]$ to $[\mu(Z')A\mu(Z)]$. In the discussion of algebraic $K$-theory we have shown: If $A$ and $B$ are controlled generators of the ideal, then $ZAZBZ - ZABZ$ is in $\mathbf{V}^G_{\mathrm{lf}}(Y)$ for $Y$ suifficiently large. Hence $p([A])p([B]) = p([AB])$. This extends by continuity to the closures. This shows that $p$ is well-defined.

We have $p \circ i = \mathtt{id}$. We have a natural inclusion $H(Z) \to H$. Again, in the algebraic $K$-theory case we have seen that if $A$ is a controlled generator of the ideal $ZAZ - A$ is in $\mathbf{V}^G_{\mathrm{lf}}(Y)$ for $Y$ suifficiently large. Hence $i(p([A])) = [A]$. □

We set

$$\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y}) := \operatorname*{\mathtt{colim}}_{Y \in \mathcal{Y}} \mathbf{V}^G_{\mathrm{lf}}(Y) \ .$$

We can consider this as a full subcategory of $\mathbf{V}^G_{\mathrm{lf}}(X)$ of objects supported on members of $\mathcal{Y}$.

In order to define the coarse homology theory we need a $K$-theory functor for $C^*$-categories.

We consider a functor

$$K : \mathbf{C^*Cat}^{\mathrm{nu}} \to \mathbf{Ab}^{\mathbb{Z}}$$

with the following properties:

1. $K$ sends unitarily isomorphic functors to equal maps (and hence unitary equivalences to isomorphisms).

2. $K$ sends exact sequences

   $$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

   to long exact sequences

   $$K(\mathbf{A}) \to K(\mathbf{B}) \to K(\mathbf{C}) \stackrel{\partial}{\to} K(\mathbf{A})[-1] \ .$$

3. $K$ preserves filtered colimits.

**Theorem 12.17.** *There exists a $K$-theory functor for $C^*$-categories.*

Here is a construction of such a functor from the standard $K$-theory functor for $C^*$-algebras. We use the adjunction

$$A^f : \mathbf{C^*Cat}^{\mathrm{nu}} \leftrightarrow \mathbf{C^*Alg}^{\mathrm{nu}} : \mathrm{incl}\ .$$

**Proposition 12.18** (M.Joachim, B-Engel)**.** *If $K$ is the usual $K$-theory functor for $C^*$-algebras, then*

$$K \circ A^f : \mathbf{C^*Cat}^{\mathrm{nu}} \to \mathbf{Ab}^{\mathbb{Z}}$$

*is a $K$-theory functor for $C^*$-categories.*

We discuss further properties of the $K$-theory functor.

Let $0[X]$ be a zero category with object set $X$. We have $K(0[X]) \cong 0$. In order to see this we note that

$$0 \to 0[X] \to 0[X] \to 0[X] \to 0$$

is an exact sequence of $C^*$-categories. Hence

$$K(0[X]) \to K(0[X]) \to K(0[X]) \to K(0[X])[-1]$$

is exact. Since the two maps induced by the identity are isomorphisms it follows that $K(0[X]) \cong 0$. We say that $K$ is reduced.

Let $K : \mathbf{C^*Cat}^{\mathrm{nu}} \to \mathbf{Ab}^{\mathbb{Z}}$ be a $K$-theory functor. Then if $\phi, \psi : \mathbf{C} \to \mathbf{D}$ and additive $\mathbf{D}$ we have

$$K(\phi \oplus \psi) = K(\phi) + K(\psi)\ .$$

This is an exercise using exactness and reducedness. As a consequence, if $\mathbf{D}$ is flasque, then $K(\mathbf{D}) \cong 0$. Indeed, if $S : \mathbf{D} \to \mathbf{D}$ implies flasqueness, then the relation $K(S) + K(\mathtt{id}) = K(S \oplus \mathtt{id}) = K(S)$ implies the assertion.

We now define the $\delta$-functor $(K\mathcal{X}, \partial)$. We let

$$K\mathcal{X}^G : \mathbf{BornCoarse}^2 \to \mathbf{Ab}^{\mathbb{Z}}\ , \quad K\mathcal{X}^G(X, \mathcal{Y}) := K(\mathbf{V}^G_{\mathrm{lf}}(X, \mathcal{X}))\ .$$

For every $(X, \mathcal{Y})$ in $\mathbf{BornCoarse}^2$ we have a natural exact sequence

$$\begin{array}{ccccccc}
K(\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X)) & \longrightarrow & K\mathcal{X}^G(X) & \longrightarrow & K\mathcal{X}(X, \mathcal{Y}) & \xrightarrow{\partial} & K(\mathbf{V}^G_{\mathrm{lf}}(\mathcal{Y} \subseteq X))[-1] \\
\cong \uparrow & & & & & & \cong \uparrow \\
K\mathcal{X}^G(\mathcal{Y}) & & & & & & K\mathcal{X}^G(\mathcal{Y})[-1]
\end{array}$$

which we take as the long exact sequence of the $\delta$-functor

**Theorem 12.19.** *The pair $(K\mathcal{X}^G, \delta)$ of is an equivariant coarse homology theory.*

*Proof.* If $f \sim g$, then $f_*$ and $g_*$ are unitarily isomorphic. Hence $K(f_*) = K(g_*)$. Hence $K\mathcal{X}^G$ is coarsely invariant.

Since $K$ preserves filtered colimits we have

$$\operatorname{\texttt{colim}}_{U \in \mathcal{C}_X^G} K(\mathbf{V}_{\mathrm{lf}}^G(X_U)) \cong K(\operatorname{\texttt{colim}}_{U \in \mathcal{C}_X^G} \mathbf{V}_{\mathrm{lf}}^G(X_U)) \cong K(\mathbf{V}_{\mathrm{lf}}^G(X)) \ .$$

Hence $K\mathcal{X}^G$ is $u$-continuous.

If $X$ is flasque, then $\mathbf{V}_{\mathrm{lf}}^G(X)$ is flasque and hence $K(\mathbf{V}_{\mathrm{lf}}^G(X)) \cong 0$. Hence $K\mathcal{X}^G$ vanishes on flasques.

Excision follows from Lemma 12.16 by applying $K$. □

For simplicity we consider the case of a trivial group $G$, but the following discussion has an equivariant generalization.

Let $(H, \mu)$ be in $\mathbf{V}(X)$.

**Definition 12.20.** *We say that $(H, \mu)$ is determined on points if $H \cong \bigoplus_{x \in X} H(x)$.*

**Definition 12.21.** *We say that $(H, \mu)$ is locally separable if $\mu(B)H$ is separable for every $B$ in $\mathcal{B}_X$.*

Let $(H, \mu)$ be in $\mathbf{V}(X)$ be locally separable and determined on points. Then we have the classical Roe algebra $C^*_{lc}(X, (H, \mu)) = \mathtt{End}_{\mathbf{V}_{lc}(X)}((H, \mu))$.

**Example 12.22.** Let $X$ be a separable proper metric space. Let $\nu$ be a regular Borel measure. Interesting operators often life on $H := L^2(M, \mathcal{B}, \nu)$. We want to turn $H$ into a locally countable $X$-controlled Hilbert space. We choose a 1-dense countable subset $L$ in $X$. Then we choose a measurable partition $(B_l)_{l \in L}$ of $X$ by subsets $B_l$ with $B_l \subseteq U[l]$. Then we define the measure

$$\mu := \sum_{l \in L} \chi_{B_l} \delta_l \ .$$

Then $(H, \mu)$ is a locally separable (by regularity of $\nu$) controlled Hilbert space which is determined on points.

The Roe algebra $C^*_{lc}(X, (H, \mu))$ is independet of the choice of $\mu$ and only depends on $\nu$.

□

**Definition 12.23.** *We say that $(H, \mu)$ is ample if it is determined on points and for every other locally separable $X$-controlled Hilbert space $(H', \mu')$ determined on points there exists an isometry $u : (H', \mu') \to (H', \mu')$ in $\mathbf{V}(X)$.*

**Definition 12.24.** *$X$ is locally countable if there exists an entourage $V$ of $X$ such that every $V$-separated subset $L$ is locally countable.*

**Proposition 12.25.** *$X$ admits an ample $X$-controlled Hilbert space if and only if $X$ is locally countable.*

*Proof.* Let $V$ be an entourage such that every $V$-separated subset is locally countable. We then choose a $V^2$-dense $V$-separated subset $L$. We then set $H := L^2(L) \otimes \ell^2$ and let $\mu$ be the counting measure. Then $(H, \mu)$ is locally separable.

We now show that it is ample. Let $(H', \mu')$ be any other $X$-controlled Hilbert space which is determined on points and locally separable. Then $\mathtt{supp}(H', \mu')$ is locally countable. We can find a map $f : \mathtt{supp}(H', \mu') \to L$ such that $(f(x), x) \in V^2$ for every $x$ in $X$. Then $f$ has countable fibres. For every $l$ in $L$ we choose an isometric embedding $u_l : \bigoplus_{x \in f^{-1}(x)} H'(x) \to H(l)$. Then $\oplus_{l \in L} u_l : H' \to H$ is $V^2$-controlled and isometric.

For the converse see [BEb, Prop.8.20]

□

The classical definition of the coarse $K$-homology is as

$$K(C^*(X, (H, \mu)))$$

for some ample $X$-controlled Hilbert space. This definition works if $X$ is locally countable.

**Definition 12.26.** *An morphism $A$ is in $\mathbf{V}(X)$ is called locally finite if it is of the form*

$$(H, \mu) \stackrel{u^*}{\to} (H_0, \mu_0) \stackrel{A'}{\to} (H'_0, \mu'_0) \stackrel{u'}{\to} (H', \mu')$$

*for $(H_0, \mu_0)$ and $(H'_0, \mu'_0)$ in $\mathbf{V}_{\mathrm{lf}}(X)$ and isometries $u, u'$ in $\mathbf{V}(X)$ and a morphism $A$ in $\mathbf{V}_{\mathrm{lf}}(X)$.*

Note that locally finite operator is locally compact. We let $\mathbf{V}_{(\mathrm{lf})}(X)$ be the wide subcategory of $\mathbf{V}_{lc}(X)$ generated by the locally finite operators. In particular we have an inclusion

$$C^*_{(\mathrm{lf})}(X, (H, \mu)) \subseteq C^*_{lc}(X, (H, \mu))$$

of Roe algebras.

Let $X$ be in **BornCoarse**.

**Definition 12.27.**

1. *$X$ is locally finite if there exists an entourage $V$ of $X$ such that every $V$-separated subset $L$ is locally finite.*

2. *X is called separable, if it admits an entourage $V$ such that there exists a countable $V$-dense subset.*

**Example 12.28.** If $X$ is coarsely equivalent to $X'$ then $X$ is locally finite iff $X'$ is locally finite. So local finiteness is a coarsely invariant notion.

If $X$ has the minimal bornology, then $X$ is locally finite. So $\mathbb{R}_{min,min}$ is locally finite. Any space $X'$ coarsely equivalent to $X$ is the also locally finite.

If $X$ is a proper metric space, then $X$ is locally finite. Indeed, if $L$ is 1-separated and $B$ is compact, then $L \cap B$ must be finite since otherwise $L \cap B$ would have an accumulation point.

An infinite dimensional Hilbert space with metric structures is not locally finite. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal family. Then $(ne_k)_{k\in\mathbb{N}}$ is $2n-1$-separated. But the bounded subset $B_{(n+1)}(0)$ contains the whole family. □

**Example 12.29.** If $X$ is coarsely equivalent to $X'$, then $X$ is separable iff $X'$ is separable. So local finiteness is a coarsely invariant notion.

A separable metric space (i.e., a metric space containing a countable $\epsilon$-net for every $\epsilon$ in $(0,\infty)$ is separable.

A Hilbert space is separable.

$\mathbb{R}_{min,min}$ is not separable. □

**Proposition 12.30** ([BEb, Prop.8.40])**.** *Assume:*

1. *$(H,\mu)$ locally separable and determined on points.*

2. *$X$ is separable and locally finite.*

*Then*

$$C^*_{(\mathrm{lf})}(X,(H,\mu)) = C^*_{lc}(X,(H,\mu)) \ .$$

*Proof.* We consider $A$ in $C^*_{lc}(X,(H,\mu))$. We must appximate $A$ by elements from $C^*_{(\mathrm{lf})}(X,(H,\mu))$.

Let $L$ be a $U$-separated and $U^2$ dense countable subset. We choose a partition $(B_l)_{l\in L}$ of $X$ by $U^2$-bounded subsets such that $B_l \subseteq U^2[l]$.

First we approximate $A$ by $A_1$ with controlled propagation up to $\epsilon/2$.

We set $B'_l := \mathtt{supp}(A_1)[B_l]$ for every $l$ in $L$. Since $A_1$ is locally compact we can choose a finite-dimensional $P_l$ on $H(B'_l)$ for every $l$ such that

$$\|P_l B'_l A_1 B_l P_l - A' B_l\| \le \epsilon 2^{-l-2} \ .$$

We well-order $L$. Then we define $H'_l := \sum_{l'\le l} \mathtt{im}(P_{l'})$ and $H' := \overline{\bigcup_{l\in L} H_l}$. Then we define $Q_l$ as the projection onto $H_l \ominus H_{l-1}$. We define $\mu' := \sum_{l\in L} Q_l\delta_l$. Then $(H',\mu')$ is locally finite.

We set $A' := \sum_{l\in L} P_l B'_l A_1 B_l P_l$. Then $A''$ has controlled propagation and

$$\|A' - A_1\| \le \epsilon/2 \ .$$

We have $\|A - A'\| \le \epsilon$ and $A'$ factorizes over $H'$ which is locally finite.

□

Thus if $X$ is separable and locally finite then can identify the classical definition of coarse $K$-theory with $K(C^*_{(\mathrm{lf})}(X,(H,\mu))$.

The following theorem holds true for the classical $K$-theory functor for $C^*$-categories. Its proof uses more properties than just the axioms.

**Theorem 12.31** ([BEb, Cor. 8.96])**.** *We assume that $X$ is separable and locally finite. Then there exists a canonical isomorphism and that $(H,\mu)$ is ample.*

$$K\mathcal{X}(X) \cong K(C^*_{(\mathrm{lf})}(X,(H,\mu))) \ .$$

*Proof.* (Sketch) We consider the category $\mathbf{V}_{\mathrm{lf}}(X)^U$ of triples $(H',\mu',U)$, where $U : (H',\mu') \to (H,\mu)$ is an isometry. We have a fully faithful functor

$$F : \mathbf{V}_{\mathrm{lf}}(X)^U \to \mathbf{V}_{\mathrm{lf}}(X) \ , \quad (H',\mu',U) \mapsto (H',\mu') \ .$$

Since $(H,\mu)$ is ample this functor is essentially surjective. Hence it is a unitary equivalence of $C^*$-categories. In particular it induces an isomorphism

$$K(F) : K(\mathbf{V}^{\mathrm{lf}}(X)^U) \cong K(\mathbf{V}^{\mathrm{lf}}(X)).$$

We have a functor

$$I : \mathbf{V}^{\mathrm{lf}}(X)^U \to C^*_{(\mathrm{lf})}(X,(H,\mu)) \ , \quad (A : (H'_0,\mu'_0,U_0) \to (H'_1,\mu'_1,U_1)) \mapsto U'AU^*$$

(note that the action on objects is clear).

Since $C^*_{(\mathrm{lf})}(X,(H,\mu))$ is generated by locally finite operators one can show that

$$K(I) : \mathbf{V}^{\mathrm{lf}}(X)^U \to K(C^*_{(\mathrm{lf})}(X,(H,\mu)))$$

is an isomorphism, too.

At this point we use more than the axioms. See the next remark.

**Remark 12.32.** In order to show this we use the picture where $K$-theory classes are given by homotopy classes of projections and unitaries.

If $p$ is a projection in $C^*_{(\mathrm{lf})}(X,(H,\mu))$, then there exists $p'$ on an object in $\mathbf{V}^{\mathrm{lf}}(X)^U$ such that $\|Up'U^* - p\| < 1/10$ by density of locally finite operators. But then we can find a projection $p''$ on the same object with $\|p'' - p\| \leq 1/4$. This projection is unique up to homotopy.

In this way we show the isomorphism on $K_0$. On $K_1$ we argue with unitaries in a similar way. □

The desired isomorphism is the the composition

$$K(\mathbf{V}^{\mathrm{lf}}(X)) \overset{\cong}{\leftarrow} K(\mathbf{V}^{\mathrm{lf}}(X)^U) \overset{\cong}{\to} K(C^*_{(\mathrm{lf})}(X,(H,\mu))) \ .$$

□

**Example 12.33.** Let $M$ be a complete Riemannian manifold. Let $D$ be a formally selfadjoint Dirac type operator on a hermitean bundle $L^2(M,E)$. Then $D$ is essentially selfadjoint, i.e. $\bar{D}$ is selfadjoint.

Then we consider $H = L^2(M,E)$. We choose a locally finite control $\mu$ using a measurable $U_1$-bounded partition. If $\dim(M) > 0$, then $(H,\mu)$ is ample.

We have the Roe algebra $C(M,(L^2(M,E),\mu))$. The coarse index $\mathtt{index}\mathcal{X}(D)$ is a class in $K_1(C(M,(L^2(M,E),\mu)))$. In order to define it we consider the homomorphism

$$C_0(\mathbb{R}) \to C(M,(L^2(M,E),\mu)) \ , \quad f \mapsto f(\bar{D})$$

of $C^*$-algebras.

The observation that $f(\bar{D})$ belongs to the Roe algebra is due to J. Roe. Here is the rough argument. Let $\hat{f}$ be the Fouriertansform of $f$. If $\mathtt{supp}(\hat{f}) \subseteq [-R,R]$, then $\mathtt{supp}(f(\bar{D})) \subseteq U_r$ (this is the finite propagation property). A general $f$ can be uniformly approximated by functions with compactly supported Fourier transform.

This gives

$$i_D : \mathbb{Z} \cong K_1(C_0(\mathbb{R})) \ \to K_1(C(M,(L^2(M,E),\mu)))$$

and we define

$$\mathtt{index}\mathcal{X}(D) := i_D(1) \ .$$

This particular simple way to define the coarse index is due to R. Zeidler. A modification including gradings works in the even case.

Using our comparision isomorphism we get a well-defined class

$$\mathtt{index}\mathcal{X}(D) \in K\mathcal{X}_1(M) \ .$$

We now discuss continuously controlled Hilbert spaces. We consider a locally compact space $X$.

**Definition 12.34.** *A continuously $X$-controlled Hilbert space is a pair $(H, \kappa)$ where $\kappa : C_0(X) \to B(H)$ is a $C^*$-homomorphism.*

**Example 12.35.** Let $X$ be a proper metric space with a regular measure $\mu$. Then we let $H := L^2(X, \mu)$ and $\kappa : C_0(X) \to B(H)$ be the action by multiplication operators. This of course extends to an action of $L^\infty(X, \mu)$. □

This is in fact a general observation. By spectral theory the homomorphism $\kappa$ extends to a homomorphism $\kappa : L^\infty(X) \to B(H)$.

**Definition 12.36.**

1. *We say that $(H, \kappa)$ is locally separable if $\kappa(\chi_B)H$ is separable for every relatively compact subset $B$.*

2. *We say that $(H, \kappa)$ is ample if $\kappa(f)$ is not compact for every $f$ in $C_0(X)$.*

**Example 12.37.** If $X$ is second countable and $L$ is a dense countable subset, then $(H, \kappa)$ $H = L^2(X) \otimes \ell^2$ with $\kappa(f) = f \otimes \mathtt{id}_{\ell^2}$ is ample.

**Example 12.38.** Let $X$ be a locally compact uniform bornological space and $(H, \mu)$ be an ample $\mathcal{O}(X)$-controlled Hilbert space. Let $[0, \infty) \times X \to X$ be the projection. We define $\kappa : C_0(X) \to B(H)$ be defined by

$$\kappa(f) := \sum_{(t,x) \in \mathcal{O}(X)} f(x)\mu(x) \ .$$

Then $(H, \kappa)$ is an ample continuously $X$-controlled Hilbert space.

**Definition 12.39.**

1. *An operator $A$ from $(H, \kappa)$ to $(H', \kappa')$ is called pseudo-local of $\kappa(f)A - A\kappa(f)$ is compact.*

2. *An operator $A$ from $(H, \kappa)$ to $(H', \kappa')$ is locally compact if $\kappa(f)A$ and $A\kappa(f)$ are compact for all $f$ in $C_0(X)$.*

We let $\Psi_1(X)$ be the $C^*$-category with objects $(H, \kappa)$ and locally compact operators. We let $\Psi_0(X)$ be the $C^*$-category with the same objects and the pseudo-local operators. We get an exact sequence of $C^*$-categories

$$0 \to \Psi_{-1}(X) \to \Psi_0(X) \to \Sigma(X)$$

where $\Sigma(X)$ is defined as the quotient. If $f : X \to X'$ is proper continuous, then it gives rose to functors

$$f_*(H, \kappa) := (H, f_*\kappa) \ , \quad f_*(A) := A \ ,$$

where $f_*\kappa := \kappa \circ f^*$.

**Definition 12.40.** *We define the analytic locally finite $K$-homology of $X$ by $K_*^{\mathrm{an}}(X) := K_{*+1}(\Sigma(X))$.*

Assume that $X \to X'$ is a proper map. Then we define the continuously $X'$-controlled Hilbert space $f_*(H,\kappa) := (H, \kappa \circ f^*)$.

One can show [HR00]:

1. $K^{\mathrm{an}}$ is homotopy invariant.

2. $K^{\mathrm{an}}$ satisfies closed excision.

3. $K^{\mathrm{an}}([0,\infty) \times X) \cong 0$.

The following uses deep functional analytic results (Voiculescu's theorem).

**Proposition 12.41.** *There exists a pseudolocal isometry $u : f_*(H,\kappa) \to (H,\kappa)$.*

**Proposition 12.42.** *The canonical map $K_*(\mathtt{End}_{\Sigma(X)}(H,\mu)) \to K_*(\Sigma(X))$ is an isomorphism*

In other words, it suffices to consider a single ample $X$-controlled $H$-Hilbert space.

Let us now assume that $X$ is a bornological coarse space.

**Definition 12.43.** *A coarse structure and a topology are compatible if there exists an open entourage.*

We assume that the bornology of $X$ consists of the relatively compact subsets and that the topology of $X$ is compatible with the bornology.

We let $\mathbf{C}(X)$ be the category with the objects $(H,\kappa)$ and the morphisms generated by the locally compact and controlled operators. We furthermore let $\mathbf{D}(X)$ be the $C^*$-category with the objects $(H,\kappa)$ and morphisms generated by controlled and pseudolocal operators. Then we get an exact sequence of $C^*$-categories

$$0 \to \mathbf{C}(X) \to \mathbf{D}(X) \to \mathbf{Q}(X) \to 0 \ .$$

We have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C}(X) & \longrightarrow & \mathbf{D}(X) & \longrightarrow & \mathbf{Q}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Psi_{-1}(X) & \longrightarrow & \Psi_0(X) & \longrightarrow & \Sigma(X) & \longrightarrow & 0 \end{array}$$

**Lemma 12.44.** *We have $K\mathcal{X}(X) \cong K(\mathbf{C}(X))$.*

*Proof.* The verification is similar as the comparison with Roe algebra. □

**Lemma 12.45.** *We have an isomorphism $\mathbf{Q}(X) \stackrel{\cong}{\to} \Psi(X)$.*

*Proof.* Here is the idea. Let $A$ be in $\Psi(X)$. Choose locally open covering by $U$-controlled subsets and partition of unity $(\chi_i^2)_{i\in I}$. Then show that $A - \sum_i \chi_i A \chi_i$ is locally compact. This shows $[A] = [\sum_i \chi_i A \chi_i]$. But $\sum_i \chi_i A \chi_i$ is controlled. □

We can now consider the index map

$$\mathtt{index}\mathcal{X} : K_*^{an}(X) \cong K_{*+1}(\Sigma(X)) \cong K_{*+1}(Q(X)) \stackrel{\delta}{\to} K_*(\mathbf{C}(X)) \cong K\mathcal{X}_*(X)\ .$$

This is the map which sends a locally symbol class of a differential operator to the coarse index.

The classical definition of the Baum-Connes assembly map is

$$\mu^{CBC} : \operatorname*{\mathtt{colim}}_U K_*^{an}(P_U(X)) \stackrel{\mathtt{colim}_U\,\mathtt{index}\mathcal{X}}{\to} \operatorname*{\mathtt{colim}}_U K\mathcal{X}_*(P_U(x)) \cong K\mathcal{X}_*(X)\ .$$

For comparison we need the square

$$\begin{array}{ccc} K\mathcal{X}(\mathcal{O}^\infty(W)) & \xrightarrow{\partial} & K\mathcal{X}(W)[-1] \\ \downarrow & & \| \\ K^{an}(W) & \xrightarrow{\mathtt{index}} & K\mathcal{X}(W)[-1] \end{array}$$

The left vertical map is given by the inclusion using the construction from Example 12.38.

**Proposition 12.46.** *This square commutes and the left vertical map is an isomorphism for nice $W$.*

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