Lecture course on C^*-algebras and C^*-categories

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Contents

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1 Algebras over $\mathbb{C}$

In this lecture course all algebras are over the field of complex numbers $\mathbb{C}$. We let $\mathbf{Vect}_{\mathbb{C}}$ denote the category of $\mathbb{C}$-vector spaces with the usual tensor product $\otimes$. We start with the definition of not necessarily unital algebras in $\mathbf{Vect}_{\mathbb{C}}$.

Let A be in $\mathbf{Vect}_{\mathbb{C}}$.

Definition 1.1. *An associative product on A is a map $\mu : A \otimes A \to A$ such that*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathtt{id}_A \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes \mathtt{id}_A} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes.

In view of the universal property of the tensor product we can interpret the product in A equivalently as a bilinear map $\mu : A \times A \to A$.

Definition 1.2. *An algebra is a pair (A, μ) of A in $\mathbf{Vect}_{\mathbb{C}}$ and an associative product μ.*

Note that we do not require the existence of a unit element.

Let (A, μ) be a algebra. For a, a' in A we will use the notation aa' instead of $\mu(a \otimes a')$. Usually we will denote algebras simply by a symbol like A or similar.

Example 1.3. If V is in $\mathbf{Vect}_{\mathbb{C}}$, then its set of endmorphisms $\mathtt{End}(V)$ has again a structure of an object of $\mathbf{Vect}_{\mathbb{C}}$. The composition of endomorphisms defines an associative and bilinear product. Hence $\mathtt{End}(V)$ becomes an algebra.

For $V = \mathbb{C}^n$, using the standard basis, we identify $\mathtt{End}(V)$ with the n-by-n-matrices $\mathtt{Mat}(n)$ with the usual matrix multiplication. $\square$

Example 1.4. If A is an algebra and X is a set, then we can form a new algebra A^X of functions from X to A with the pointwise vector space structure and product. $\square$

Example 1.5. Let G be a magma (a set with an associative product). Then we consider the vector space $\mathbb{C}[G]$ generated by G. The element in $\mathbb{C}[G]$ corresponding to g in G will be denoted by $[g]$. The associative product $G \times G \to G$ induces an associative product

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \cong \mathbb{C}[G \times G] \to \mathbb{C}[G] \ .$$

We have $[g][g'] = [gg']$. The algebra $\mathbb{C}[G]$ is called the magma ring of G.

We have $\mathbb{C}[\emptyset] = 0$.

We can identity $\mathbb{C}[\mathbb{N}]$ with the polynomial ring $\mathbb{C}[x]$ by sending $[n]$ to x^n. Similarly we have $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$. $\square$

Example 1.6. Let A be an algebra with product μ and I be a subvectorspace of A. If the composition $I \otimes I \to A \otimes A \xrightarrow{\mu} A$ takes values in I, then we get a subalgebra $(I, \mu_{|I \otimes I})$. $\square$

Definition 1.7.

1. *A linear subspace I of A is called a left ideal if for all a in A and i in I we have $ai \in I$.*

2. *A linear subspace I of A is called a right ideal if for all a in A and i in I we have $ia \in I$.*

3. *A linear subspace I of A is called an (two-sided) ideal if it is a left- and right ideal.*

If I is a (possibly one-sided) ideal in an algebra A, then it is in particular a subalgebra.

Example 1.8. Let V be in $\mathbf{Vect}_{\mathbb{C}}$. Then the subset $\mathtt{End}^{\mathrm{fr}}(V)$ of $\mathtt{End}(V)$ of finite rank endomorphisms is an ideal. $\square$

Example 1.9. Let $(A_i)_{i \in I}$ be a family of algebras. Then we can form the sum $\bigoplus_{i \in I} A_i$ of underlying vector spaces. It carries an algebra structure with the component wise product $\oplus_i a_i \oplus_i a'_i := \oplus_i a_i a'_i$.

If $(A)_{x \in X}$ is a constant family indexed by a set X, then $\bigoplus_{x \in X} A$ can be identified with the ideal in the algebra A^X consisting of the functions with finite support. $\square$

Example 1.10. Let A be an algebra with product μ. If I is an ideal in A, then the quotient vector space A/I has an algebra structure which will be denoted by $\bar{\mu}$. It is given by the unique factorization

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow \\ A/I \otimes A/I & \xrightarrow{\bar{\mu}} & A/I \end{array}$$

of μ which is easily seen to exist by the condition on I being a two-sided ideal. Here we use the identification $A/I \otimes A/I \cong A \otimes A/(A \otimes I + I \otimes A)$. □

Example 1.11. Let H be a Hilbert space. The bounded operators $B(H)$ form an algebra with respect to the composition of operators. The subspace of compact operators $K(H)$ is an ideal in $B(H)$. The quotient algebra $Q(H) := B(H)/K(H)$ is called the Calkin algebra of H. □

Example 1.12. We can consider the algebra $D(\mathbb{C})$ of differential operators with polynomial coefficients on $\mathbb{C}$. Typical elements are x (multiplication by x) and ∂ (differentiation by x). The element $e = x\partial$ is called the Euler operator. We have the relations $\partial x - x\partial = 1$ and $ex^n - x^n e = nx^n$. □

We now describe the category of algebras. We consider two algebras A, B with products μ_A and μ_B.

Definition 1.13. *A homomorphism $f : A \to B$ of algebras is morphism $f : A \to B$ in* $\mathbf{Vect}_{\mathbb{C}}$ *such that*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow{\scriptstyle f \otimes f} & & \downarrow{\scriptstyle f} \\ B \otimes B & \xrightarrow{\mu_B} & B \end{array}$$

commutes.

We get the category $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of algebras and homomorphisms. The superscript nu stands for non-unital since we do not require the existence of units nor that maps preserve units, if they exist.

Example 1.14. Let V', V'' be in $\mathbf{Vect}_{\mathbb{C}}$ and consider their sum $V := V' \oplus V''$. Then we get a homomorphism of algebras $\mathtt{End}(V') \to \mathtt{End}(V)$ which sends ϕ to $\phi \oplus 0$. This is called the left-upper-corner inclusion.

For n, m in $\mathbb{N}$ with $n \geq m$ we have $\mathbb{C}^n \cong \mathbb{C}^m \oplus \mathbb{C}^{n-m}$. This gives the left-upper-corner inclusion $\mathtt{Mat}(m) \to \mathtt{Mat}(n)$. □

Example 1.15. Let A be an algebra and $f : X \to Y$ be a map of sets. We get a homomorphism $f^* : A^Y \to A^X$ given by restriction of functions along f. □

Example 1.16. If $G \to H$ is a morphism of magmas, then the induced map $\mathbb{C}[G] \to \mathbb{C}[H]$ is a homomorphism of algebras.

The inclusion $\mathbb{N} \to \mathbb{Z}$ induces an inclusion $\mathbb{C}[\mathbb{N}] \to \mathbb{C}[\mathbb{Z}]$ which can be identified with the inclusion $\mathbb{C}[x] \to \mathbb{C}[x, x^{-1}]$. $\square$

Example 1.17. Let $f : A \to B$ be a homomorphism, I be an ideal in A and J be an ideal in B such that $f(I) \subseteq J$. Then the natural factorization $\bar{f}$ in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\bar{f}} & B/J \end{array}$$

is a homomorphism. On elements it is given by $[f]([a]) := [f(a)]$. $\square$

Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$

Definition 1.18. *A is called unital if it admits an element 1_A such that $1_A a = a = a 1_A$ for all a in A.*

Such an element 1_A is called a unit of A.

Lemma 1.19. *If A is unital, then the unit of A is uniquely determined.*

Proof. Let 1_A and $1'_A$ be two units. Then using the defining relation for $1'_A$ and 1_A we have $1_A = 1_A 1'_A = 1_{A'}$. $\square$

So units in algebras are a property, not data.

Example 1.20. If V is in $\mathbf{Vect}_{\mathbb{C}}$, then $\mathtt{End}(V)$ is unital with unit $1_{\mathtt{End}(V)} = \mathtt{id}_V$. If $\dim(V) = \infty$, then the algebra of finite rank endmorphisms $\mathtt{End}^{\mathrm{fr}}(V)$ is not unital. $\square$

Example 1.21. If A is unital and X is a set, then A^X is unital, where the unit 1_{A^X} is the constant function with value 1_A.

If X is infinite, then the subalgebra $\bigoplus_{x \in X} A$ of A^X is not unital. $\square$

Example 1.22. The algebra $D(\mathbb{C})$ is unital with unit 1. $\square$

Example 1.23. A monoid is a magma with an identity element. For a magma G the magma algebra $\mathbb{C}[G]$ is unital if and only of G is a monoid. In this case the unit is given by $1_{\mathbb{C}[G]} := [e]$, where e is the unit of G. $\square$

Example 1.24. For a Hilbert space H the algebra $B(H)$ is unital. The algebra $K(H)$ is unital if and only if H is finite-dimensional. $\square$

Let A and B be unital algebras and $f : A \to B$ be a homomorphism of algebras.

Definition 1.25. *f is called unital if $f(1_A) = 1_B$.*

We get the category $\mathbf{Alg}_{\mathbb{C}}$ of unital algebras and unital homomorphisms.

Example 1.26. Note that $\mathtt{Mat}(n)$ and $\mathtt{Mat}(n+1)$ are unital. But the left upper corner inclusion $\mathtt{Mat}(n) \to \mathtt{Mat}(n+1)$ is a morphism between algebras which does not preserve units. $\square$

We have an inclusion functor

$$\mathrm{incl} : \mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}} \to \mathbf{Alg}_{\mathbb{C}} \ .$$

This functor is faithful, but not full and not essentially surjective. In the following lemma we show that it has a left-adjoint. The latter will be called the unitalization functor.

Lemma 1.27. *We have an adjunction $(-)^u : \mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : \mathrm{incl}$.*

Proof. We show the existence of the adjunction by providing an explicit construction of the left adjoint and of the unit and counit of the adjunction. We start with constructing the unitalization functor $(-)^u$.

1. objects: Let A be in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$. Then the underlying vector space of A^u is $A \oplus \mathbb{C}$. The product is defined by

   $$(a, \lambda)(a', \lambda') := (aa' + \lambda a' + a\lambda', \lambda\lambda') \ .$$

   One checks associativity by calculation. The algebra A^u has a unit which is given by $1_{A^u} = (0, 1)$.

2. morphisms: If $f : A \to B$ is a homomorphism, then we define $f^u : A^u \to B^u$ by $f^u(a, \lambda) := (f(a), \lambda)$. One checks by calculation that this is a unital homomorphism.

One checks by calculation that this construction defines a functor.

In order to define the adjunction we provide the unit and counit transformations:

1. unit: $A \to \mathrm{incl}(A^u)$ is given by $a \mapsto (a, 0)$.

2. counit: $\mathrm{incl}(B)^u \to B$ is given by $(b, \lambda) \mapsto b + \lambda 1_B$.

One checks that these formulas define morphisms in the respective categories and are natural. One further checks by calculation that the following map of sets

$$\mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u, B) \stackrel{\mathrm{incl}}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}}(\mathrm{incl}(A^u), \mathrm{incl}(B)) \stackrel{unit^*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}}(A, \mathrm{incl}(B))$$

is a bijection with inverse

$$\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(A,\mathrm{incl}(B)) \stackrel{(-)^u}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u,\mathrm{incl}(B)^u) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A^u,B)\ .$$

□

For now on we will usually omit incl from the notation.

Remark 1.28. For A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ we have a split exact sequence

$$0 \to A \stackrel{unit}{\to} A^u \stackrel{(a,\lambda)\mapsto\lambda}{\to} \mathbb{C} \to 0$$

with split $\mathbb{C} \to A^u$ given by $\lambda \mapsto (0,\lambda)$. If A is unital, then we have an isomorphism of algebras

$$A^u \cong A \oplus \mathbb{C}\ , \quad (a,\lambda) \mapsto (a+\lambda 1_A,\lambda)\ .$$

□

We have a functor $U : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{C}}$ which forgets the product and takes the underlying vector space. It restricts to a functor $U : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Vect}$.

Lemma 1.29. *We have adjunctions*

$$T : \mathbf{Vect}_{\mathbb{C}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : U$$

and

$$T^{\geq 1} : \mathbf{Vect}_{\mathbb{C}} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : U\ .$$

Proof. (sketch) We again provide explicit constructions of the left-adjoints and the units and counits. The functor T associates to a vector space V the tensor algebra

$$T(V) := \bigoplus_{n\geq 0} V^{n\otimes}$$

with the concatenation product. The unit $V \to U(T(V))$ of the adjunction is the inclusion $V \to T(V)$ into the summand for $n = 1$, and the counit $T(U(A)) \to A$ sends $a_1 \otimes \dots \otimes a_n$ in the n'th summand to the product $a_1 \dots a_n$ in A. It further sends λ in the 0'th summand $\mathbb{C} = V^{0\otimes}$ to $\lambda 1_A$.

The functor $T^{\geq 1}$ is the subfunctor of T which sends V to

$$T^{\geq 1}(V) := \bigoplus_{n\geq 1} V^{n\otimes}\ .$$

The unit and counit are given by the same description. One checks that for A in $\mathbf{Alg}_{\mathbb{C}}$

$$\mathtt{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V,U(A)) \stackrel{T}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(T(V),T(U(A)) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(T(V),A)$$

and for A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$

$$\mathtt{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V,U(A)) \stackrel{T^{\geq 1}}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(T^{\geq 1}(V),T^{\geq 1}(U(A)) \stackrel{counit_*}{\to} \mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(T^{\geq 1}(V),A)$$

are bijections. The inverses can be constructed explicitly using the units. □

Let $S : \mathbf{Vect}_{\mathbb{C}} \to \mathbf{Set}$ be the functor which takes the underlying set. We have an adjunction

$$\mathbb{C}[-] : \mathbf{Set} \leftrightarrows \mathbf{Vect}_{\mathbb{C}} : S \ ,$$

where the right-adjoint takes the underlying set of a vector space. The vector space $\mathbb{C}[X]$ is the vector space generated by the set X. Composing this adjunction with the adjunctions in Lemma 1.29 we get:

Corollary 1.30. *We have adjunctions*

$$\mathrm{Free} : \mathbf{Set} \leftrightarrows \mathbf{Alg}_{\mathbb{C}} : S \ , \quad \mathrm{Free}^{\mathrm{nu}} : \mathbf{Set} \leftrightarrows \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : S \ , \tag{1.1}$$

where

$$\mathrm{Free}(X) := T(\mathbb{C}[X]) \quad \textit{and} \quad \mathrm{Free}^{\mathrm{nu}}(X) := T^{\geq 1}(\mathbb{C}[X])$$

The right-adjoints take the underlying sets of an algebra.

The algebras $\mathrm{Free}(X)$ and $\mathrm{Free}^{\mathrm{nu}}(X)$ are called the free unital and non-unital algebras generated by the set X.

Recall that a category is complete if it admits limits for all small diagrams. Similarly it is called cocomplete if it admits colimits for all small diagrams.

Proposition 1.31. *The categories $\mathbf{Alg}_{\mathbb{C}}$ and $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are complete and cocomplete.*

Proof. (sketch) In view of the adjunctions (1.1) the forgetful functor from algebras to sets preserves limits. Thus limits in $\mathbf{Alg}_{\mathbb{C}}$ and $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are obtained by forming the limits in $\mathbf{Set}$ and equipping the results with the induced vector space and algebra structures. In order to show completeness it suffices to show the existences of products of small families and equalizers. We discuss products and equalizers in the Examples 1.32 and 1.33 below.

The explicit description of colimits is more complicated. But to show cocompleteness it suffices to show the existence of coproducts of small families and coequalizers which are discussed in Examples 1.34 and 1.35 below. □

Example 1.32. Let $(A_i)_{i\in I}$ be a family of algebras. Then the product of the family is given by the product of the underlying sets $\prod_{i\in I} A_i$ with the factorwise operations. The structure maps are the projections to the factors.

If the family consists of unital algebras, then $(1_{A_i})_{i\in I}$ is the unit of the product.

If $(A_i)_{i\in I}$ is a finite family, then $\prod_{i\in I} A_i \cong \bigoplus_{i\in I} A_i$.

If $(A)_{x\in X}$ is a constant family, then $\prod_{x\in X} A \cong A^X$. □

Example 1.33. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a diagram in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. The limit of this diagram is called the equalizer of f and g and given by the subalgebra

$$\mathrm{Eq}(f,g) := \{a \in A \mid f(a) = g(a)\}$$

of A. The structure map is the canonical inclusion $\mathrm{Eq}(f,g) \to A$.

If A, B are unital and f and g preserve units, then $1_A \in \mathrm{Eq}(f,g)$ so that the equalizer is unital. □

Example 1.34. In this example we describe coproducts in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. For every A in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ we have an exact sequence

$$0 \to I_A \to \mathrm{Free}^{\mathrm{nu}}(S(A)) \stackrel{counit}{\to} A \to 0 ,$$

where the ideal I_A is defined as the kernel of the counit.

Let A, B be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Then we form $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$. The map $S(A) \to S(A) \sqcup S(B)$ induces a map

$$I_A \to \mathrm{Free}^{\mathrm{nu}}(S(A)) \to \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$$

and similarly for B. We let I be the two-sided ideal in $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))$ generated by the images of I_A and I_B. Then we have factorizations

$$\begin{array}{ccc} \mathrm{Free}^{\mathrm{nu}}(S(A)) & \longrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B)) \\ \downarrow{\scriptstyle counit} & & \downarrow \\ A & \dashrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Free}^{\mathrm{nu}}(S(B)) & \longrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B)) \\ \downarrow{\scriptstyle counit} & & \downarrow \\ B & \dashrightarrow & \mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I \end{array}$$

These maps present $\mathrm{Free}^{\mathrm{nu}}(S(A) \sqcup S(B))/I$ as the coproduct $A \sqcup B$ in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

It is often denoted by $A * B$ and called the free product of A and B.

Note that for commutative algebras A, B the coproduct $A * B$ in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ is non-commutative and differs from the coproduct in commutative algebras which is given by the vector space $A \otimes B$ with the induced algebra structure.

E.g. in $\mathbb{C}[x] * \mathbb{C}[y]$ we have $xy \neq yx$.

A similar construction works for $\mathbf{Alg}_{\mathbb{C}}$ and for coproducts of arbitrary families of objects. □

Example 1.35. In this example we describe coequalizers. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a coequalizer diagram in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Then the elements $f(a) - g(a)$ for all a in A generate a two-sided ideal I in B. The projection $B \to B/I$ presents B/I as the coequalizer $\mathrm{Coeq}(f, g)$.

The same construction works in the unital case. □

Example 1.36. The initial algebra in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ is the coproduct of the empty family. It is the zero algebra 0. Note that the zero algebra happens to be unital with unit $1_0 = 0$. But it is not the initial object in $\mathbf{Alg}_{\mathbb{C}}$. E.g. there is no unital morphism $0 \to \mathbb{C}$. In fact, $\mathbb{C}$ is an initial algebra in $\mathbf{Alg}_{\mathbb{C}}$.

The zero algebra is the final algebra in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and $\mathbf{Alg}_{\mathbb{C}}$. □

Example 1.37. An exact sequence $0 \to I \to A \to A/I \to 0$ of algebras in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ can be interpreted as diagram

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A/I \end{array}$$

which is a push-out and a pull-back at the same time. □

We finally introduce the concept of the spectrum of an element in an algebra. Let A be in $\mathbf{Alg}_{\mathbb{C}}$ and a be in A.

Definition 1.38. *a is invertible if there exists and element b in A such that $ba = 1_A$ and $ab = 1_A$.*

The element b is called and inverse of a.

Lemma 1.39. *An inverse is uniquely determined.*

Proof. Let b, b' be two inverses of a. Then we have $b = b1_A = bab' = 1_A b' = b'$. □

We usually use the notation a^{-1} for the inverse of a.

Let $f : A \to B$ be a morphism in $\mathbf{Alg}_{\mathbb{C}}$ and a be in A.

Lemma 1.40. *If a is invertible, then $f(a)$ is invertible and $f(a)^{-1} = f(a^{-1})$.*

Proof. We calculate $f(a)f(a^{-1}) = f(aa^{-1}) = f(1_A) = 1_B$ and similarly $f(a^{-1})f(a) = 1_B$. $\square$

Note that we use in this proof that f preserves units.

In the following for A in $\mathbf{Alg}_{\mathbb{C}}$ we use the notation

$$\lambda := \lambda 1_A$$

Let a be in A.

Definition 1.41. *The spectrum of a is the set*

$$\sigma(a) := \{\lambda \in \mathbb{C} \mid (\lambda - a) \text{ is not invertible in } A\} .$$

The complement $\rho(a) := \mathbb{C} \setminus \sigma(a)$ is called the resolvent set.

Example 1.42. Let a be in $\mathtt{Mat}(n)$. Then the spectrum $\sigma(a)$ is set of eigenvalues of a. $\square$

Example 1.43. For a in $\mathbb{C}^X$ we have $\sigma(a) = a(X)$ $\square$

Let A be in $\mathbf{Alg}_{\mathbb{C}}$ and a be in A. Let p be in $\mathbb{C}[x]$. Then we can form $p(a)$ in A in the obvious way.

Lemma 1.44. *We have $p(\sigma(a)) \subseteq \sigma(p(a))$.*

Proof. Since $p(\lambda) - p(x)$ vanishes at $x = \lambda$ we can write $p(\lambda) - p(x) = (\lambda - x)q(x)$ for some q in $\mathbb{C}[x]$. The equality

$$p(\lambda) - p(a) = (\lambda - a)q(a)$$

implies that if the left-hand side is invertible, so the two factors on the right-hand side. $\square$

Let $f : A \to B$ be morphism in $\mathbf{Alg}_{\mathbb{C}}$ and a in A.

Lemma 1.45. *We have $\sigma(f(a)) \subseteq \sigma(a)$.*

Proof. Consider λ in $\mathbb{C}$. If $\lambda \notin \sigma(a)$, then $(\lambda - a)^{-1}$ exists in A and hence $f((\lambda - a)^{-1}) = (\lambda - f(a))^{-1}$ exists in B. Hence $\lambda \notin \sigma(f(a))$ $\square$

Example 1.47 shows that $\sigma(f(a))$ can be strictly smaller than $\sigma(a)$.

We consider A in $\mathbf{Alg}_{\mathbb{C}}$ and a in A.

Definition 1.46. *We define the spectral radius* $r(a) := \sup |\sigma(a)|$.

It is an element of $[-\infty, \infty]$. We have $-\infty$ iff $\sigma(a) = \emptyset$ and ∞ if $\sigma(a)$ is unbounded

Example 1.47. We consider x in $\mathbb{C}[x]$. Then $\sigma(x) = \mathbb{C}$. We have $r(x) = \infty$.

If we consider x in $\mathbb{C}(x)$ (the quotient field of $\mathbb{C}[x]$), then $\sigma(x) = \emptyset$. In this case $r(x) = -\infty$. □

We consider a morphism $f : A \to B$ in $\mathbf{Alg}_{\mathbb{C}}$ and a in A. The following is immediate from Lemma 1.45.

Corollary 1.48. $r(f(a)) \leq r(a)$

By Example 1.47 the inequality in Corollary 1.48 can be strict.

The notion of the spectrum is extended to the non-unital case as follows. Let A be in $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ and a be in A. Then we consider $(a, 0) \in A^u$

Definition 1.49. *We define* $\sigma^u(a) := \sigma((a, 0))$.

We always have $0 \in \sigma^u(a)$.

Lemma 1.50. *If A is unital, then* $\sigma^u(a) = \sigma(a) \cup \{0\}$.

Proof. We use that $A^u \cong A \oplus \mathbb{C}$ is given by $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$. Under this identification $\lambda 1_{A^u} - (a, 0) \mapsto (\lambda 1_A - a, \lambda)$. We read off that λ in $\sigma^u(a)$ iff $\lambda \in \sigma(a)$ or $\lambda = 0$ □

2 Banach algebras

We consider a norm $\|-\|$ on a vector space V. The pair $(V, \|-\|)$ is called a normed vector space. The norm induces a metric $d(v, v') := \|v - v'\|$ on V. A normed vector space is called complete if it is complete (in the sense of metric spaces) with respect to this metric.

A Banach space is a topological vector space whose topology is induced from a norm and which is complete with respect to the induced metric. This condition does not depend on the choice of the metric. In other words, when we talk about Banach spaces, then we only care about the topology, but not about the specific norm generating the topology.

For a Banach space we let B^* denote the space of continuous linear maps $B \to \mathbb{C}$. It will be equipped with the topology of uniform convergence on bounded subsets of B (defined using some norm). It is again a Banach space. By the Hahn-Banach theorem we know that the canonical map $B \to (B^*)^*$ is injective.

Let B be a Banach space, U be an open subset of $\mathbb{C}$, and $f : U \to B$ be a continuous function. We consider a curve $\gamma : [0,1] \to U$. Then we can consider the Riemann integral

$$\int_\gamma f(z)dz \ .$$

It is an element of B defined as the limit over Riemann sums

$$\sum_{i=1}^{n} f(\gamma(t_i))(t_i - t_{i-1})$$

over the filtered poset of finite partitions of the interval $[0,1]$ here given as sequences

$$0 = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = 1 \ .$$

Thereby the partitions are are partially ordered by refinement. One shows convergence using uniform continuity of $f \circ \gamma$ in the same way as in scalar case. Note that here we use the completeness of B in order to ensure existence of the limit.

If $A : B \to B'$ is a linear continuous map between Banach spaces, then it preserves integrals:

$$A(\int_\gamma f(z)dz) = \int_\gamma A(f(z))dz \ .$$

One can talk about holomorphic functions $f : U \to B$. But for simplicity in the course we will only use the notion of weak holomorphy which reduces everything to the scalar case.

Definition 2.1. *f is called weakly holomorphic if $\phi(f) : U \to \mathbb{C}$ is holomorphic for every continuous functional ϕ in B^**

Lemma 2.2. *If f is weakly holomorphic and γ is closed and contractible in U, then we have $\int_\gamma f(z)dz = 0$.*

Proof. Since $B \to (B^*)^*$ is injective it suffices to show that

$$\phi(\int_\gamma f(z)dz) = 0$$

for all ϕ in B^*. Since ϕ is continuous we have

$$\phi(\int_\gamma f(z)dz) = \int_\gamma \phi(f(z))dz \ .$$

Finally by the Cauchy integral theorem we have

$$\int_\gamma \phi(f(z))dz = 0 \ .$$

□

We now consider the interplay between norms and the product on an algebra. Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and $\|-\|$ be a norm on the underlying vector space.

Definition 2.3. *We say that $\|-\|$ is sub-multiplicative if $\|aa'\| \le \|a\|\|a'\|$ for all a, a' in A.*

Definition 2.4. *A normed algebra $(A, \|-\|)$ is pair of an algebra and a sub-multiplicative norm.*

We add the adjective unital in order to express the compatibility of the norm with the unit of the algebra.

Definition 2.5. *A unital normed algebra $(A, \|-\|)$ is a normed algebra such that $A \in \mathbf{Alg}_{\mathbb{C}}$ and $\|1_A\| = 1$.*

Example 2.6. Let $(V, \|-\|_V)$ be a finite-dimensional normed vector space. Then $\mathtt{End}(V)$ has the operator norm given by

$$\|A\| := \sup_{v \in V, \|v\|_V = 1} \|Av\|_V \ .$$

Note that we use finite-dimensionality in order to ensure that the norm is finite. Then $(\mathtt{End}(V), \|-\|)$ is a unital normed algebra.

If V is not finite-dimensional, then the set of endomorphisms with finite norm forms a unital subalgebra of $\mathtt{End}(V)$ which is unitally normed. □

Example 2.7. We consider a normed algebra $(A, \|-\|)$ and assume that A is unital. We explicitly do not require that A is unitally normed. In general we then have

$$\|1_A\| \ge 1 \ .$$

In order to see this we start with

$$1_A^n = 1_A \ .$$

Using the sub-multiplicativity of the norm we get

$$\|1_A\| = \|1_A^n\| \le \|1_A\|^n \ .$$

We insert $n = 2$ and conclude the desired inequality. In Example 2.8 we show that this inequality may be strict. □

Example 2.8. We consider a unital normed algebra $(A, \|-\|)$. Then we set $\|-\|' := 2\|-\|$. Then $(A, \|-\|')$ is normed algebra, but $\|1_A\| = 2 \neq 1$ □

Recall that two norms $\|-\|$ and $\|-\|'$ on a vector space V are called equivalent if there exists C in $(0, \infty)$ such that

$$C^{-1}\|v\| \le \|v\|' \le C\|v\|$$

for all v in V. The equivalence class of norms defining the topology of a Banach space is uniquely determined.

Let $(A, \|-\|)$ be a normed algebra and A be unital.

Lemma 2.9. *There is an equivalent sub-multiplicative norm $\|-\|'$ on A with $\|1_A\|' = 1$.*

Proof. We have a map

$$A \to \mathtt{End}_{\mathbf{Vect}_{\mathbb{C}}}(A) \ , \quad b \mapsto (a \mapsto ba) \ .$$

The operator norm on $\mathtt{End}_{\mathbf{Vect}_{\mathbb{C}}}(A)$ induces a sub-multiplicative seminorm (i.e. we allow that $\|a\|' = 0$ for non-zero a) $\|-\|'$ on A such that $\|1_A\|' = 1$.

We must show that $\|-\|'$ and $\|-\|$ are equivalent. This also implies that $\|-\|'$ is a norm.

For all b in A we have

$$\|b\|' = \sup_{a, \|a\|=1} \|ba\| \le \|b\|$$

and

$$\|b\| = \|b1_A\| \le \|b\|' \|1_A\| \ .$$

□

Definition 2.10. *A (unital) Banach algebra is a (unital) algebra A with admits a norm $\|-\|$ such that the underlying normed vector space is a Banach space.*

Example 2.11. Let H be a Hilbert space. Then $\mathbb{B}(H)$ (with the operator norm) is a unital Banach algebra.

If we specialize H to $\mathbb{C}^n$ with the standard scalar product and associated norm, then we see that $\mathtt{Mat}(n, \mathbb{C}) \cong \mathbb{B}(\mathbb{C}^n)$ is a Banach algebra. □

Example 2.12. We consider a topological space X and a Banach algebra A with norm $\|-\|$. The subalgebra $C_b(X, A)$ of the space $C(X, A)$ of bounded continuous functions from X to A is a Banach algebra with respect to sup-norm

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\| \ .$$

If A is unital, then so is $C_b(X, A)$.

We can also consider the closure $\overline{C}_c(X, A)$ of the compactly supported functions with respect to $\|-\|_\infty$. If X is not compact, then this a is non-unital Banach algebra. Note that if X is not locally compact, then $\overline{C}_c(X, A)$ might be very small. □

Next we show that unitalization preserves Banach algebras. Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Lemma 2.13. *If A is Banach, then A^u is unital Banach.*

Proof. We define a norm $\|(a, \lambda)\| := \|a\| + |\lambda|$ and check the submultiplicativity

$$\|(a, \lambda)(a', \lambda')\| = \|aa' + \lambda a' + \lambda' a\| + |\lambda\lambda'| \le (\|a\| + |\lambda|)(\|a'\| + |\lambda'|) \ .$$

It is clear that A^u with this norm is Banach.

Furthermore we have $\|(0,1)\| = 1$ □

We now study the spectrum of elements in a Banach algebra using analytic arguments. Let A be a unital Banach algebra.

Lemma 2.14. *The set $GL_1(A)$ of invertible elements in A is open. Furthermore the map $a \mapsto a^{-1}$ is continuous.*

Proof. Let a be in $GL_1(A)$ and set $r := \|a^{-1}\|$. If b is in A such that $\|a-b\| < r$, then b is in $GL_1(A)$. In order to see this we write

$$b = a + (b-a) = a(1_A + a^{-1}(b-a)) \ .$$

Then

$$b^{-1} = (1_A + a^{-1}(b-a))^{-1}a^{-1} \ ,$$

where

$$(1_A + a^{-1}(b-a))^{-1} = \sum_{n=0}^{\infty} (-1)^n [a^{-1}(b-a)]^n \ .$$

The sum (Neumann series) converges absolutely since $\|a^{-1}(b-a)\| \leq r\|b-a\| < 1$. The identity

$$(1_A + a^{-1}(b-a)) \sum_{n=0}^{n} (-1)^n [a^{-1}(b-a)]^n = 1_A + (-1)^n [a^{-1}(b-a)]^{n+1}$$

shows by considering the limit as $n \to \infty$ that the infinite sum represents the inverse of $1_A + a^{-1}(b-a)$.

We furthermore get

$$b^{-1} - a^{-1} = \sum_{n=1}^{\infty} (-1)^n [a^{-1}(b-a)]^n a^{-1} \ .$$

As long as $\|b-a\| \leq r/2$ this gives an estimate

$$b^{-1} - a^{-1} \leq \|b-a\| C$$

for some constant which does not depend on b. This shows the continuity of the inverse map. □

We consider a unital Banach algebra A with norm $\|-\|$ and a in A. Recall from Definition 1.46 that $r(a)$ denotes the spectral radius of a.

Corollary 2.15. $r(a) \leq \|a\|$.

Proof. Assume that λ is in $\mathbb{C}$ and $|\lambda| > \|a\|$. Then $\lambda - a = \lambda(1 - \lambda^{-1}a)$ is invertible since $\|\lambda^{-1}a\| < 1$. $\square$

We consider a unital Banach algebra A and a in A.

Lemma 2.16. *$\rho(a)$ is open and*

$$\rho(a) \ni \lambda \mapsto (\lambda - a)^{-1} \in A$$

is continuous and weakly holomorphic.

Proof. The map

$$\mathbb{C} \ni \lambda \mapsto \lambda - a \in A$$

is continuous. Hence the set $\rho(a)$ is open since it is the preimage of the open subset $GL_1(A)$ of A under this map. Furthermore, $\lambda \mapsto (\lambda - a)^{-1}$ is continuous on $\rho(a)$. It remains to show weak holomorphy. Let ϕ be in A^*. Then for μ in $\rho(a)$ and λ close to μ the Neumann series implies the formula

$$(\lambda - a)^{-1} = \sum_{k=0}^{\infty} (\mu - a)^{-k-1}(\mu - \lambda)^k$$

and hence

$$\phi((\lambda - a)^{-1}) = \sum_{k=0}^{\infty} \phi((\mu - a)^{-k-1})(\mu - \lambda)^k$$

where we have used the continuity of ϕ in order to bring it inside of the sum. We already know from the proof of Lemma 2.14 that this sum converges absolutely for λ near μ and therefore defines a holomorphic function in λ. $\square$

We consider a non-zero unital Banach algebra A and a in A.

Lemma 2.17 (Formula for spectral radius)**.** *We have the equality*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} .$$

Proof. Note that this includes the assertion that the limit exists.

If λ is in $\sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by Lemma 1.44 applied to $p(x) = x^n$. This implies

$$|\lambda|^n \leq \|a^n\|$$

for all n in $\mathbb{N}$ and hence

$$|\lambda| \leq \liminf_{n \in \mathbb{N}} \|a^n\|^{1/n} .$$

We conclude that

$$r(a) \leq \liminf_{n\in\mathbb{N}} \|a^n\|^{1/n} \ . \tag{2.1}$$

We now take s in $(0,\infty)$ and use the notation $B_s := \{|z| < s\}$ for the s-ball and $S_s := \partial B_s$ for its boundary. We assume that $r(a) < s$. Then $\lambda \mapsto (\lambda - a)^{-1}$ exists and is continuous and weakly holomorphic on an open neighbourhood of $\mathbb{C} \setminus B_s$. We claim that

$$a^n = \frac{1}{2\pi i} \int_{S_{s'}} \frac{\lambda^n d\lambda}{\lambda - a} \tag{2.2}$$

for all s' in $(0,\infty)$ with $s' \geq s$. Indeed, the right-hand side is independent of s' (apply arbitrary ϕ in A^* and the Cachy integral theorem) and for $s' > \limsup_{n\to\infty} \|a^n\|^{1/n}$ it is equal to

$$\frac{1}{2\pi i} \int_{S_{s'}} \sum_{k=0}^{\infty} \lambda^{n-1-k} a^k d\lambda = a^n \ .$$

We estimate the norm of the integral by the integral of the norm of the integrand. Then we get the estimate

$$\|a^n\| \leq \sup_{z\in S_s} \|\frac{1}{z-a}\| s^{\prime,n}$$

We now take the n'th root and consider the limit of the right-hand side as $n \to \infty$. Since we can choose s' arbitrary close to s we get

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq s \ .$$

We now vary s and get

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq \max\{0, r(a)\} \ . \tag{2.3}$$

We exclude the case $r(a) = -\infty$ as follows. In this case $(\lambda - a)^{-1}$ is holomorphic on all of $\mathbb{C}$. By (2.2) for $n = 1$ get get $a = 0$. But in this case, since $A \neq 0$, we get $r(0) = 0$ a contradiction.

Combining (2.1) and (2.3) we get the desired assertion. □

Corollary 2.18. *If A is a non-zero unital Banach algebra and a is in A, then $\sigma(a) \neq \emptyset$*

Proof. This follows from $0 \leq r(a)$. □

Example 2.19. If $A = 0$, then $r(0) = -\infty$ since $\sigma(0) = \emptyset$. In view of Corollary 2.18 the zero algebra is the only Banach algebra which contains elements with empty spectrum. □

Corollary 2.20 (Gelfand-Mazur)**.** *Assume that A is unital Banach algebra such that every non-zero element is invertible. Then $A \cong \mathbb{C}$.*

Proof. We argue by contradiction. Let a be in A and assume that $a \neq \lambda$ for all λ in $\mathbb{C}$. Then $\lambda - a$ is invertible for all λ in $\mathbb{C}$ This implies that $\sigma(a) = \emptyset$ and hence $r(a) = -\infty$. This is a contradiction.

□

We consider a unital Banach algebra A and an ideal I in A. Then also $\bar{I}$ is an ideal.

Lemma 2.21. *If I is proper, then also $\bar{I}$.*

Proof. We show that for every b in I we have $\|1_A - b\| \geq 1$. This then implies that $1_A \notin \bar{I}$.

Assume by contradiction that $\|1 - b\| < 1$. Then b is invertible and $I = A$, a contradiction. □

3 $*$-algebras and C^*-algebras

In this section in consider algebras with an involution called $*$-algebras. We investigate the categories unital and non-unital of $*$-algebras. We then introduce the condition of compatibility of a norm with the involution called the C^*-equality and study its consequences.

Let A be in $\mathbf{Alg}_{\mathbb{C}}$.

Definition 3.1. *An involution on A is a complex antilinear map $* : A \to A$ (written on elements as $a \mapsto a^*$) such that:*

1. $* \circ * = \mathtt{id}$
2. $(aa')^* = a'^* a^*$

Definition 3.2. *A $*$-algebra is an algebra over $\mathbb{C}$ with an involution. A $*$-homomorphism between $*$-algebras is a homomorphism between algebras which preserves the involution.*

A $*$-homomorphism f between $*$-algebras is thus a homomorphism between algebras which in addition satisfies the identity $f(a^*) = f(a)^*$. By ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ we denote the categories of unital and not-necessarily unital $*$-algebras and corresponding $*$-homomorphisms.

Example 3.3. Complex conjugation is an involution on the algebra $\mathbb{C}$ turning it into a unital $*$-algebra.

If A is a $*$-algebra and X is a set, then A^X is a $*$-algebra with the involution defined pointwise by $(a^*)(x) := a(x)^*$.

If B is a subalgebra of a $*$-algebra which is invariant (as a set) by the involution, then B with the restriction of the involution to B is a $*$-algebra. $\square$

Example 3.4. If H is a Hilbert space, then $B(H)$ is a $*$-algebra with the involution which sends an operator to its adjoint. The algebra of compact operators $K(H)$ is a $*$-subalgebra.

For $H = \mathbb{C}^n$ we get an involution on $\mathtt{Mat}(n)$. In this case $a^* = \bar{a}^t$, the complex conguate of the transposed matrix. $\square$

Example 3.5. If G is a group, then on the group ring $\mathbb{C}[G]$ we have the involution given by

$$(\sum_{g\in G} \lambda_g[g])^* := \sum_{g\in G} \bar{\lambda}_g[g^{-1}] \ .$$

Note that here we need inverses in G and this definition does not work for general monoids. $\square$

Example 3.6. The polynomial ring $\mathbb{C}[z]$ has an involution extending $z^* = z$. This extends further to the quotient field $\mathbb{C}(z)$

The differential operators $D(\mathbb{C})$ have the involution determined by $\partial^* := -\partial$ and $z^* := z$. Indeed the defining relation $\partial z - z\partial = 1$ is preserved:

$$1^* = 1 \ , \quad (\partial z - z\partial)^* = z^*\partial^* - \partial^* z* = -z\partial - (-\partial z) = \partial z - z\partial \ .$$

$\square$

Let I be a two-sided ideal in a $*$-algebra A. For a subset S of A we write $S^* = \{s^* | s \in S\}$.

Definition 3.7. *I is a $*$-ideal if $I = I^*$.*

Example 3.8. If I is a $*$-subalgebra and a left or right-ideal, then it is automatically a $*$-ideal. Assume that I is right ideal and a $*$-subalgebra. Then for every a in A we have $ai = (i^*a^*)^* \in I^* = I$, hence I is also a left ideal. $\square$

Example 3.9. Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and I be a $*$-ideal in A. Then A/I is $*$-algebra with involution given $[a]^* := [a^*]$. The map $A \to A/I$ is initial for $*$-homomorphisms from A to $*$-algebras which send the elements of I to zero.

Example 3.10. If S is a a subset of A, then we can form the smallest $*$-ideal

$${}^*(S) := \bigcap_{S\subseteq I} I$$

containing S, where the intersection runs over all $*$-ideals I of A containing S. In generaal ${}^*(S)$ can be larger than the two-sided ideal (S) generated by S. It is easy to see that ${}^*(S) = (S \cup S^*)$. The quotient map

$$A \to A/{}^*(S)$$

is initial for $*$-homomorphisms from A to $*$-algebras which send the elements of S to zero. $\square$

The unitalization adjunction extends to $*$-algebras.

Lemma 3.11. *We have an adjunction*

$$(-)^u : {}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : \mathrm{incl}$$

Proof. Let A in ${}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$. Then A^u has an involution given by $(a,\lambda)^* := (a^*, \bar{\lambda})$. One checks that the unit and counit of the adjunction from Lemma 1.27 are morphisms of $*$-algebras. This implies the assertion. $\square$

The underlying set $S(A)$ of a $*$-algebra carries an action of the group C_2 such that the non-trivial element σ in C_2 acts by $\sigma a := a^*$. We therefore have a forgetful functor $S : {}^*\mathbf{Alg}_{\mathbb{C}} \to C_2\mathbf{Set}$.

Lemma 3.12. *We have adjunctions*

$$\mathrm{Free}^{*,\mathrm{nu}} : C_2\mathbf{Set} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}} : S \ , \quad \mathrm{Free}^* : C_2\mathbf{Set} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : S \ .$$

Proof. (sketch) We give an explicit construction of the left-adjoinst. For X in $C_2\mathbf{Set}$ we first consider the vector space $\mathbb{C}[X]$ in with the anti-linear action of C_2 which extends the action on X. So

$$\sigma(\lambda_1[x_1] + \cdots + \lambda_n[x_n]) = \bar{\lambda}_1[\sigma x_1] + \cdots + \bar{\lambda}_n[\sigma x_n] \ .$$

Then we equip the tensor algebra $T(\mathbb{C}[X])$ with the anti-linear involution characterized by

$$(v_1 \otimes \cdots \otimes v_n)^* := \sigma(v_n) \otimes \cdots \otimes \sigma(v_1)$$

The resulting $*$-algebra in ${}^*\mathbf{Alg}_{\mathbb{C}}$ will be denoted by $\mathrm{Free}^*(X)$. It is the free unital $*$-algebra generated by the set X. It is straightforward to extend this construction to morphisms of C_2-sets so that we obtain the functor $\mathrm{Free}^* : \mathbf{Set} \to {}^*\mathbf{Alg}_{\mathbb{C}}$.

The unit of the adjunction is given by the canonical inclusion of C_2-sets $X \mapsto S(\mathrm{Free}^*(X))$ sending x in X to the basis vector $[x]$ of $\mathbb{C}[X]$ considered as a summand of $\mathrm{Free}^*(X)$. One then checks that for every A in ${}^*\mathbf{Alg}_{\mathbb{C}}$ the composition

$$\mathtt{Hom}_{{}^*\mathbf{Alg}_{\mathbb{C}}}(\mathrm{Free}^*(X), A) \stackrel{S}{\to} \mathtt{Hom}_{C_2\mathbf{Set}}(S(\mathrm{Free}^*(X)), S(A)) \stackrel{unit^*}{\to} \mathtt{Hom}_{C_2\mathbf{Set}}(X, S(A))$$

is a bijection.

The non-unital case is similar with $\mathrm{Free}^{*,\mathrm{nu}} := T^{\geq 1}(\mathbb{C}[X])$. $\square$

Let $\mathcal{F} : {}^*\mathbf{Alg}_{\mathbb{C}} \to \mathbf{Alg}_{\mathbb{C}}$ or $\mathcal{F}^{\mathrm{nu}} : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ denote the functors which forget the $*$-operation.

Lemma 3.13. *We have adjunctions*

$$L : \mathbf{Alg}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : \mathcal{F} \ , \quad L^{\mathrm{nu}} : \mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : \mathcal{F}^{\mathrm{nu}} \ .$$

Proof. (sketch) Let A be in $\mathbf{Alg}_{\mathbb{C}}$ and consider the exact sequence

$$0 \to I \to \mathrm{Free}(A) \to A \to 0 \ .$$

We form the C_2-set $A \sqcup A$ with the action of C_2 flipping the components. Let $i_0, i_1 : A \to A \sqcup A$ denote the two inclusions. We have a homomorphisms

$$\mathrm{Free}(i_0) : \mathrm{Free}(A) \to \mathrm{Free}(A \sqcup A)$$

induced by the inclusion of the first copy of A.

We now note that $\mathrm{Free}(A \sqcup A)$ has the structure of a $*$-algebra $\mathrm{Free}^*(A \sqcup A)$ as in Lemma 3.12. We let J be the $*$-ideal generated by $\mathrm{Free}(i_0)(I)$ and form $L(A) := \mathrm{Free}(A \sqcup A)/J$. We then have the diagram in $C^*\mathbf{Alg}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathrm{Free}(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Free}(i_0)} & & \downarrow{\scriptstyle unit} & & \\
0 & \longrightarrow & \mathcal{F}(J) & \longrightarrow & \mathcal{F}(\mathrm{Free}^*(A \sqcup A)) & \longrightarrow & \mathcal{F}(L(A)) & \longrightarrow & 0
\end{array}$$

defining the unit map.

It is straightforward to define L on morphisms so that one gets a functor $L : \mathbf{Alg}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$. Finally one checks that for every B in ${}^*\mathbf{Alg}_{\mathbb{C}}$ the composition

$$\mathtt{Hom}_{{}^*\mathbf{Alg}_{\mathbb{C}}}(L(A), B) \stackrel{\mathcal{F}}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(\mathcal{F}(L(A)), \mathcal{F}(B)) \stackrel{unit^*}{\to} \mathtt{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(A, \mathcal{F}(B))$$

is a bijection.

The non-unital case is analoguous using $\mathrm{Free}^{\mathrm{nu}}$. $\square$

Proposition 3.14. *The categories ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are complete and cocomplete.*

Proof. By Lemma 3.13 and Corollary 1.30 the forgetful functors ${}^*\mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to \mathbf{Set}$ are right-adjoints. Therefore limits in ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are calculated on the level of underlying sets, respectively. The results then equipped with an involution induced by functoriality.

The argument for colimits and is similar as in Prop. 1.31. It is enough to show the existence of coequalizers and coproducts. This is done in Examples 3.15 and 3.16 below. $\square$

Example 3.15. Let $(A_i)_{i\in I}$ be a family in ${}^*\mathbf{Alg}_{\mathbb{C}}$. Then we consider for every i the exact sequence

$$0 \to I_i \to \mathrm{Free}^*(A_i) \to A_i \to 0 \ .$$

Then we form the $*$-algebra $\mathrm{Free}^*(\bigsqcup_{i\in I} A_i)$ and the $*$-ideal J generated by the images of the ideals I_i. The quotient $\mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J$ together with the family $(e_i)_{i\in I}$ of $*$-homomorphisms $e_i : A_i \to \mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J$ induced by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_i & \longrightarrow & \mathrm{Free}^*(A_i) & \longrightarrow & A_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle e_i} & & \\
0 & \longrightarrow & J & \longrightarrow & \mathrm{Free}^*(\bigsqcup_{i\in I} A_i) & \longrightarrow & \mathrm{Free}^*(\bigsqcup_{i\in I} A_i)/J & \longrightarrow & 0
\end{array}$$

represent the coproduct of the family.

The same construction works in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ using $\mathrm{Free}^{*,\mathrm{nu}}$ in place of Free^*.

Example 3.16. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a coequalizer diagram in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ or ${}^*\mathbf{Alg}_{\mathbb{C}}$. Then the elements $f(a) - g(a)$ for all a in A generate a $*$-ideal I in B. The projection $B \to B/I$ presents B/I as the coequalizer $\mathrm{Coeq}(f, g)$ in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ or ${}^*\mathbf{Alg}_{\mathbb{C}}$, respectively.

We consider an algebra A with involution $*$ and a norm $\|-\|$.

Definition 3.17. $(A, *, \|-\|)$ *is a normed $*$-algebra if* $\|a^*\| = \|a\|$ *for all a in A.*

In other words, for a normed $*$-algebra we require that $*$ acts isometrically.

Example 3.18. If $\|-\|$ is any norm on a $*$-algebra, then we can form a new norm $\|-\|'$ by

$$\|-\|' = \max\{\|a\|, \|a^*\|\} \ .$$

Then $(A, *, \|-\|')$ is a normed $*$-algebra.

In general it is not clear that $\|-\|'$ is equivalent to $\|-\|$. But if A is a Banach algebra with norm $\|-\|$ and $*$ is continuous, then $\|-\|'$ is equivalent to $\|-\|$. □

Usually we use the symbol A in order to denote a $*$-algebra or a normed $*$-algebra.

Let A be a normed $*$-algebra

Definition 3.19. $\|-\|$ *has the C^*-property if* $\|a^*a\| = \|a\|^2$ *for all a in A.*

We call such a norm a C^*-norm

Definition 3.20. *A C^*-$*$-algebra is a $*$-algebra which admits a norm turning it into a normed Banach algebra and which satisfies the C^*-property.*

Remark 3.21. Being C^*-algebra is property of a $*$-algebra. Note that the norm is not part of the data for a C^*-algebra. We just require existence. But it will turn out later that it is actually uniquely determined.

In order to show that a given $*$-algebra is a C^*-algebra one usually produces a norm and shows that it has the required properties.

Our definition is not the standard definition of a C^*-algebra, but equivalent to it, as we shall see below. $\square$

We let $C^*\mathbf{Alg}^{\mathrm{nu}}$ be the full subcategory of ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of C^*-algebras and $C^*\mathbf{Alg}$ be the full subcategory of ${}^*\mathbf{Alg}_{\mathbb{C}}$ of unital C^*-algebras.

Example 3.22. If A is a C^*-algebra and B is $*$-subalgebra of A which is closed w.r.t a norm exhibiting A as a C^*-algebra, then B is a C^*-algebra, too.

Any closed $*$-subalgebra of $\mathbb{B}(H)$ is a C^*-algebra.

Actually, one classical definition of the notion of a C^*-algebra is as a closed $*$-subalgebra of $B(H)$ form some Hilbert space H. One can show that every C^*-algebra is isomorphic to such a subalgebra. $\square$

Example 3.23. Let X be a topological space and A be a C^*-algebra. Then $C_b(X, A)$ is again a C^*-algebra exhibited by the norm $\|a\|_\infty := \sup_{x\in X} \|a(x)\|_A$. Furthermore, $\overline{C}_c(X, A)$ is a closed subalgebra of $C_b(X, A)$ and hence a C^*-algebra. $\square$

We consider a C^*-algebra with norm $\|-\|$. Let a be in A. Note that $r(a^*a)$ (the spectral radius of a^*a) only depends on the algebra A and not on the norm.

Lemma 3.24 (C^*-norm completely determined by $*$-algebra structure)**.** *We have $\|a\|^2 = r(a^*a)$ for all a in A.*

Proof. We use the formula for the spectral radius of a^*a given in Lemma 2.17. For k in $\mathbb{N}$, using the C^*-property repeatedly, we have

$$\|a\|^{2k+1} = \|a^*a\|^{2^k} = \|(a^*a)^2\|^{2^{k-1}} = \cdots = \|(a^*a)^{2^k}\| \ .$$

This gives $\|a\|^2 = \|(a^*a)^{2^k}\|^{2^{-k}}$ for all k in $\mathbb{N}$. We take the limit as $k \to \infty$ and get $\|a\|^2 = r(a^*a)$. $\square$

Corollary 3.25. *A C^*-algebra has a unique C^*-norm.*

We now consider a morphism $f : A \to B$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Note that by definition f is just a homomorphism of algebras which is compatible with the $*$-operation. The following Lemma shows that it is a contraction (in particular continuous) provided we equip the algebras with their unique C^*-norms $\|-\|_A$ and $\|-\|_B$.

Corollary 3.26 (automatic continuity of morphisms). *For all a in A we have $\|f(a)\|_B \leq \|a\|_A$.*

Proof. We use Corollary 1.48 and Lemma 3.24 in order to calculate

$$\|f(a)\|_B^2 = r(f(a)^* f(a)) = r(f(a^*a)) \leq r(a^*a) = \|a\|_A^2 \ .$$

□

Next we show that unitalization is compatible with C^*-algebras.

Lemma 3.27. *We have an adjunction*

$$(-)^u : C^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Alg} : \mathrm{incl} \ .$$

Proof. We restrict the adjunction from Lemma 3.11 to C^*-algebras. It suffices to show that $(-)^u$ preserves C^*-algebras.

First assume that A is unital. Then we have an isomorphism of $*$-algebras $A^u \xrightarrow{\cong} A \oplus \mathbb{C}$ given by $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$. Therefore the norm $\|(a, \lambda)\| := \max\{|\lambda|, \|a + \lambda 1_A\|_A\}$ is a C^*-norm on A^u.

We now assume that A is non-unital. Note that the obvious norm on A^u is not a C^*-norm: We have

$$\|(a,\lambda)^*(a,\lambda)\| = \|(a^*a + \bar{\lambda}a + \lambda a^*, |\lambda|^2)\| = \|a^*a + \bar{\lambda}a + \lambda a^*\| + |\lambda|^2$$

which is in general not equal to

$$(\|a\| + |\lambda|)^2 = \|a^*a\| + 2|\lambda|\|a\| + |\lambda|^2 \ .$$

Note that A^u acts on A be left multiplication $(b, \lambda)a = (ba + \lambda a)$. We let $\|-\|'$ be the operator norm, i.e.

$$\|(b,\lambda)\|' := \sup_{a \in A \ , \|a\|=1} \|ba + \lambda a\| \ .$$

It is a $*$-norm.

We first verify the C^*-property for $\| - \|'$. We have

$$\begin{aligned}
\|(b,\lambda)^*(b,\lambda)\|' &= \sup_{\|a\|=1} \|b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a\| \\
&= \sup_{\|a\|=1} \|a^*\| \|b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a\| \\
&\geq \sup_{\|a\|=1} \|a^*(b^*ba + \bar{\lambda}ba + \lambda b^*a + |\lambda|^2 a)\| \\
&= \sup_{\|a\|=1} \|(a^*b^* + a^*\bar{\lambda})(ba + \lambda a)\| \\
&= \sup_{\|a\|=1} \|ba + \lambda a\|^2 = \|(b,\lambda)\|'^{,2}.
\end{aligned}$$

We get the reverse inequality

$$\|(b,\lambda)^*(b,\lambda)\|' \leq \|(b,\lambda)\|'^{,2}$$

using the sub-multiplicativity of the norm $\| - \|'$ (a general property of the operator norm) and the fact that it is a $*$-norm.

It remains to show that $\| - \|$ is equivalent to the obvious norm. This implies that A^u is complete w.r.t to $\| - \|'$. The inequality

$$\|(a,\lambda)\|' \leq \|a\| + |\lambda|$$

is clear. We claim that $\|(a,\lambda)\|' = 0$ implies $(a,\lambda) = 0$. This claim implies that $\| - \|'$ is equivalent to $\| - \|$. Indeed, if A' denotes the Banach-closure of A with respect to $\| - \|'$, then the map $A \to A'$ is a continuous surjective map of Banach spaces. The condition implies injectivity. Hence $A \to A'$ is an isomorphism by the bounded inverse theorem.

We now show the claim. Assume that $\lambda \neq 0$. Then $ab + \lambda b = 0$ for all b in A implies $\lambda^{-1}a$ is a unit of A. Since A was non-unital this is impossible.

If $\lambda = 0$, then in particular $aa^* = 0$ and hence $\|a\|^2 = \|aa^*\| = 0$ which implies that $a = 0$. □

Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a be in A.

Definition 3.28. *a is called:*

1. *selfadjoint if $a^* = a$*
2. *normal if $[a^*, a] = 0$*
3. *a projection if $a^2 = a$*
4. *an orthogonal projection if $a^2 = a$ and $a^* = a$*

5. *a partial isometry if a^*a and aa^* are (necessarily orthogonal) projections*

6. *an isometry if $a^*a = 1_A$ (for A in ${}^*\mathbf{Alg}_{\mathbb{C}}$)*

7. *a unitary if $a^*a = 1_A$ and $aa^* = 1_A$ (for A in ${}^*\mathbf{Alg}_{\mathbb{C}}$)*

Let A be a C^*-algebra and a be in A.

Lemma 3.29. *If a is a partial isometry or an orthogonal projection, then $\|a\| \in \{0, 1\}$.*

Proof. Let a be an orthogonal projection. Then we have

$$\|a\|^2 = r(a^*a) = r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a\|^{1/n} \in \{0, 1\} \ .$$

Assume now that a is a partial isometry. Then

$$\|a\|^2 = \|a^*a\| \in \{0, 1\}$$

by the first case since a^*a is an orthogonal projection. □

Let A be in $C^*\mathbf{Alg}$ and u be in A.

Lemma 3.30. *If u is unitary, then $\sigma(u) \subseteq U(1)$.*

Proof. We consider λ in $\mathbb{C}$. We first assume that $|\lambda| > 1$. Then we have

$$(\lambda - u) = \lambda(1 - \lambda^{-1}u)$$

and the right-hand side is invertible since $\|\lambda^{-1}u\| < 1$. We now assume that $|\lambda| < 1$. Then

$$(\lambda - u) = -u(1 - \lambda u^*)$$

and the right-hand side is invertible since $\|\lambda u^*\| < 1$.

In both cases we see that $\lambda \in \rho(u)$. □

Example 3.31. We consider the C^*-algebra $B(L^2(\mathbb{R}))$ and u in $B(L^2(\mathbb{R}))$ given by $(uf)(x) = f(x + 1)$. This operator is unitary. In order to calculate the spectrum of u we use the Fouriertransformation $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

(this formula makes sense for f in $C_c(\mathbb{R})$ and extends by continuity). Then $\mathcal{F}$ is a unitary isomorphism with inverse given by

$$\mathcal{F}^{-1}(\hat{f})(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi \ .$$

One calculates that $\mathcal{F} \circ u \circ \mathcal{F}^{-1}$ is the multiplication operator by the function $x \mapsto e^{-ix}$. The image of this function is all of $U(1)$. It follows that $\sigma(u) = U(1)$.

Example 3.32. We consider u in $B(L^2(S^1))$ given by $(uf)(z) = f(ze^{2\pi i\alpha})$. We let $\mathcal{F} : L^2(S^1) \to L^2(\mathbb{Z})$ be the Fouriertransformation given by

$$\mathcal{F}(f)(n) := \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(e^{\pi i x}) dx \ .$$

The inverse is given by

$$\mathcal{F}^{-1}(\hat{f})(x) := \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \hat{f}(\xi) \ .$$

The operator $\mathcal{F} \circ u \circ \mathcal{F}^{-1}$ is the multiplication by the function $n \mapsto e^{-2\pi i n\alpha}$. The closure of its image is $U(1)$ if α is irrational or a finite subset of $U(1)$ if α is rational. Hence $\sigma(u)$ is $U(1)$ or a finite subset of $U(1)$, respectively.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a be in A.

Lemma 3.33. *If a is self-adjoint, then $\sigma(a) \subseteq \mathbb{R}$.*

Proof. By considering the image $(a, 0)$ of a in A^u we can assume that A is unital. We assume that λ in $\sigma(a)$. Then we can define e^{ia} by a convergent power series. We claim that $e^{i\lambda} \in \sigma(e^{ia})$ and that $e^{i\lambda} \in U(1)$. These two assertions imply that $\lambda \in \mathbb{R}$.

We now show the claim. We consider

$$b := \sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^n}{n!} \ .$$

Then

$$e^{ia} - e^{-\lambda} = e^{i\lambda}(e^{i(a-\lambda)} - 1) = (a-\lambda)e^{i\lambda}b$$

Since $(a - \lambda)$ not invertible also $e^{ia} - e^{-\lambda}$ is not invertible. Hence $e^{i\lambda} \in \sigma(a)$.

In general, if c, d are in A and $[c, d] = 0$, then we have $e^c e^d = e^{c+d}$ by a calculation with the power series. We furthermore have the relation$(e^c)^* = e^{c^*}$.

We have $(e^{ia})^* = e^{-ia}$. This implies that $e^{ia}(e^{ia})^* = 1_A = (e^{ia})^* e^{ia}$. Thus e^{ia} is unitary and we have $\sigma(e^{ia}) \subseteq U(1)$ by Lemma 3.30. Hence $e^{i\lambda} \in U(1)$ as claimed. □

4 Gelfand duality

In this section we consider the structure of the subcategory of $C^*\mathbf{Alg}$ of commutative C^*-algebras.

Let $L : \mathcal{C} \to \mathcal{D}$ be a functor.

Definition 4.1. *We say that L is a reflective localization if it fits into an adjunction*

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

where R is fully faithful.

Example 4.2. Here we present some examples of reflective localizations for the purpose of illustration.

Let **Metr** be the category of metric spaces and isometries, and $\mathbf{Metr}_{\mathrm{compl}}$ be the full subcategory of complete metric spaces. Then

$$\mathrm{compl} : \mathbf{Metr} \leftrightarrows \mathbf{Metr}_{\mathrm{compl}} : \mathrm{incl}$$

is a reflective localization, where compl sends a metric space to its completion.

Let **Hausd** be the full subcategory of **Top** of Hausdorff spaces. Then we have a reflective localization

$$(-)_{\mathrm{Hausd}} : \mathbf{Top} \leftrightarrows \mathbf{Hausd} : \mathrm{incl} \ ,$$

where for X in **Top** we denote by X_{Hausd} is the maximal Hausdorf quotient of X.

We have a reflective localization

$$K_0 : \mathbf{Monoids} \leftrightarrows \mathbf{Groups} : \mathrm{incl} \ ,$$

where K_0 sends a monoid to its group completion (Grothendieck construction). □

For every topological space X we have a commutative C^*-algebra $C_b(X)$ of continuous bounded $\mathbb{C}$-valued functions on X. If $f : X \to X'$ is a continuous map, then we have a homomorphism

$$f^* : C_b(X') \to C_b(X) \ , \quad a \mapsto a \circ f \ .$$

Let $C^*\mathbf{Alg}^{\mathrm{comm}}$ and $C^*\mathbf{Alg}^{\mathrm{nu,comm}}$ be the full subcategories of $C^*\mathbf{Alg}$ and $C^*\mathbf{Alg}^{\mathrm{nu}}$ of unital and not necessarily unital commutative C^*-algebras. Then this construction determines a functor

$$C_b : \mathbf{Top}^{\mathrm{op}} \to C^*\mathbf{Alg}^{\mathrm{comm}} \ .$$

The main theorem of this section is:

Theorem 4.3. *There is a reflective localization*

$$C_b : \mathbf{Top} \leftrightarrows (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}} : G$$

where G identifies $(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}$ with the full subcategory of $\mathbf{Hausd}^{\mathrm{comp}}$ of **Top** *of compact Hausdorff spaces.*

The functor G is called the Gelfand transformation. It sends a commutative C^*-algebra A to its space $G(A)$ of characters, i.e., the space of non-zero homomorphisms $A \to \mathbb{C}$. The proof of the theorem will be given after some preparations about characters.

Let A be in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Definition 4.4. *A character of A is a homomorphism $A \to \mathbb{C}$.*

The set of characters of A is thus the morphism set $\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(A, \mathbb{C})$. We actually have a functor

$$\mathtt{Hom}_{\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(-, \mathbb{C}) : (\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}})^{\mathrm{op}} \to \mathbf{Set}$$

represented by $\mathbb{C}$.

Non-zero characters on a unital algebra are automatically unital. Assume that A is in $\mathbf{Alg}_{\mathbb{C}}$ and that $\phi : A \to \mathbb{C}$ is a character.

Lemma 4.5. *If $\phi \neq 0$, then $\phi(1_A) = 1$.*

Proof. Indeed, let a in A be such that $\phi(a) \neq 0$. Then we have $\phi(a) = \phi(1_A a) = \phi(1_A)\phi(a)$. This implies that $\phi(1_A) = 1$. □

Characters of C^*-algebras are automatically morphisms of $*$-algebras. Indeed, let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and ψ be a character on A.

Lemma 4.6. *ψ is a $*$-homomorphism.*

Proof. We can extend ψ to a homomorphism $\psi^u : A^u \to \mathbb{C}$ such that $\psi^u(a, 0) = \psi(a)$. We have $\psi^u(a, 0) \in \sigma(a, 0)$ by Lemma 1.45.

First assume that a is selfadjoint. Then $(a, 0)$ is selfadjoint in A^u. Using Lemma 3.33 (this lemma applies since we assume that A is C^*-algebra and not only a $*$-algebra) we get $\psi(a) = \psi^u(a, 0) \in \sigma(a, 0) \subseteq \mathbb{R}$.

If a is general, then we write a as a sum of selfadjoints

$$a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i} \ .$$

We then have

$$\psi(a) = \psi(\frac{a + a^*}{2}) + i\psi(\frac{a - a^*}{2i}) \ , \quad \psi(a^*) = \psi(\frac{a + a^*}{2}) + i\psi(\frac{a^* - a}{2i}) = \overline{\psi(a)}$$

□

Recall that $*$-homomorphisms between C^*-algebras are automatically continuous (Corollary 3.26). So characters on a C^*-algebra are automatically continuous.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$. By A^* we denote the dual of A in the sense of Banach spaces. The C^*-norm on A induces a norm in A^*.

On A^* we also have the weak topology generated by the maps $(\phi \mapsto \phi(a)) : A^* \to \mathbb{C}$ for all a in A. The unit ball $B(A^*)$ is compact w.r.t. to the weak topology by the theorem of Banach-Alaoglou.

We let $\hat{A}$ denote the set of non-zero characters of A. If ϕ in $\hat{A}$, then $\phi \in B(A^*)$, since a $*$-homomorphism is contractive by Corollary 3.26. We equip $\hat{A}$ with the weak topology induced from the weak topology on A^*. The set $\hat{A} \cup \{0\}$ is a closed subset of $B(A^*)$. In fact, if $(\phi_i)_{i\in I}$ is a converging net in $\hat{A}$, then $\mathtt{lim}_{i\in I}\, \phi_i$ is again a character. Indeed, for a, b in A we have

$$\mathop{\mathtt{lim}}_{i\in I} \phi_i(ab) = \mathop{\mathtt{lim}}_{i\in I} \phi_i(a)\phi_i(b) = \mathop{\mathtt{lim}}_{i\in I} \phi_i(a) \mathop{\mathtt{lim}}_{i\in I} \phi_i(b) \ .$$

Note that it may happen that $\mathtt{lim}_{i\in I}\, \phi_i = 0$.

It follows that $\hat{A} \cup \{0\}$ is compact. Consequently, $\hat{A}$ is locally compact.

Lemma 4.7. *If A is unital, then $\hat{A}$ is compact.*

Proof. For every ϕ in $\hat{A}$ we have $\phi(1_A) = 1$. Hence any limit point of $\hat{A}$ in $B(A^*)$ satisfies this condition, too. It follows that 0 is isolated in $\hat{A} \cup \{0\}$ and hence $\hat{A}$ itself is closed and hence compact. □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider a in A. Then we define a function

$$g_A(a) : \hat{A} \to \mathbb{C} \ , \quad \phi \mapsto g_A(a)(\phi) := \phi(a) \ .$$

Definition 4.8. *The function $g_A(a) : \hat{A} \to \mathbb{C}$ is called the Gelfand transform of a.*

Lemma 4.9. *We have $g_A(a) \in C_b(\hat{A})$ and $\|g_A(a)\|_\infty \le \|a\|$.*

Proof. The function $g_A(a)$ is continuous by the very definition of the weak topology. Furthermore, since a $*$-homomorphism between C^*-algebras is contractive by Corollary 3.26 we have

$$\|g_A(a)\|_\infty = \sup_{\phi\in\hat{A}} |\phi(a)| \le \sup_{\phi\in\hat{A}} \|\phi\| \|a\| \le \|a\| \ .$$

□

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider a in A. Then we consider the set of values of the Gelfand transformation $g_A(a)$. Recall that $\sigma^u(a) = \sigma(a, 0)$, where $(a, 0)$ is the image of a in the unitalization A^u of A.

Lemma 4.10. *We have $g_A(a)(\hat{A}) \cup \{0\} \subseteq \sigma^u(a)$. If A is commutative, then this is an equality.*

Proof. We have already seen (in the proof of Lemma 4.6) that

$$g_A(a)(\hat{A}) \subseteq \sigma^u(a) \ .$$

Indeed, by Lemma 1.45 for ϕ in $\hat{A}$ we have $g_A(a)(\phi) = \phi(a) = \phi^u(a, 0) \in \sigma(a, 0) = \sigma^u(a)$.

We now assume that A is commutative and show that

$$\sigma^u(a) \subseteq g_A(a)(\hat{A}) \cup \{0\} \ .$$

We assume that λ is in $\mathbb{C} \setminus \{0\}$ and $\lambda \in \sigma^u(a)$. Then $(\lambda - (a, 0))A^u$ is proper ideal in A^u since otherwise $(\lambda - (a, 0))$ would be invertible. Here we use commutativity in order to conclude that the ideal is two-sided. Using Zorn's Lemma we find a maximal proper ideal I with $(\lambda - (a, 0))A^u \subseteq I$

We claim that I is closed. If it is not closed, $\bar{I}$ would larger and also proper by Lemma 2.21. Then A^u/I is a field and a Banach algebra. It follows from Corollary 2.20 (Gelfand-Mazur) that $A^u/I \cong \mathbb{C}$. We define the character

$$\psi : A \to A^u \to A^u/I = \mathbb{C} \ .$$

By construction $\psi(a) = \lambda \neq 0$ so that $\psi \in \hat{A}$. Hence $\lambda \in g_a(\hat{A})$. □

We consider A in $C^*\mathbf{Alg}^{\mathrm{comm}}$. The following is the key result leading to the main theorem of this section.

Theorem 4.11 (Gelfand)**.** *The Gelfand transform $g_A : A \to C_b(\hat{A})$ is an isomorphism of C^*-algebras .*

Proof. We first show that g_A is a homomorphism of $*$-algebras. It is linear since

$$g_A(a + \lambda b)(\phi) = \phi(a + \lambda\phi) = \phi(a) + \lambda\phi(b) = (g_A(a) + \lambda g_A(b))(\phi)$$

and multiplicative since

$$g_A(ab)(\phi) = \phi(ab) = \phi(a)\phi(b) = g_A(a)(\phi)g_A(b)(\phi) = (g_A(a)g_A(b))(\phi) \ .$$

Finally we show that g_A is compatible with the involution:

$$g_A(a^*)(\phi) = \phi(a^*) = \overline{\phi(a)} = g_A(a)^*(\phi) \ .$$

Next we show that g_A is isometric. We indeed have

$$\|g_A(a)\|_\infty^2 = \sup_{\phi \in \hat{A}} |\phi(a)|^2 = \sup_{\phi \in \hat{A}} |\phi(a)^*\phi(a)| = \sup_{\phi \in \hat{A}} |\phi(a^*a)| \overset{!}{=} r(a^*a) \overset{!!}{=} \|a\|^2 \ .$$

Here we use the equality assertion in Lemma 4.10 at the equality marked by !, and Lemma 3.24 at the equality marked by !!.

This implies that g_A is injective.

We now show that g_A is surjective. We are going to apply the Stone-Weierstrass theorem to the compact space $\hat{A}$ (at this be use the assumption that A is unital). We observe that $g_A(A)$ is C^*-subalgebra of $C_b(\hat{A})$. It clearly separates points. Finally for every ϕ in $\hat{A}$ there exists a in A such that $g_A(a)(\phi) = \phi(a) \neq 0$ (since $\hat{A}$ consists of non-zero characters). By the Stone-Weierstrass we conclude that $g_A(A)$ is dense in $C_b(\hat{A})$. Since g_A is an isometry we have $g_A(A) = C_b(\hat{A})$. □

Proof of Theorem 4.3. We define the functor

$$G : (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}} \to \mathbf{Top} \ , \quad G(A) := \hat{A} \ .$$

If $f : A \to B$ is a morphism in $C^*\mathbf{Alg}^{\mathrm{comm}}$, then $f^* : B^* \to A^*$ is continuous for the weak topology. Indeed, for every a in A the function $f^*(-)(a) = (-)(f(a)) : B^* \to \mathbb{C}$ is continuous. Since $f(1_A) = 1_B$ the pull-back preserves non-zero characters. This implies that the restriction of f^* to non-zero characters is a continuous map $G(f) : G(B) \to G(A)$.

In order to construct the adjunction claimed in Theorem 4.3 we construct the unit and counit.

The counit $u : C_b \circ G \to \mathtt{id}$ will be given by the family $(g_A^{\mathrm{op}})_{A \in (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}$. Note that without opping we have $g_A : A \to C_b(G(A))$, where g_A sends a in A to its Gelfand transformation $g_A(a) \in C_b(\hat{A})$.

We check naturality of u. Let $f : A \to B$ be a morphism in $C^*\mathbf{Alg}^{\mathrm{comm}}$. We must check that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g_A} & & \downarrow{\scriptstyle g_B} \\ C_b(G(A)) & \xrightarrow{C_b(G(f))} & C_b(G(B)) \end{array}$$

commutes. For a in A and ϕ in $G(B)$ we indeed have

$$g_B(f(a))(\phi) = \phi(f(a)) = G(f)(\phi)(a) = g_A(a)(G(f)(\phi)) = (C_b(G(f))(g_A)(a))(\phi) \ .$$

The unit of the adjunction is $h : \mathtt{id} \to G \circ C_b$ given by the family $(h_X)_{X \in \mathbf{Top}}$, where

$$h_X : X \to G(C_b(X)) \ , \quad h_X(x) := (a \mapsto a(x)) \ .$$

We must check that h_X is continuous. To this end we observe that for every a in $C_b(X)$ the function $x \mapsto h_X(x)(a) = a(x)$ is continuous. In view of the definition of the weak topology on $G(C_b(X))$ this implies that $X \to G(C_b(X))$ is continuous.

We check naturality of h. Let $f : X \to Y$ be a morphism in **Top**. Then we must check that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h_X & & \downarrow h_Y \\ G(C_b(X)) & \xrightarrow{G(C_b(f))} & G(C_b(Y)) \end{array}$$

commutes. For x in X and b in $C_b(Y)$ we have

$$h_Y(f(x))(b) = b(f(x))$$

and

$$G(C_b(f))(h_X(x))(b) = h_X(x)(C_b(f)(b)) = b(f(x)) \ .$$

We next show that u, h define an adjunction. To this end we consider the maps

$$\alpha : \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), B) \stackrel{G}{\to} \mathtt{Hom}_{\mathbf{Top}}(G(C_b(X)), G(B)) \stackrel{h_X^*}{\to} \mathtt{Hom}_{\mathbf{Top}}(X, G(B)) \ .$$

$$\beta : \mathtt{Hom}_{\mathbf{Top}}(X, G(B)) \stackrel{C_b}{\to} \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), C_b(G(B))) \stackrel{u_{B,*}}{\to} \mathtt{Hom}_{(C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}}(C_b(X), B) \ .$$

We show that these maps are inverse to each other. We have for $r : B \to C_b(X)$, b in B, and x in X that

$$\begin{aligned} (\beta(\alpha(r))(b))(x) &= (C_b(\alpha(r)) \circ u_B)(b)(x) \\ &= C_b(\alpha(r))(u_B(b))(x) \\ &= u_B(b)(\alpha(r)(x)) \\ &= \alpha(r)(x)(b) \\ &= (G(r) \circ h_X)(x)(b) \\ &= r(b)(x) \ , \end{aligned}$$

hence $\beta \circ \alpha = \mathtt{id}$. We furthermore calculate for $s : X \to G(B)$, x in X and b in B

$$\begin{aligned} \alpha(\beta(s))(x)(b) &= (G(\beta(s)) \circ h_X)(x)(b) \\ &= G(\beta(s))(\mathtt{ev}_x)(b) \\ &= \beta(s)(b)(x) \\ &= (C_b(s) \circ u_B)(b)(x) \\ &= C_b(s)(u_B(b))(x) \\ &= u_B(b)(s(x)) = s(x)(b), \end{aligned}$$

hence $\alpha \circ \beta = \mathtt{id}$. This finishes the construction of the adjunction. By Theorem 4.11 the counit of the adjunction is an isomorphism. Consequently, the right-adjoint is fully faithful.

Its image consists of compact Hausdorff spaces. We must show that all compact Hausdorff spaces belong to the essential image.

Let X be a compact Hausdorff space. Then we have a map $h_X : X \to G(C_b(X))$. In order to show that it is an isomorphism it suffices to show that it is a bijection.

We first show that h_X is injective. Let x, x' be X be distinct points. Since $C_b(X)$ separates points (by the Urysohn Lemma since compact Hausdorff spaces are normal) there exists f in $C_b(X)$ such that $f(x) \neq f(x')$. Then $h_X(x)(f) = f(x) \neq f(x') = h_X(x')(f)$. Hence $h_X(x) \neq h_X(x')$.

We now show surjectivity of h_X. We consider a non-zero character $\psi : C_b(X) \to \mathbb{C}$. Then $\mathtt{ker}(\psi)$ is a proper closed $*$-subalgebra of $C_b(X)$. This subalgebra also separates points. Indeed let x, x' in X be distinct and choose a in $C_b(X)$ such that $a(x) \neq a(x')$. Then $a - \psi(a) \in \mathtt{ker}(\psi)$ and separates x, x'.

By the Stone-Weierstraß Theorem and since $\mathtt{ker}(\psi)$ is a closed proper ideal of $C_b(X)$ there exists a point x in X such that $a(x) = 0$ for all a in $\mathtt{ker}(\psi)$ (because otherwise $\mathtt{ker}(\psi) = C_b(X)$ by SWT). We now show that $\psi = h_X(x)$. Let f be in $C_b(X)$. Then we have $f = \psi(f) + (f - \psi(f))$, where $f - \psi(f) \in \mathtt{ker}(\psi)$ and hence $(f - \psi(f))(x) = 0$. We get

$$f(x) = \psi(f) + (f - \psi(f))(x) = \psi(f) \ .$$

This finishes the verification that h_X is an isomorphism for compact Hausdorff spaces X. This finishes the proof of Theorem 4.3. □

Corollary 4.12 (Gelfand duality)**.** *The functors C_b restricts to an equivalence of categories*

$$C_b : \mathbf{Hausd}^{\mathrm{comp}} \stackrel{\simeq}{\rightrightarrows} (C^*\mathbf{Alg}^{\mathrm{comm}})^{\mathrm{op}}$$

with inverse G.

5 The non-unital case

We extend Gelfand duality to the non-unital case.

For a category $\mathcal{C}$ and object c in $\mathcal{C}$ we can consider the slice category $\mathcal{C}_{/c}$. An object in $\mathcal{C}_{/c}$ is a morphism $c' \to c$ in $\mathcal{C}$. A morphism $(c' \to c) \to (c'' \to c)$ in $\mathcal{C}_{/c}$ is a commutative triangle

$$\begin{array}{ccc} c' & \longrightarrow & c'' \\ & \searrow \quad \swarrow & \\ & c & \end{array} \ .$$

Analoguously we define $\mathcal{C}_{c/}$ such that $\mathcal{C}_{c/} \cong (\mathcal{C}^{\mathrm{op}}_{/c})^{\mathrm{op}}$. We extend the unitalization functor to a functor

$$U : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}_{/\mathbb{C}} \ , \quad A \mapsto (A^u \to \mathbb{C}) \ .$$

Lemma 5.1. *The functor U is an equivalence of categories.*

Proof. The inverse $K : C^*\mathbf{Alg}_{/\mathbb{C}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ sends $(\phi : B \to \mathbb{C})$ to $\mathtt{ker}(\phi)$. A morphism in $C^*\mathbf{Alg}_{/\mathbb{C}}$ is a commuting triangle

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & \mathbb{C} & \end{array} \ .$$

We get an induced morphism $K(h) : K(\phi) \to K(\phi')$, where $K(h)$ is the restriction of h to $\mathtt{ker}(\phi)$ considered as a homomorphism with values in $\mathtt{ker}(\phi')$.

We now provide the natural isomorphisms $\alpha : \mathtt{id} \to K \circ U$ and $\beta : U \circ K \to \mathtt{id}$ exhibiting U and K as inverses to each other. The canonical inclusion $A \to A^u$ identifies A with $K(U(A))$. We let $\alpha_A : A \to K(U(A))$ be this canonical inclusion.

For $\phi : B \to \mathbb{C}$ in $C^*\mathbf{Alg}_{/\mathbb{C}}$ we have the canonical inclusion $K(\phi) \to B$. Since B is unital we can extend it uniquely to $U(K(\phi)) \to B$. Then we let $\beta_B : (\mathtt{ker}(\phi)^u \to \mathbb{C}) \to (\phi : B \to \mathbb{C})$ be the resulting morphism in the slice category. $\square$

An object in $(\mathbf{Hausd}^{\mathrm{comp}})_{*/}$ is a pointed compact Hausdorff space (X, x). We let $\mathbf{Hausd}^{\mathrm{lcomp}}$ be the a category whose objects are the locally compact topological spaces, and whose morphisms are partially defined proper maps $X \supseteq U \stackrel{f}{\to} X'$. Here the condition that f is proper means that preimages of compact subsets of X' are compact in U, or equivalently, that f has a continuous extension $f^+ : U^+ \to X^+$ to the one-point compactifications. The composition of such a map with $X' \supseteq U' \stackrel{f'}{\to} X''$ is given by $X \supseteq (U \cap f^{-1}(U')) \stackrel{f' \circ f}{\to} X''$. Using the characterization with preimages of compact subsets it is easy to check that this map is proper.

We can define a functor

$$R : (\mathbf{Hausd}^{\mathrm{comp}})_{*/} \to \mathbf{Hausd}^{\mathrm{lcomp}} \ , \quad (X, x) \mapsto X \setminus \{x\} \ .$$

A morphism $f : (X, x) \to (X', x')$ is send by R to a partially defined map

$$L(f) : X \supseteq (X \setminus f^{-1}(x')) \to X' \setminus \{x\} \ .$$

Lemma 5.2. *The functor L fits into an equivalence of categories*

$$((-)^+, +) : \mathbf{Hausd}^{\mathrm{lcomp}} \leftrightarrows (\mathbf{Hausd}^{\mathrm{comp}})_{*/} : R \ ,$$

Proof. The left-adjoint $((-)^+, +)$ sends a locally compact space X to its one-point compactification $(X^+, +)$ pointed by the additional point. If $U \to X'$ is a map defined on an open subset U of X, then its image under the left-adjoint is its extension to a map $X^+ \to U^+ \to X'^{,+}$ under $+$, where the first map is the collapse map sending every point outside U to $+$.

On the one hand, we have a natural isomorphism $L(X^+,+) \cong X$. On the other hand, the canonical map $(X \setminus \{x\}) \to X$ extends to a map $(X^+,+) \to (X,x)$. This map is a bijection between compact Hausdorff spaces and hence an isomorphism. □

Note that $C_b(*) \cong \mathbb{C}$.

We consider the functor

$$C_0 : \mathbf{Hausd}^{\mathrm{lcomp}} \stackrel{X \mapsto (X^+,+)}{\to} \mathbf{Hausd}^{\mathrm{comp}}_{*/} \stackrel{C_b}{\to} (C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}})^{\mathrm{op}} \stackrel{K}{\simeq} (C^*\mathbf{Alg}^{\mathrm{nu,comm}})^{\mathrm{op}} .$$

Thus $C_0(X)$ is the algebra of the continuous functions on X^+ which vanish at $+$.

Remark 5.3. In general $C_0(X)$ is bigger than the closure of the subsspace of $C_b(X)$ of functions of compact support. □

We let

$$G_0 : (C^*\mathbf{Alg}^{\mathrm{nu,comm}})^{\mathrm{op}} \stackrel{U}{\to} (C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}})^{\mathrm{op}} \stackrel{G}{\to} \mathbf{Hausd}^{\mathrm{comp}}_{*/} \stackrel{(X,x)\mapsto X\setminus\{*\})}{\to} \mathbf{Hausd}^{\mathrm{lcomp}}$$

be the functor which sends A to $G(A^u) \setminus \{\epsilon_A\}$, where $\epsilon_A : A^u \to \mathbb{C}$ is the canonical character. All functors in these compositions are equivalences.

Corollary 5.4. *We have an equivalence of categories*

$$C_0 : \mathbf{Hausd}^{\mathrm{lcomp}} \leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu,comm}} : G_0$$

Example 5.5. We consider the functor

$$(-)^u : C^*\mathbf{Alg}^{\mathrm{nu,comm}} \stackrel{U}{\to} C^*\mathbf{Alg}^{\mathrm{comm}}_{/\mathbb{C}} \stackrel{(B\to\mathbb{C})\mapsto B}{\to} C^*\mathbf{Alg}^{\mathrm{comm}} .$$

This functor sends a commutative C^*-algebra to its unitaliztion A^u considered as an object in $C^*\mathbf{Alg}^{\mathrm{comm}}$. Under Gelfand duality this functor of $C^*\mathbf{Alg}^{\mathrm{nu,comm}} \to C^*\mathbf{Alg}^{\mathrm{comm}}$ corresponds to the functor

$$(-)^+ : \mathbf{Hausd}^{\mathrm{lcomp}} \stackrel{((-)^+,+)}{\to} \mathbf{Hausd}_{*/} \stackrel{(X,x)\mapsto X}{\to} \mathbf{Hausd}^{\mathrm{comp}} .$$

This functor sends a locally compact topological space X to its one-point compactification X^+. □

We consider the functor

$$\beta : \mathbf{Top} \stackrel{C_b}{\to} C^*\mathbf{Alg}^{\mathrm{comm}} \stackrel{G}{\to} \mathbf{Hausd}^{\mathrm{comp}} .$$

The unit of the adjunction from Theorem 4.3 is a natural transformation

$$\iota : \mathtt{id} \to \beta .$$

Its evaluation at X is given by the map

$$X \to \beta(X) = G(C_b(X)) , \quad x \mapsto (a \mapsto a(x)) .$$

Definition 5.6. *The map $X \to \beta(X)$ is called the Stone-Čech compactification of X.*

The following is an immediate consequence of Theorem 4.3. We just compose the functor C_b with the equivalence $G_{|\mathbf{Hausd}^{\mathrm{comp}}}$.

Corollary 5.7. *We have a reflective localization*

$$\beta : \mathbf{Top} \to \mathbf{Hausd}^{\mathrm{comp}} : \mathrm{incl}$$

whose unit is given by β.

Example 5.8. Assume that X is locally compact. Then the natural map $X \to X^+$ extends uniquely to a map $\beta(X) \to X^+$ such that the composition $X \to \beta(X) \to X^+$ is the canonical inclusion. This shows that $X \to \beta(X)$ is injective. Thus X can be considered as a subspace of $\beta(X)$. Since the restriction of functions along $X \to \beta(X)$ is the identity $C_b(\beta(X)) = C_b(X))$ the subspace X is dense in $\beta(X)$.

A C^*-algebra is separable A if it has a countable dense subset A_0. We assume that A is separable. We call a topological space second countable if its topology has a countable base.

Note that the weak topology on A^* is generated by the functions $\mathtt{ev}_a : B(A^*) \to \mathbb{C}$ for all a in A. Since a uniform limit of continuous functions is continuous and the functions $\mathtt{ev}_a$ for a_0 in A_0 are dense in all such evaluation functions the topology is also generated by the countable family functions $(\mathtt{ev}_a)_{a \in A_0}$. Since the topology of $\mathbb{C}$ is second countable it follows that the weak topology on $B(A^*)$ is second countable. On the other hand if X is in $\mathbf{Hausd}^{\mathrm{comp}}$ and has a countable base, then $C_b(X)$ is separable.

We let $\mathbf{Hausd}^{\mathrm{comp}}_{\mathrm{sep}}$ denote the full subcategory of $\mathbf{Hausd}^{\mathrm{comp}}$ of second countable compact Hausdorff spaces and $C^*\mathbf{Alg}^{\mathrm{comm}}_{\mathrm{sep}}$ be the full subcategory of $C^*\mathbf{Alg}^{\mathrm{comm}}$ of separable algebras.

Corollary 5.9. *The Gelfand duality restricts to an equivalence*

$$C_b : \mathbf{Hausd}^{\mathrm{comp}}_{\mathrm{sep}} \leftrightarrows (C^*\mathbf{Alg}^{\mathrm{comm}}_{\mathrm{sep}})^{\mathrm{op}} : G \ .$$

A locally compact Hausdorff space X is second countable if and only if its one-point compactification X^+ is second countable.

Similarly A in $C^*\mathbf{Alg}^{\mathrm{nu,comm}}$ is separable if and only if A^+ is separable.

Corollary 5.10. *Gelfand duality restricts to an equivalence of categories*

$$C_0 : \mathbf{Hausd}^{\mathrm{lcomp}}_{\mathrm{sep}} \leftrightarrows (C^*\mathbf{Alg}^{\mathrm{nu,comm}}_{\mathrm{sep}})^{\mathrm{op}} : G_0 \ .$$

6 Function calculus, positivity, and approximate units

Let A be a unital C^*-algebra and consider a in A. We let $C_A(a)$ be the sub-C^*-algebra of A generated by a.

Definition 6.1. *a is called normal if $[a^*, a] = 0$.*

Lemma 6.2. *If a is normal, then $C_A(a)$ is commutative.*

Proof. We describe $C_A(a)$ explicitly. We consider $\mathbb{C}[x, y]$ as a $*$-algebra with the involution determined by $x^* = y$. Formally we have $\mathbb{C}[x, y] := \mathrm{Free}^*(\{x, y\})/I$, where $\{x, y\}$ is a C_2-set with the flip action, and I is generated by $[x, y]$. By the universal property of $\mathbb{C}[x, y]$ we have a unique $*$-homomorphism $\mathbb{C}[x, y] \to A$ which sends x to a and y to a^*. A general element p in $\mathbb{C}[x, y]$ is then sent to $p(a, a^*)$. Since $\mathbb{C}[x, y]$ is commutative, the image of this homomorphism is a commutative $*$-subalgebra of A. Then

$$C_A(a) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[x, y]\}} \ .$$

It is a closure of a commutative subalgebra and hence itself commutative. □

We define a continuous map

$$c : \widehat{C_A(a)} \to \sigma(a) \ , \quad \phi \mapsto \phi(a) \ .$$

.

Lemma 6.3. *The map $c : \widehat{C_A(a)} \to \sigma(a)$ is a homeomorphism.*

Proof. c is a continuous map between compact Hausdorff spaces. It suffices to show that it injective and surjective.

Surjectivity follows from Lemma 4.10. If ϕ, ϕ' are in $\widehat{C_A(a)}$ and $\phi(a) = \phi'(a)$, then $\phi = \phi'$ on the image of $\mathbb{C}[x, y]$ which is dense in $C_A(a)$. Hence $\phi = \phi'$ by continuity. □

Corollary 6.4. *We have an isomorphism $c^* : C_b(\sigma(a)) \xrightarrow{\cong} C_A(a)$.*

Let A be in $C^*\mathbf{Alg}$ and a be in A normal.

Definition 6.5 (continuous function calculus)**.** *For f in $C_b(\sigma(a))$ we define $f(a) := c^*f \in C_A(a)$.*

Proposition 6.6.

1. *(calculus) For f, g in $C_b(\sigma(a))$ and λ in $\mathbb{C}$ we have $(f + \lambda g)(a) = f(a) + \lambda g(a)$ and $(fg)(a) = f(a)g(a)$, $f(a)^* = \bar{f}(a)$.*

2. *(spectral mapping principle) For f in $C_b(\sigma(a))$ we have $\sigma(f(a)) = f(\sigma(a))$.*

3. *(composition rule) For f in $C_b(\sigma(a))$ and g in $C_b(\sigma(f(a))$ we have $g(f(a)) = (g \circ f)(a)$.*

Proof. The calculus properties follow from the fact that c^* is a homomorphism.

Since c^* is an isomorphism we have $\sigma(f(a)) = \sigma(f) = f(\sigma(a))$.

For the composition rule we argue as follows. Since $f(a) \in C_A(a)$ we have $C_A(f(a)) \subseteq C_A(a)$. Consequently, $g(f(a)) \in C_A(a)$. For every ψ in $\widehat{C_A(f(a))}$ we have $\psi(g(f(a)) = g(\psi(f(a))$. Now assume that $\psi = \phi_{|C_A(f(a))}$ for some $\phi \in \widehat{C_A(a)}$ Then

$$\phi(g(f(a))) = g(\phi(f(a))) = g(f(\phi)) = (g \circ f)(\phi) = \phi((g \circ f)(a))$$

Since ϕ is arbitrary we conclude that $g(f(a)) = (g \circ f)(a)$. □

We now extend the function calculus to non-unital algebras. Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a be in A normal. Then we consider $(a, 0)$ in A^u. For every p in $\mathbb{C}[x, y]$ we have

$$p((a, 0), (a^*, 0)) = (p(a, a^*), p(0, 0)) \ .$$

In the following corollary we consider A as a subalgebra of A^u in the natural way. We conclude that $p((a, 0), (a^*, 0)) \in A$ provided $p(0, 0) = 0$.

Corollary 6.7.

1. *For f in $C_b(\sigma^u(a))$ with $f(0) = 0$ we have $f(a, 0) \in A$. We set $f(a) := f(a, 0)$.*

2. *(calculus) For f, g in $C_b(\sigma^u(a))$ with $f(0) = g(0) = 0$ and λ in $\mathbb{C}$ we have $(f + \lambda g)(a) = f(a) + \lambda g(a)$ and $(fg)(a) = f(a)g(a)$, $f(a)^* = \bar{f}(a)$.*

3. *(spectral mapping principle) We have $\sigma^u(f(a)) = f(\sigma^u(a))$.*

4. *(composition rule) For g in $C_b(\sigma^u(f(a))$ with $g(0) = 0$ we have $g(f(a)) = (g \circ f)(a)$.*

We consider A in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a in A.

Definition 6.8. *a is called positive if $a = a^*$ and $\sigma^u(a) \subseteq [0, \infty)$.*

In formulas the assertion that A is positive is written as $a \geq 0$. The set of positive elements of A is denoted by $A^+ := \{a \in A \mid a \geq 0\}$.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and assume that a is in A^+

Lemma 6.9. *There exists a unique b in A^+ such that $b^2 = a$.*

Proof. In order to show existence we apply the function calculus. Note that $\sqrt{0} = 0$. Since $\sigma(a) \subseteq [0, \infty)$ the function $\sqrt{-}$ is continuous on $\sigma(a)$. We set $b := \sqrt{a}$. Since $\sqrt{-}$ is a real function we have $b = b^*$. Furthermore, by the composition rule since $\sqrt{-}^2 = \mathtt{id}_{[0,\infty)}$, we have $b^2 = (\sqrt{a})^2 = (\sqrt{-})^2(a) = a$.

In order to show uniqueness, we consider $c \in A^+$ such that $c^2 = a$. Then again by the composition rule since $\sqrt{(-)^2} = \mathtt{id}_{[0,\infty)}$ we have $c = \sqrt{a}$. Hence $c = b$. □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Corollary 6.10. *A is generated linearly by A^+*

Proof. We first write $a = (a^* + a)/2 + i(a - a^*)/2i$. Since $(a^* + a)/2$ and $(a - a^*)/2i$ are selfadjoint we conclude that A is linearly generated by selfadjoints.

We now consider a selfadjoint a in A. We must write it as a linear combination of positive elements. We can apply the function calculus to the restriction of the continuous function $|-|$ to $\sigma^u(a)$. Note that $|0| = 0$. We have

$$\mathtt{id} = (|-| + \mathtt{id})/2 - (|-| - \mathtt{id})/2$$

and $|-| + \mathtt{id}$ and $|-| - \mathtt{id}$ are non-negative functions. We therefore define the positive and negative parts of a by

$$a^+ := (|a| + a)/2 \ , \quad a^- := (|a| - a)/2 \ .$$

Note that $(|a|+a)/2$ and $(|a|-a)/2$ belong to A^+, and that $a = a^+ - a^-$ and $a^+a^- = 0$. □

In the next lemma we characterize positivity using the norm. Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a be in A selfadjoint.

Lemma 6.11.

1. *If $\|a - t\| \le t$ for some t in $\mathbb{R}$, then a in A^+.*
2. *If $\|a\| \le t$ and $a \in A^+$, then $\|a - t\| \le t$.*

Proof. By considering $(a, 0)$ in A^u instead of a we can assume that A is unital.

We now show Assertion 1. Since a is selfadjoint we have $\sigma(a) \subseteq \mathbb{R}$. The subalgebra $C_A(a)$ of A is isomorphic to $C_b(\sigma(a))$ such that a corresponds to the function x. The inequality

$\|a-t\| \le t$ means that $|x-t| \le t$ for all x in $\sigma(a)$. If there existed s in $\sigma(a)$ with $s<0$, then $|s-t|>t$ which is a contradiction. Hence $\sigma(a) \subseteq [0,\infty)$.

We show Assertion 2. By assumption we have $\sigma(a) \subseteq [0,t]$. But then $|x-t| \le t$ for all x in $\sigma(a)$. □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Lemma 6.12. *A^+ is a closed cone in A.*

Proof. We first show that A^+ is a closed subset. Using the embedding $A \to A^u$ we reduce to the unital case. We now assume that A is unital. Let λ be in $(-\infty,0)$. Then $\lambda \in \rho(a)$ for every a in A^+. Moreover, by the spectral mapping princple $\|(\lambda-a)^{-1}\| \le \frac{1}{|\lambda|}$. This implies (use Neumann series, see Lemma 2.14) that $\lambda \in \rho(b)$ for every b in A with $\|a-b\| < |\lambda|$.

Let now b in A be an accumulation point of A^+. Then there exists a in A^+ with $\|a-b\| \le |\lambda|/2$. Then $\lambda \notin \sigma(b)$. Since λ is arbitrary we conclude that $b \ge 0$.

For $r \in [0,\infty)$ and a in A^+ we have $\sigma(ra) = r\sigma(a) \subseteq \mathbb{R}$ by the spectral mapping principle. Hence $ra \in A^+$.

Finally we consider a,b in A^+ and show that then also $a+b \in A^+$. We know from Lemma 6.11.2 that

$$\|a-\|a\|\| \le \|a\| \ , \quad \|b-\|b\|\| \le \|b\| \ .$$

We get

$$\|a+b-\|a\|-\|b\|\| \le \|a-\|a\|\| + \|b-\|b\|\| \le \|a\|+\|b\| \ .$$

Again by Lemma 6.11.1 we conclude that $a+b \ge 0$ □

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a,b be in A.

Lemma 6.13. *We have $\sigma^u(ab) = \sigma^u(ba)$.*

Proof. Assume that λ is in $\rho^u(ab)$. Then we will show that $\lambda \in \rho^u(ba)$. Note that $\lambda \ne 0$: We consider

$$c := \frac{1}{\lambda}(1+b(\lambda-ab)^{-1}a) \ .$$

We calculate

$$\begin{aligned} c(\lambda-ba) &= \frac{1}{\lambda}(1+b(\lambda-ab)^{-1}a)(\lambda-ba) \\ &= \frac{1}{\lambda}(\lambda-ba+b(\lambda-ab)^{-1}a\lambda-b(\lambda-ab)^{-1}(\lambda-\lambda+ab)a) \\ &= \frac{1}{\lambda}(\lambda-ba+b(\lambda-ab)^{-1}a\lambda-b(\lambda-ab)^{-1}\lambda a+ba) \\ &= 1_{A^u} \end{aligned}$$

and analogously $(\lambda - ba)c = 1_{A^u}$. $\square$

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a be in A.

Lemma 6.14. *We have $a^*a \in A^+$*

Proof. We set
$$b := a^*a \ .$$
Then we decompose b into the positive and negative parts:
$$b = b^+ - b^- \ .$$
We set
$$c := ab^- \ .$$
Then we calculate
$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \in A^+ \ ,$$
where we use the spectral mapping principle in order to conclude that $(b^-)^3 \in A^+$.

We now claim that that this implies that $c = 0$. The claim implies that $b^- = 0$ and hence $a^*a = b = b^+ \in A^+$.

It remains to show the claim. Since $\sigma^u(c^*c) = \sigma^u(cc^*)$ by Lemma 6.13 we also have $-cc^* \in A^+$.

We write
$$c = e + if$$
with selfadjoint e, f. Then we have
$$cc^* + c^*c = (e + if)(e - if) + (e - if)(e + if) = 2e^2 + 2f^2 \in A^+$$
But then $cc^* = (cc^* + c^*c) - c^*c \in A^+$. Since $\sigma^u(c^*c) = \sigma^u(cc^*)$ we conclude that $\sigma^u(c^*c) = \sigma^u(cc^*) = \{0\}$. But then $c = 0$, since $\|c\|^2 = r(c^*c)$. $\square$

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and a, b, c be in A

Lemma 6.15.

1. *If $a \leq b$, then $c^*ac \leq c^*bc$.*

2. *If $a, b \in A^+$ are invertible and $a \leq b$, then $b^{-1} \leq a^{-1}$.*

Proof. For the first assertion we write $b-a=d^*d$. Then $c^*(b-a)c^*=c^*d^*dc\in A^+$.

For the second assertion we first observe that $1\le c$ implies $c^{-1}\le 1$ by the spectral mapping principle.

Using Assertion 1 we have $1=\sqrt{a}^{-1}a\sqrt{a}^{-1}\le\sqrt{a}^{-1}b\sqrt{a}^{-1}$. Hence $\sqrt{a}b^{-1}\sqrt{a}\le 1$ which implies $b^{-1}\le a^{-1}$.

□

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Definition 6.16. *An approximative unit is a net $(u_\nu)_{\nu\in N}$ such that*

1. *N is a filtered poset*
2. *$u_\nu\in A^+$*
3. *$N\ni\nu\mapsto u_\nu\in A^+$ is order preserving*
4. *$\|u_\nu\|\le 1$ for all ν in N*
5. *$\mathtt{lim}_N\, au_\nu=a$ for every a in A.*

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Lemma 6.17. *A admits an approximative unit.*

Proof. If A is unital, then it has even a unit.

So we now assume that A is not unital. We consider the poset

$$N:=\{a\in A^+\mid \|a\|<1\}$$

with the induced partial order and the tautological family $(u)_{u\in N}$.

We first show that N is filtered. To this end we consider a,b in N. We then let $a':=a(1-a)^{-1}$ and $b':=b(1-b)$ in N. Note that $a=a'(1+a')^{-1}$ and $b=b'(1+b')^{-1}$. We consider

$$\frac{a'+b'}{1+a'+b'}\in N\ .$$

We now show that

$$a=a'(1+a')^{-1}\le\frac{a'+b'}{1+a'+b'}\quad\text{ and }\quad b=b'(1+b')^{-1}\le\frac{a'+b'}{1+a'+b'}\ .$$

We claim that

$$0 \le c \le d \quad \text{implies} \quad \frac{c}{1+c} \le \frac{d}{1+d} \ .$$

By the claim the inequality $a' \le a' + b'$ indeed implies that $\frac{a'}{1+a'} \le \frac{a'+b'}{1+a'+b'}$.

First of all we have $1+c \le 1+d$. Since $(-)^{-1}$ reverses order by Lemma 6 we get

$$\frac{1}{1+d} \ge \frac{1}{1+c} \ .$$

We conclude that

$$1 - \frac{1}{1+c} \le 1 - \frac{1}{1+d}$$

which is the claim.

We thus have shown that N is filtered.

In order to verify Condition 5 it suffices to show $\mathtt{lim}_N \, au = a$ for a selfadjoint since A is spanned by selfadjoints. Let $\Omega := \sigma^u(a) \setminus \{0\}$.

Let $\epsilon \in (0,\infty)$ and $\phi : C_A(a) \to C_0(\Omega)$ be Gelfand isomorphism. The set

$$K := \{\omega \in \Omega \mid |\omega(a)| \ge \epsilon\}$$

is compact. We can find $g : \Omega \to [0,1]$ such that $g_{|K} = 1$ and such that $\mathtt{supp}(g)$ is compact. Let δ be in $(1-\epsilon, 1)$. Then

$$\|\phi(a) - \delta g \phi(a)\| \le \epsilon \ .$$

We set $u := \phi^{-1}(\delta g)$. Then $u \in N$ and $\|a - au\| \le \epsilon$. For any u' in N with $u \le u'$ we have $1 - u' \le 1 - u$ and therefore $a(1-u')a \le a(1-u)a$. This implies

$$\|a - au'\|^2 = \|a(1-u')^{1/2}(1-u')^{1/2}\|^2 \le \|a(1-u')^{1/2}\|^2 = \|a(1-u')a\| \le \|a(1-u)a\| \le \epsilon \ .$$

□

7 The maximal norm- from ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ to $C^*\mathbf{Alg}^{\mathrm{nu}}$

Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. In this section we study the question how to decide whether A is a C^*-algebra or can at least be completed to a C^*-algebra. This leads to the notion of the maximal norm and of a pre-C^*-algebra. We then study the adjunction between the categories of $*$-algebras, pre-C^*-algebras and C^*-algebras and deduce completeness and cocompleteness of the latter two.

Let A in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a be an element A.

Definition 7.1. *We define the maximal norm*

$$\|a\|_{\max} := \sup_{\rho: A \to B} \|\rho(a)\|_B$$

where the supremum runs over all homomorphisms $\rho : A \to B$ in ${}^*\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ with B a C^*-algebra.

Here $\| - \|_B$ denotes the unique C^*-norm on B. We have $\|a\|_{\max} \in [0, \infty]$ since we always have the zero representation $0 : A \to 0$.

Example 7.2. We consider $\mathbb{C}[z]$ in ${}^*\mathbf{Alg}_{\mathbb{C}}$ with involution determined by $z^* = z$. In other words $\mathbb{C}[z] \cong \mathrm{Free}^*(\{z\})$, where C_2 acts trivially on the one-element set $\{z\}$.

Lemma 7.3. *We have $\|z\|_{\max} = \infty$.*

Proof. For every t in $\mathbb{R}$ we have the $*$-representation $\rho_t : \mathbb{C}[z] \to \mathbb{C}$ uniquely determined by $z \mapsto t$. Then $\|\rho_t(z)\| = |t|$. □

We now consider the $\mathbb{C}(z)$ in ${}^*\mathbf{Alg}_{\mathbb{C}}$ with the involution again determined by $z^* = z$. It is the quotient field of $\mathbb{C}[z]$ to which the involution is extends using the universal property of the quotient field.

Lemma 7.4. *The maximal norm $\| - \|_{\max}$ on $\mathbb{C}(z)$ vanishes.*

Proof. Let $\rho : \mathbb{C}(z) \to B$ be a homomorphism. We claim that $\rho = 0$.

Assume that $B \neq 0$. Replacing B by $\rho(1_{\mathbb{C}[z]})B\rho(1_{\mathbb{C}[z]})$ we can assume that ρ is unital and $B \neq 0$. Let f be in $\mathbb{C}(z) \setminus \mathbb{C}$ (i.e., f is not constant). Since $\mathbb{C}(z)$ is a field we conclude that $(\lambda - f)^{-1}$ exists for every λ in $\mathbb{C}$. Therefore $\sigma(f) = \emptyset$. Since $\sigma(\rho(f)) \subseteq \sigma(f)$ by Lemma 1.45 we conclude that $\sigma(\rho(f)) = \emptyset$. In view of Example 2.19 we conclude that $B = 0$. This is a contradiction. □

Example 7.5. We consider the $*$-algebra of differential operators $D(\mathbb{C})$ on $\mathbb{C}$ from Example 3.6.

Lemma 7.6. *The maximal norm on $D(\mathbb{C})$ vanishes.*

Proof. We show that every homomorphism $\rho : D(\mathbb{C}) \to B$ with a C^*-algebra B is trivial.

Assume by contradiction that it is non-trivial. Then we can assume that it is unital and $B \neq 0$. Otherwise we replace B by $\rho(1_{D(\mathbb{C})})B\rho(1_{D(C)})$. Then $\rho(z) \neq 0$ since $[\partial, z] = 1_{D(\mathbb{C})}$ and hence $[\rho(\partial), \rho(z)] = \rho(1_{D(\mathbb{C})}) = 1_B \neq 0$. Let $x := z/\|\rho(z)\|_B$. Then $\|\rho(x)\|_B = 1$. Since x is selfadjoint we have $\|\rho(x^k)\|_B = 1$ for all k in $\mathbb{N}$. We have the relation

$$[\partial, x^k] = \frac{k}{\|\rho(z)\|_B} x^{k-1} .$$

This implies

$$\|[\rho(\partial), \rho(x^k)]\|_B = \frac{k}{\|\rho(z)\|_B} .$$

On the other hand we have $\|[\rho(\partial), \rho(x^k)]\|_B \leq 2\|\rho(\partial)\|_B$ for all k in $\mathbb{N}$. This is a contradiction. $\square$

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ with norm $\| - \|_A$. Note that $\| - \|_A$ is the unique C^*-norm on A such that A is complete.

Lemma 7.7. *We have $\| - \|_{\max} = \| - \|_A$.*

Proof. Let a be in A. For every $\rho : A \to B$ we have

$$\|\rho(a)\|_B \leq \|a\|_A$$

by the automatic continuity result Corollary 3.26. This implies $\|a\|_{\max} \leq \|a\|_A$. Since we can consider the identity representation $\mathtt{id}_A : A \to A$ in place of ρ we get the reverse inequality $\|a\|_A \leq \|a\|_{\max}$. $\square$

Let A in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Lemma 7.8. *The maximal norm on A has the following properties:*

1. *For all a in A and λ in $\mathbb{C}$ we have $\|\lambda a\|_{\max} = |\lambda|\|a\|_{\max}$.*
2. *For all a in A we have $\|a^*\|_{\max} = \|a\|_{\max}$.*
3. *For all a, a' in A we have $\|aa'\|_{\max} \leq \|a\|_{\max}\|a'\|_{\max}$.*
4. *For all a, a' in A we have $\|a + a'\|_{\max} \leq \|a\|_{\max} + \|a'\|_{\max}$.*
5. *For all a, a' in A we have $\|a^*a\|_{\max} = \|a\|^2_{\max}$ (C^*-property).*
6. *For every morphism $\phi : A \to A'$ in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a in A we have $\|\phi(a)\|_{\max} \leq \|a\|_{\max}$.*

Proof. We use that the norm of any C^*-algebra has these properties.

$$\begin{aligned} \|\lambda a\|_{\max} &= \sup_{\rho:A\to B} \|\rho(\lambda a)\|_B \\ &= \sup_{\rho:A\to B} \|\lambda\rho(a)\|_B \\ &= \sup_{\rho:A\to B} |\lambda|\|\rho(a)\|_B \\ &= |\lambda|\|a\|_{\max} . \end{aligned}$$

$$
\begin{aligned}
\|a^*\|_{\max} &= \sup_{\rho:A\to B} \|\rho(a^*)\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)^*\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)\|_B \\
&= \|a\|_{\max} \ .
\end{aligned}
$$

$$
\begin{aligned}
\|a+a'\|_{\max} &= \sup_{\rho:A\to B} \|\rho(a+a')\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)+\rho(a')\|_B \\
&\leq \sup_{\rho:A\to B} \|\rho(a)\|_B + \|\rho(a')\|_B \\
&\leq \sup_{\rho:A\to B} \|\rho(a)\|_B + \sup_{\rho:A\to B} \|\rho(a')\|_B \\
&= \|a\|_{\max} + \|a'\|_{\max} \ .
\end{aligned}
$$

$$
\begin{aligned}
\|aa'\|_{\max} &= \sup_{\rho:A\to B} \|\rho(aa')\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)\rho(a')\|_B \\
&\leq \sup_{\rho:A\to B} \|\rho(a)\|_B\|\rho(a')\|_B \\
&\leq \sup_{\rho:A\to B} \|\rho(a)\|_B \sup_{\rho:A\to B} \|\rho(a')\|_B \\
&= \|a\|_{\max}\|a'\|_{\max}
\end{aligned}
$$

$$
\begin{aligned}
\|aa^*\|_{\max} &= \sup_{\rho:A\to B} \|\rho(aa^*)\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)\rho(a)^*\|_B \\
&= \sup_{\rho:A\to B} \|\rho(a)\|_B^2 \\
&= \|a\|_{\max}^2
\end{aligned}
$$

For last assertion note that

$$
\|\phi(a)\|_{\max} = \sup_{\rho':A'\to B} \|\rho'(\phi(a))\|_B \leq \sup_{\rho:A\to B} \|\rho(a)\|_B = \|a\|_{\max} \ , \tag{7.1}
$$

where the first inequality comes from the fact that the right supremum amounts to take the supremum of $\|\rho(a)\|_B$ over representations $\rho : A \to B$ which factor over ϕ. □

We consider A in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$.

Definition 7.9. *A is called a pre-C^*-algebra if $\|-\|_{\max}$ is finite on A.*

We let ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ denote the full subcategory of ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of pre-C^*-algebras.

By Lemma 7.7 we have the following fact.

Corollary 7.10. *We have an inclusion $C^*\mathbf{Alg}^{\mathrm{nu}} \subseteq {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$.*

Let A be in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and consider a subset S of A. Let $\langle S\rangle_A$ in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ denote the subalgebra of A generated by S.

Lemma 7.11. *If S consists of orthogonal projections and partial isometries, then $\langle S\rangle_A \in {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$.*

Proof. We consider s in S. Then for for every $\rho : \langle S\rangle_A \to B$ the image $\rho(s)$ is a partial isometry or a projection. Hence $\|\rho(s)\|_B \le 1$. This implies that $\|s\|_{\max} \le 1$. Since every elements of $\langle S\rangle_A$ is a finite linear combination of finite products of elements of $S \cup S^*$ we conclude, using Lemma 7.8, that $\|a\|_{\max} < \infty$ for all a in $\langle S\rangle_A$. □

Example 7.12. Let A and B be C^*-algebras. Then we can form the algebraic tensor product $A \otimes^{\mathrm{alg}} B$ with involution $(a \otimes b)^* = a^* \otimes b^*$.

Lemma 7.13. *The algebraic tensor product $A \otimes^{\mathrm{alg}} B$ is a pre-C^*-algebra.*

Proof. (Sketch) Let $\pi : A \otimes^{\mathrm{alg}} B \to C$ be a homomorphism into a C^*-algebra. We show that

$$\|\pi(a \otimes b)\|_C \le \|a\|_A \|b\|_B \ .$$

If A and B are unital, then we can define homomorphisms $\pi_A : A \to C$ by $\pi_A(a) := \pi(a \otimes 1_B)$ and $\pi_B : B \to C$ by $\pi_B(b) := \pi(1_A \otimes b)$. We then conclude

$$\|\pi(a \otimes b)\| = \|\pi_A(a)\pi_b(B)\|_C \le \|\pi_A(a)\|_C \|\pi_B(b)\|_C \le \|a\|_A \|b\|_B \ .$$

This implies

$$\|a \otimes b\|_{\max} \le \|a\|_A \|b\|_B \ .$$

Since every element of $A \otimes^{\mathrm{alg}} B$ is a finite linear combination of elements of the form $a \otimes b$ we conclude, using Lemma 7.8, that all these elements have finite maximal norm.

The non-unital case is considerably more complicated. Using the notion of the multiplier algebra $M(C)$ of C we argue as follows. Without loss of generality we can assume that the image of π is dense in C (otherwise replace C by the closure of the image). Then there exists homomorphisms $\pi_A : A \to M(C)$ and $\pi_B : B \to M(C)$ such that $\pi(a \otimes b) = \pi_A(a)\pi_B(a)$. Using these representations and the fact that $C \to M(C)$

is isometric we then argue as in the unital case. The representations π_A and π_B are characterized by $\pi_A(a)\pi(a'\otimes b') := \pi(aa'\otimes b')$ and $\pi_B(b)\pi(a'\otimes b') := \pi(a'\otimes bb')$. The difficulty consists in showing that these representations are still well-defined. One first shows that for every a in A the linear map $B\mapsto \pi(a\otimes b)\in C$ is continuous, and similarly, for every b in B the linear map $A\mapsto \pi(a\otimes b)\in C$ is continuous. Then one uses an approximative unit $(u_\nu)_\nu$ of B and observes that

$$\lim_{\nu} \pi(a\otimes u_n)\pi(a'\otimes b') = \lim_{\nu}\pi(aa'\otimes u_\nu b') = \pi(aa'\otimes b')\ .$$

This implies that the multiplier $\pi_A(a)$ is well-defined on $\pi(A\otimes^{\mathrm{alg}} B)$ and bounded by $\|\pi(a\otimes -)\|_{B\to C}$. □

We now study the relations between the categories $C^*\mathbf{Alg}^{\mathrm{nu}}$, ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and their unital versions.

Lemma 7.14. *We have colocalizations*

$$\mathrm{incl} : {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} : \mathrm{Bd}^\infty\ ,\quad \mathrm{incl} : {}_{\mathrm{pre}}C^*\mathbf{Alg} \leftrightarrows {}^*\mathbf{Alg}_{\mathbb{C}} : \mathrm{Bd}^\infty$$

Proof. We give the argument in the non-unital case. The unital case is similar. We first provide an explicit formula for the right-adjoint. Let A be ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Then we define the subset

$$\mathrm{Bd}(A) := \{a\in A\mid \|a\|_{\max}<\infty\}$$

of A. Using the properties of the maximal norm listed in Lemma 7.8 we see that $\mathrm{Bd}(A)$ is a subalgebra of A. If $\phi : A\to B$ is a homomorphismus, then

$$\mathrm{Bd}(\phi) : \mathrm{Bd}(A)\to \mathrm{Bd}(B)$$

is defined by restriction of ϕ using (7.1). This turns Bd into a functor

$$\mathrm{Bd} : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}\ .$$

Note that A is in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ if and only if $\mathrm{Bd}(A) = A$.

We now define by transfinite induction a decreasing family $(\mathrm{Bd}^\alpha(A))_\alpha$ indexed by ordinals.

1. $\mathrm{Bd}^0(A) := A$

2. $\mathrm{Bd}^{\alpha+1}(A) := \mathrm{Bd}(\mathrm{Bd}^\alpha(A))$

3. $\mathrm{Bd}^\alpha(A) := \bigcap_{\beta<\alpha}\mathrm{Bd}^\beta(A)$ if α is limit ordinal.

By construction, if $\alpha\le\alpha'$, then we have $\mathrm{Bd}^{\alpha'}(A)\subseteq \mathrm{Bd}^\alpha(A)$. If $\alpha'\ge\alpha\ge|A|$ then $\mathrm{Bd}^{\alpha'}(A) = \mathrm{Bd}^\alpha(A)$. So the decreasing family eventually stabilizes. We define

$$\mathrm{Bd}^\infty(A) := \bigcap_\alpha \mathrm{Bd}^\alpha(A)\ .$$

We claim that $\mathrm{Bd}^\infty(A) \in {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$. Indeed this follows from

$$A = \mathrm{Bd}(\mathrm{Bd}^\infty(A)) = \mathrm{Bd}^{\infty+1}(A) = \mathrm{Bd}^\infty(A) \ .$$

We have defined a functor

$$\mathrm{Bd}^\infty : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}} \ .$$

We now construct the adjunction. The counit $c : \mathrm{incl} \circ \mathrm{Bd}^\infty \Rightarrow \mathtt{id}$ of the adjunction is given by the canonical inclusions

$$c_A : \mathrm{Bd}^\infty(A) \to A$$

for all A in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Naturality is straightforward to check. For every B in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ we consider the map

$$\beta : \mathtt{Hom}_{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}}(B, \mathrm{Bd}^\infty(A)) \stackrel{\mathrm{incl}}{\cong} \mathtt{Hom}_{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(B, \mathrm{Bd}^\infty(A)) \stackrel{c_{A,*}}{\to} \mathtt{Hom}_{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(B, A) \ .$$

The first isomorphism reflects the fact that the functor incl is fully faithful by definition. We claim that β is a bijection. Since c_A is injective, β is injective. In order to show that β is also surjective, we consider a homomorphism $\phi : B \to A$, where B is in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$. Using that ϕ is contractive w.r.t. to the maximal norms (see (7.1)) we conclude that ϕ takes values in $\mathrm{Bd}^\infty(A)$. Hence ϕ factorizes over c_A and is therefore in the image of β. □

Example 7.15. We have $\mathrm{Bd}^\infty(\mathbb{C}[z]) = \mathbb{C}$. In fact, already $\mathrm{Bd}(\mathbb{C}[z]) = \mathbb{C}$. In order to see this let p be in $\mathbb{C}[z]$ non-constant. Then $\|\rho_t(p)\| = |p(t)|$. We have $\sup_{t\in\mathbb{R}} \|\rho_t(p)\| = \sup_{t\in\mathbb{R}} |p(t)| = \infty$. Hence $\|p\|_{\max} = \infty$. See Example 7.2 for notation.

Lemma 7.8 has the following consequence.

Corollary 7.16. *If A is in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$, then $\| - \|_{\max}$ is a semi norm*

Lemma 7.17. *We have localizations*

$$\mathrm{compl} : C^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl} \ , \quad \mathrm{compl} : C^*\mathbf{Alg} \leftrightarrows {}_{\mathrm{pre}}C^*\mathbf{Alg} : \mathrm{incl}$$

Proof. We discuss the non-unital case. The unital case is analoguous.

We first construct the left-adjoint which is called the completion functor. Let A be in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$. We first observe that

$$I := \{a \in A \mid \|a\|_{\max} = 0\}$$

is $*$-ideal. Indeed, if i is in I, then also $i^* \in I$ since $\|i^*\|_{\max} = \|i\|_{\max} = 0$. If $i, i' \in I$ and $\lambda \in \mathbb{C}$, then

$$\|i + \lambda i'\|_{\max} \le \|i\|_{\max} + |\lambda| \|i'\|_{\max} = 0 \ ,$$

hence $i + \lambda i' \in I$. Finally, if i is in I and a is in A, then $\|ai\|_{\max} \le \|a\|_{\max}\|i\|_{\max} = 0$, hence $ai \in I$.

We form the quotient A/I in ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. We define a seminorm

$$\|[a]\| := \|a\|_{\max} \ .$$

This seminorm is well-defined since

$$\|a+i\|_{\max} \le \|a\|_{\max} + \|i\|_{\max} = \|a\|_{\max} = \|a+i-i\|_{\max} \le \|a+i\|_{\max} + \|i\|_{\max} = \|a+i\|_{\max}$$

for all i, and therefore the inequalities are equalities. If $\|[a]\| = 0$, then $a \in I$ and hence $[a] = 0$. Therefore $\| - \|$ is a norm. on A/I. One checks using Lemma 7.8 that $\| - \|$ is a submultiplicative $*$-norm and satisfies the C^*-equality.

We let $\mathrm{compl}(A)$ denote the completion of A/I in the sense of normed vector spaces. Note that $A/I \to \mathrm{compl}(A)$ is injective. Since $\| - \|$ is submultiplicative the multiplication extends by continuity. The extension of the norm to the completion exhibits $\mathrm{compl}(A)$ as a C^*-algebra.

Let $\phi : A \to B$ be a homomorphism. We write I_A and I_B for the ideals of zero elements on A and B. By (7.1) we see that $\phi(I_A) \subseteq I_B$. Hence we get an induced homomorphism represented by the dotted arrow in

$$\begin{array}{ccc} I_A & \xrightarrow{\phi_{|I_B}} & I_B \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A/I_A & \dashrightarrow & B/I_B \\ \downarrow & & \downarrow \\ \mathrm{compl}(A) & \xrightarrow{\mathrm{compl}(\phi)} & \mathrm{compl}(B) \end{array} \ .$$

The dashed arrow is obtained from the universal property of the completion.

We now have constructed a functor

$$\mathrm{compl} : {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}} \ .$$

The unit of the adjunction $\mathtt{id} \Rightarrow \mathrm{incl} \circ \mathrm{compl}$ is given by the canonical homomorphisms

$$i_A : A \to \mathrm{compl}(A) \ .$$

For A in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ and B in $C^*\mathbf{Alg}^{\mathrm{nu}}$ we consider the map

$$\alpha : \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{compl}(A), B) \stackrel{\mathrm{incl}}{\cong} \mathtt{Hom}_{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{incl}(\mathrm{compl}(A)), \mathrm{incl}(B)) \stackrel{\iota_A^*}{\to} \mathtt{Hom}_{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}}(A, \mathrm{incl}(B)) \ .$$

We show that α is a bijection. We first show that α is injective. Assume that ϕ, ϕ' are in $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{compl}(A), B)$ such that $\alpha(\phi) = \alpha(\phi')$. Then ϕ and ϕ' coincide on the image of i_A. Since this is dense in $\mathrm{compl}(A)$, we conclude $\phi = \phi'$ by continuity.

We now show that α is surjective. Let ϕ be in $\mathtt{Hom}_{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}}(A, \mathrm{incl}(B))$. Then $\|\phi(a)\|_B \leq \|a\|_{\max}$. Therefore $\phi(I_A) = 0$. Hence we have a factorization given by the dotted arrow in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow & \nearrow & \uparrow \phi \\ A/I & \longrightarrow & \mathrm{compl}(A) \end{array} \quad .$$

The dashed arrow is the continuous extension of the dotted arrow to the completion which exists by the universal property of the latter and since B is complete. By construction we have $\alpha(\phi) = \psi$. Hence α is surjective. □

Example 7.18. Let A and B be in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Then we have $A \otimes^{\mathrm{alg}} B$ in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ by Lemma 7.13.

Definition 7.19. *The maximal tensor product is defined by*

$$A \otimes_{\max} B := \mathrm{compl}(A \otimes^{\mathrm{alg}} B) \ .$$

One can check that for C in $C^*\mathbf{Alg}^{\mathrm{nu}}$ the set $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A \otimes B, C)$ is in bijection with the set of bilinear maps $A \times B \to C$ which are compatible with the involutions. □

Remark 7.20. We have a functor

$$\mathrm{compl} \circ \mathrm{Bd}^{\infty} : {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}} \to C^*\mathbf{Alg}^{\mathrm{nu}} \ .$$

It is a composition of a right-adjoint and a left adjoint.

1. $\mathrm{compl} \circ \mathrm{Bd}^{\infty}(\mathbb{C}[z])) \cong \mathbb{C}$.

2. $\mathrm{compl}(\mathbb{C}(z)) = 0$

3. $\mathrm{compl}(D(z)) = 0$

□

We now want to show that the categories of pre-C^*-algebras and C^*-algebras are complete and cocomplete. We will use the following fact. Assume that

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

is a reflective localization, i.e., an adjunction such that R is fully faithful.

Proposition 7.21.

1. *If $\mathcal{C}$ is complete, then so is $\mathcal{D}$. The functor R preserves and detects limits.*

2. *If $\mathcal{C}$ is cocomplete, then so is $\mathcal{D}$. If $D : \mathbf{I} \to \mathcal{D}$ is a diagram in $\mathcal{D}$, then*

$$\operatorname{colim}_{\mathbf{I}} D \cong L(\operatorname{colim}_{\mathbf{I}} R(D)) \ .$$

Proof. Since R is fully faithful, we can identify $\mathcal{D}$ with the essential image of R. We will omit the inclusion from the notation.

We first consider limits. Let W be the class of morphisms in $\mathcal{C}$ which are send to isomorphisms by L. An object C of $\mathcal{C}$ is called W-local if for every w in W the morphism $\mathtt{Hom}_{\mathcal{C}}(w, C)$ is an isomorphism. We claim that $\mathcal{D}$ consists exactly of the W-local objects. Indeed, if D is in $\mathcal{D}$, then it is W-local since $\mathtt{Hom}_{\mathcal{C}}(w, D) \cong \mathtt{Hom}_{\mathcal{C}}(L(w), D)$.

Assume now that C is W-local. Let $\eta : C \to L(C)$ be the unit of the adjunction. We show that η is an isomorphism. This implies that $D \in \mathcal{D}$. The map η itself belongs to W since

$$L(C) \stackrel{L(\eta)}{\to} L(\mathrm{incl}(L(C))) \stackrel{counit \circ L}{\to} L(C)$$

is an isomorphism by the triple identity of the adjunction (here it is useful to write the inclusion), and the counit is an isomorphism since R is fully faithful. Since we assume that C is W-local

$$\mathtt{Hom}_{\mathcal{C}}(L(C), C) \stackrel{\eta^*}{\to} \mathtt{Hom}_{\mathcal{C}}(C, C)$$

is an isomorphism. We let $\kappa : L(C) \to C$ be the preimage of $\mathtt{id}_C$. Then by definition $\kappa \circ \eta = \mathtt{id}_C$. This implies that $\kappa \in W$ since $L(\kappa) \circ L(\eta) = \mathtt{id}_{L(C)}$ and $L(\eta)$ is an isomorphism. Furthermore

$$\mathtt{Hom}_{\mathcal{C}}(C, L(C)) \stackrel{\kappa^*}{\to} \mathtt{Hom}_{\mathcal{C}}(L(C), L(C)) \ .$$

is an isomorphism. Hence there exists $\delta : C \to L(C)$ such that $\delta \circ \kappa = \mathtt{id}_{L(C)}$. Both equalities togther imply that $\delta = \eta$ and hence η is invertible.

We now show that $\mathcal{D}$ is closed under limits. Let $D : \mathbf{I} \to \mathcal{D}$ be a diagram. Then for every w in W we have $\mathtt{Hom}_{\mathcal{C}}(w, \mathtt{lim}_{\mathbf{I}} D) \cong \mathtt{lim}_{\mathbf{I}} \mathtt{Hom}(w, D)$. Since a limit of isomorphisms is an isomorphism we conclude that $\mathtt{Hom}_{\mathcal{C}}(w, \mathtt{lim}_{\mathbf{I}} D)$ is an isomorphism. Since w is arbitrary we conclude that $\mathtt{lim}_{\mathbf{I}} D$ is W-local and hence in $\mathcal{D}$.

Since $\mathcal{D} \to \mathcal{C}$ is fully faithful, we can conclude that $\mathcal{D}$ has all limits. They are calculated in $\mathcal{C}$. This finishes the proof of Assertion 1.

Let $D : \mathbf{I} \to \mathcal{D}$ be a diagram in $\mathcal{D}$. Since $\mathbf{C}$ is cocomplete we can form the colimit $\mathtt{colim}_{\mathbf{I}} R(D)$. Its structure maps $(\iota_i : R(D_i) \to \mathtt{colim}_{\mathbf{I}} R(D))$ induce a bijection

$$\mathtt{Hom}_{\mathcal{C}}(\operatorname{colim}_{\mathbf{I}} R(D), R(D')) \cong \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathcal{D}}(R(D), R(D')) \ .$$

Using the adjunction and the fact that R is fully faithful (and therefore that the counit $L \circ R \stackrel{\cong}{\to} \mathtt{id}$ is an isomorphism) we conclude that the structure maps

$$D_i \cong L(R(D_i)) \stackrel{L(\iota_i)}{\to} L(\mathtt{colim}_{\mathbf{I}} R(D)) \tag{7.2}$$

induce a bijection

$$\mathtt{Hom}_{\mathcal{D}}(L(\mathtt{colim}_{\mathbf{I}} R(D)), D') \cong \mathtt{lim}_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathcal{D}}(D, D') \ .$$

Hence the family of structure maps (7.2) for i in $\mathbf{I}$ presents $L(\mathtt{colim}_{\mathbf{I}} R(D))$ as the colimit of the diagram D in $\mathcal{D}$.

We can conclude that $\mathcal{D}$ is cocomplete. □

Corollary 7.22.

1. *The categories ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ and ${}_{\mathrm{pre}}C^*\mathbf{Alg}$ are cocomplete. The inclusion ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}} \to {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ (and ${}_{\mathrm{pre}}C^*\mathbf{Alg} \to {}^*\mathbf{Alg}_{\mathbb{C}}$, respectively) detects and preserves colimits.*

2. *The categories ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ and ${}_{\mathrm{pre}}C^*\mathbf{Alg}$ are complete. The limit of a diagram $C : \mathbf{I} \to {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ (or $C : \mathbf{I} \to {}_{\mathrm{pre}}C^*\mathbf{Alg}$, respectively) is calculated by*

$$\mathtt{lim}_{\mathbf{I}} C \cong \mathrm{Bd}^{\infty}(\mathtt{lim}_{\mathbf{I}} \mathrm{incl}(C)) \ .$$

Proof. Use the analog of Proposition 7.21 for colocalizations obtained by taking opposites and apply it to the colocalizations from Lemma 7.14. □

Corollary 7.23.

1. *The categories $C^*\mathbf{Alg}^{\mathrm{nu}}$ and $C^*\mathbf{Alg}$ are complete. The inclusion $C^*\mathbf{Alg}^{\mathrm{nu}} \to {}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ (and $C^*\mathbf{Alg} \to {}_{\mathrm{pre}}C^*\mathbf{Alg}$, respectively) detects and preserves limits.*

2. *The categories $C^*\mathbf{Alg}^{\mathrm{nu}}$ and $C^*\mathbf{Alg}$ are cocomplete. The colimit of a diagram $C : \mathbf{I} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ (or $C : \mathbf{I} \to C^*\mathbf{Alg}$, respectively) is calculated by*

$$\mathtt{colim}_{\mathbf{I}} C \cong \mathrm{compl}(\mathtt{colim}_{\mathbf{I}} \mathrm{incl}(C)) \ .$$

Proof. Use Proposition 7.21 for the localizations from Lemma 7.17. □

8 Some limits and colimits in $C^*\mathrm{Alg}$ and $C^*\mathrm{Alg}^{\mathrm{nu}}$

Let $(A_i)_{i \in I}$ be a family of C^*-algebras in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Lemma 8.1. *The product* $\prod_{i\in I}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_i$ *is the subalgebra of the product* $\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i$ *consisting of sequences* $(a_i)_{i\in I}$ *with*

$$\|(a_i)_i\| := \sup_{i\in I} \|a_i\|_{A_i} < \infty \ .$$

Proof. By Corollary 7.22 the product of the family in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ is given by

$$\prod_{i\in I}^{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}} A_i = \mathrm{Bd}^{\infty}(\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i) \ .$$

If fact if a sequence $(a_i)_i$ in $\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i$ is in $\mathrm{Bd}(\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i)$ then $\|(a_i)_i\| < \infty$ since we can using the projections to a components as test representations. We now observe that the subalgebra of bounded sequences with the norm $\| - \|$ is actually a C^*-algebra. This immediately implies that $\mathrm{Bd}(\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i) = \mathrm{Bd}^{\infty}(\prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i)$ is exactly the subalgebra of bounded sequences. By Corollary 7.22 this subalgebra also represents the product in $C^*\mathbf{Alg}^{\mathrm{nu}}$. $\square$

If the family $(A_i)_{i\in I}$ has infinitely many members, then the inclusion

$$\prod_{i\in I}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_i \to \prod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i$$

is proper.

Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \ .$$

be an equalizer diagram in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Lemma 8.2. *The equalizer* $\mathrm{Eq}^{C^*\mathbf{Alg}^{\mathrm{nu}}}(f,g)$ *is the subalgebra* $\{a \in A \mid f(a) = g(a)\}$ *of* A.

Proof. The equalizer $\mathrm{Eq}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(f,g) = \{a \in A \mid f(a) = g(a)\}$ is a closed subalgebra of A and hence a C^*-algebra. It follows that

$$\mathrm{Eq}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}(f,g) = \mathrm{Eq}^{{}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}}(f,g) = \mathrm{Eq}^{C^*\mathbf{Alg}^{\mathrm{nu}}}(f,g) \ .$$

$\square$

Let $(A_i)_{i\in I}$ be a family of C^*-algebras in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Lemma 8.3. *We have*

$$\coprod_{i\in I}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_i \cong \mathrm{compl}(\coprod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i)\ .$$

Proof. We have

$$\coprod_{i\in I}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_i \cong \mathrm{compl}(\coprod_{i\in I}^{{}^{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}} A_i) = \mathrm{compl}(\coprod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i)\ .$$

□

Note that $\coprod_{i\in I}^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} A_i$ is the algebraic free product. The completion is usually strictly bigger.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$, and let I be a closed $*$-ideal in A. The quotient $A/^{C^*\mathbf{Alg}^{\mathrm{nu}}}I$ is defined as the push-out

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A/^{C^*\mathbf{Alg}^{\mathrm{nu}}}I \end{array} \tag{8.1}$$

in $C^*\mathbf{Alg}^{\mathrm{nu}}$. By construction it is the completion of the quotient $A/^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}I$ taken in the sense of algebras. In order to work with quotients we must understand this completion explicitly.

On $A/^{{}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}}I$ can consider the norm obtained by taking the quotient in the sense of normed vector spaces, i.e., $\|[a]\| := \inf_{i\in I}\|a+i\|$. Our next goal is to show that this is exactly the norm on $A/^{C^*\mathbf{Alg}^{\mathrm{nu}}}I$.

Let $(u_\nu)_{\mu\in N}$ be an approximate unit of I.

Lemma 8.4. *We have*

$$\|[a]\| = \lim_{\nu\in N}\|a-u_\nu a\|\ . \tag{8.2}$$

Proof. Since $u_\nu a\in I$ for every ν in N we have $\|[a]\|\le\|a-u_\nu a\|$. This implies

$$\|[a]\| \le \liminf_{\nu\in N}\|a-u_\nu a\|\ . \tag{8.3}$$

We fix ϵ in $(0,\infty)$. Then there exists i in I such that $\|a+i\|\le\|[a]\|+\epsilon/2$. There exists ν_0 in N such that $\|i-u_\nu i\|\le\epsilon/2$ for all ν in N with $\nu\ge\nu_0$. Then (working in A^u if A is not unital)

$$\begin{aligned} \|a - u_\nu a\| &= \|(a+i) - u_\nu(a+i) - (i - u_\nu i)\| \\ &\le \|(1_A - u_\nu)(a+i)\| + \|i - u_\nu i\| \\ &\le \|a+i\| + \|i - u_\nu i\| \\ &\le \|[a]\| + \epsilon \end{aligned}$$

for all such ν. Since we can chose ϵ arbitrary small this implies

$$\limsup_{\nu \in N} \|a - u_\mu a\| \le \|[a]\| \ . \tag{8.4}$$

The inequalities (8.3) and (8.4) together imply the assertion. ☐

Lemma 8.5. *$A/^{C^*\mathbf{Alg}^{\mathrm{nu}}} I$ is isomorphic to the algebra A/I with the Banach-quotient norm.*

Note that this lemma says in particular that forming quotients does not involve any completion.

Proof. We use Corollary 7.23 saying that

$$A/^{C^*\mathbf{Alg}^{\mathrm{nu}}} I \cong \mathrm{compl}(A/^{*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} I) \ ,$$

where A/I is already known to be a pre-C^*-algebra by Corollary 7.22. We know that $\|-\|$ presents $A/^{*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}} I$ as a Banach space. It suffices to show that $\|-\|$ is submultiplicative and satisfies the C^*-identity. We have, writing $(a+i)(b+j) = ab + ib + aj + ij$ with $ib + aj + ij \in I$,

$$\|[a][b]\| = \|[ab]\| = \inf_{i \in I} \|ab + i\| \le \inf_i \|a + i\| \inf_j \|b + j\| = \|[a]\|\|[b]\| \ .$$

We now calculate, using that $I = I^*$,

$$\|[a]^*\| = \|[a^*]\| = \inf_{i \in I} \|a^* + i\| = \inf_{i \in I} \|a^* + i^*\| = \inf_{i \in I} \|a + i\| = \|[a]\| \ .$$

We finally verify the C^*-equality. We can assume that A is unital. Otherwise work in A^u. By Lemma 8.4 we have

$$\begin{aligned} \|[a]\|^2 &= \lim_{\nu \in N} \|a - u_\nu a\|^2 \\ &= \lim_{\nu \in N} \|(a - u_\nu a)(a^* - a^* u_\nu)\| \\ &= \lim_{\nu \in N} \|(1 - u_\nu) a a^* (1 - u_\nu)\| \\ &\le \lim_{\nu \in N} \|(1 - u_\nu) a a^*\| \\ &= \|[aa^*]\| \ . \end{aligned}$$

On the other hand we have

$$\|[aa^*] \le \|[a]\|\|[a^*]\| = \|[a]\|^2 \ .$$

Both inequalities togther imply

$$\|[a]\|^2 = \|[aa^*]\| \ .$$

□

Corollary 8.6. *The square* (8.1) *is also a pull-back square.*

Proof. The point here is that I is the kernel of the quotient map $A \to A/^{C^*\mathbf{Alg}^{\mathrm{nu}}} I$. This is not apriori clear, but follows from Lemma 8.5. □

From now on we will just write A/I for the quotient of a C^*-algebra by a closed ideal.

Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \ .$$

be a coequalizer diagram in $C^*\mathbf{Alg}^{\mathrm{nu}}$. We let I be a closed ideal generated by the elements $f(a) - g(a)$ for all a in A.

Corollary 8.7. *The quotient map* $\pi : B \to B/I$ *presents* B/I *as the coequalizer* $\mathrm{Coeq}^{C^*\mathbf{Alg}^{\mathrm{nu}}}(f, g)$.

Proof. First of all it is clear that $\pi \circ f = \pi \circ g$. Let C be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider a homomorphism $h : B \to C$ such that $h \circ f = h \circ g$. Then $h(f(a) - g(a)) = 0$ for all a in A. Hence h factorizes uniquely through a $*$-algebra morphism $\bar{h} : B/I \to C$. This is then also a morphism in $C^*\mathbf{Alg}^{\mathrm{nu}}$. □

The following examples demonstrate constructions of C^*-algebras by generators and relations. In general we first construct the free $*$-algebra on the generators, and then implement the relations by forming a quotient. Then we must ensure that the result is a pre-C^*-algebra so that we can form the closure. Sometimes there may be further closures with respect non-maximal norms.

Example 8.8. Let G be a group and consider the group $*$-algebra $\mathbb{C}[G]$ from Example 3.5.

Lemma 8.9. $\mathbb{C}[G]$ *is in* ${}_{\mathrm{pre}}C^*\mathbf{Alg}$.

Proof. Every element in $\mathbb{C}[G]$ is a finite linear combinations of elements $[g]$ for g in G. It suffices to show that $\|g\|_{\max} < \infty$. But since $[g]^* = [g^{-1}] = [g]^{-1}$ we see that $[g]$ is unitary and hence $\|[g]\|_{\max} \leq 1$. □

Definition 8.10. *The C^*-algebra $C^*(G) := \mathrm{compl}(\mathbb{C}[G])$ is called the maximal group C^*-algebra.*

We can consider $C^*(G)$ as the C^*-algebra which generated the C_2-set G (whose elements we write as $[g]$) with the action $[g] \mapsto [g^{-1}]$ subject to relations $[g][g'] = [gg']$.

This C^*-algebra has a natural representation ρ on the Hilbert space $L^2(G)$. It is determined by $(\rho([g])f)(h) := f(g^{-1}h)$.

$$\begin{aligned}(\rho([g][g'])f)(h) &= (\rho([gg'])f)(h) = f((gg')^{-1}h) \\ &= f(g'^{,-1}g^{-1}h) = \rho([g])(f(g'^{,-1}h)) = (\rho([g])\rho([g'])(f))(h)\end{aligned}$$

hence $\rho([g][g']) = \rho([g])\rho([g'])$. Furthermore

$$\langle \rho([g])(f), f' \rangle = \sum_{h \in G} \bar{f}(g^{-1}h)f'(h) = \sum_{h \in G} \bar{f}(h)f'(gh) = \langle f, \rho([g]^*)(f') \rangle \ ,$$

hence $\rho([g])^* = \rho([g]^*)$. We define the reduced norm $\| - \|_r$ on $\mathbb{C}[G]$ by $\|x\|_r := \|\rho(x)\|_{B(L^2(G))}$.

Definition 8.11. *We define the reduced group C^*-algebra $C^*_r(G)$ as the completion of $\mathbb{C}[G]$ with respect to the norm $\| - \|_r$.*

Equivalently, $C^*_r(G)$ is the sub-C^*-algebra of $B(L^2(G))$ generated by the elements $\rho([g])$ for all g in G which satisfy the relations as above.

Since $\| - \|_r \leq \| - \|_{\max}$ we have a canonical homomorphism

$$C^*(G) \to C^*_r(G) \ .$$

In general, this is not an isomorphism. It is one if G is amenable.

If $\kappa : G \to U(H)$ is any unitary representation of G on a Hilbert space, then we get an extension to a $*$-homomorphism $\kappa : \mathbb{C}[G] \to B(H)$ and hence a homomorphism $\kappa : C^*(G) \to B(H)$. We apply this to the trivial representation of G and get a character

$$\kappa_1 : C^*(G) \to \mathbb{C}$$

characterized by $\kappa_1([g]) = 1$ for all G. In general this homomorphism does not factorize through $C^*(G) \to C^*_r(G)$.

For example, let F_2 be the free group on two generators. Then $C^*_r(F)$ does not have any non-trivial finite-dimensional representation. But as the example above shows, $C^*(F_2)$ has one.

Example 8.12. We consider a finite set S and let S' be a second copy. We write s' for the copy of s in S'. Then $S \sqcup S'$ has a natural C_2-action sending s to s'. We can form the free unital $*$-algebra algebra $\mathrm{Free}^*(S \sqcup S')$. Inside this free algebra we consider the ideal I generated by the elements generated by

1. $s' - s^*$
2. $\sum_{s \in S} ss' - 1$
3. $s's - 1_A$.

Then $A_S := \mathrm{Free}^*(S \sqcup S')/I$. The $*$-algebra A_S is generated as a $*$-algebra by the image of the set S in A_S. The relations imply that $\|\rho(s)\| \le 1$ for any representation of A_S in a C^*-algebra. It follows that A_S is in ${}_{\mathrm{pre}}C^*\mathbf{Alg}$.

Definition 8.13. *The C^*-algebra $\mathcal{O}_S := \mathrm{compl}(A)$ is called the Cuntz algebra on S.*

One can show that A has a unique C^*-norm. Therefore any unital representation $\rho : \mathcal{O}_S \to B$ is injective. It is determined by fixing a collection of isometries $(b_s)_{s \in S}$ such that $\sum_s b_s b_s^* = 1_B$ and $\rho(s) = b_s$.

9 C^*-categories

In this lecture course we do not require that categories have identity morphisms. A category which has identity morphisms will be called unital. We have a category $\mathbf{Cat}^{\mathrm{nu}}$ of categories and functors. It contains the subcategory $\mathbf{Cat}$ of unital categories and identity preserving (unital) functors.

Example 9.1. A magma M can be considered as an object BM of $\mathbf{Cat}^{\mathrm{nu}}$. It has one object $*$, and its endmorphisms are given by $\mathtt{End}_{BM}(*) = M$ with the composition given by the magma structure on M.

If M is a monoid, then we have $BM \in \mathbf{Cat}$. □

We have an adjunction

$$(-)^u : \mathbf{Cat}^{\mathrm{nu}} \leftrightarrows \mathbf{Cat} : \mathrm{incl}$$

where the left-adjoint adds units to all endomorphism sets (a new one even if there was already one).

Definition 9.2. *A $\mathbf{Vect}_{\mathbb{C}}$-enriched category is a category is a category together with $\mathbb{C}$-vector space structures on the morphism sets such that the composition is bilinear.*

A functor between $\mathbf{Vect}_{\mathbb{C}}$-enriched categories is a functor which induces linear maps between the morphism sets.

We get the category $\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ of $\mathbf{Vect}_{\mathbb{C}}$-enriched categories and functors and the full subcategory $\mathbf{Cat}_{\mathbb{C}}$ of unital $\mathbf{Vect}_{\mathbb{C}}$-enriched categories and unital functors.

Example 9.3. If A is an algebra over $\mathbb{C}$, then the category of A-modules $\mathbf{Mod}(A)$ is naturally an object of $\mathbf{Cat}_{\mathbb{C}}$. The wide subcategory $\mathbf{Mod}(A)_c$ of $\mathbf{Mod}(A)$ of morphisms with finite-dimensional range belongs to $\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$. □

Example 9.4. The categories $\mathbf{Alg}_{\mathbb{C}}$ and $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$ are full subcategories of $\mathbf{Cat}_{\mathbb{C}}$ and $\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$ consisting of the categories with a single object.

If $\mathbf{C}$ is in $\mathbf{Cat}_{\mathbb{C}}$ and $\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$, then $\mathtt{End}_{\mathbf{C}}(C)$ belongs to $\mathbf{Alg}_{\mathbb{C}}$ or $\mathbf{Alg}_{\mathbb{C}}^{\mathrm{nu}}$, respectively, for every object C of $\mathbf{C}$. □

We have adjunctions

$$\mathbb{C}[-] : \mathbf{Cat}^{\mathrm{nu}} \leftrightarrows \mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} : \mathrm{forget}\ , \quad \mathbb{C}[-] : \mathbf{Cat} \leftrightarrows \mathbf{Cat}_{\mathbb{C}} : \mathrm{forget}\ ,$$

where the right-adjoints forget the enrichments, and the left-adjoints replace the morphism sets by free $\mathbb{C}$-vector spaces generated by them.

Example 9.5. For a magma M the magma algebra $\mathbb{C}[M]$ is the same as $\mathbb{C}[BM]$. □

We have a unitalization adjunction

$$(-)^u : \mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} \leftrightarrows \mathbf{Cat}_{\mathbb{C}} : \mathrm{incl}\ .$$

Applying the left-adjoint $(-)^u$ to $\mathbf{C}$ amounts to replace $\mathtt{End}_{\mathbf{C}}(C)$ by $\mathtt{End}_{\mathbf{C}}(C)^u$ for every object C in $\mathbf{C}$ and extending the composition in the obvious way.

The following has no analogue in the case of algebras.

Lemma 9.6. *We have an adjunction*

$$\mathrm{incl} : \mathbf{Cat}_{\mathbb{C}} \leftrightarrows \mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} : U\ .$$

Proof. We construct the functor U. Let $\mathbf{D}$ be in $\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$. Then $U(\mathbf{D})$ is given as follows.

1. objects: The objects of $U(\mathbf{D})$ are pairs (D, p) of an object D in $\mathbf{D}$ and a projection p in $\mathtt{End}_{\mathbf{D}}(D)$.

2. morphisms: We define $\mathtt{Hom}_{U(\mathbf{D})}((D, p), (D', p')) := p'\mathtt{Hom}_{\mathbf{D}}(D, D')p$.

3. composition and enrichment: The composition and the $\mathbb{C}$-enrichment is inheritied from $\mathbf{C}$.

Note that $U(\mathbf{D})$ is unital. The unit of the object (D, p) is given by p.

If $\phi : \mathbf{D} \to \mathbf{D}'$ is a functor, then $U(\phi) : U(\mathbf{D}) \to U(\mathbf{D}')$ is given as follows:

1. objects: $U(\phi)(D,p) := (\phi(D), \phi(p))$.

2. morphisms: $U(\phi)(f) := \phi(f)$.

Note that $U(\phi)$ is unital. The counit of the adjunction evaluated at $\mathbf{D}$ is the functor $\mathrm{incl}(U(\mathbf{D})) \to \mathbf{D}$ which sends (D,p) to D and is the canonical inclusion on the level of morphisms. We now check the universal property. We consider for $\mathbf{C}$ in $C^*\mathbf{Cat}$ the map

$$\alpha : \mathtt{Hom}_{C^*\mathbf{Cat}}(\mathbf{C}, U(\mathbf{D})) \stackrel{\mathrm{incl}}{\to} \mathtt{Hom}_{C^*\mathbf{Cat}^{\mathrm{nu}}}(\mathrm{incl}(\mathbf{C}), \mathrm{incl}(U(\mathbf{D}))) \stackrel{counit}{\to} \mathtt{Hom}_{C^*\mathbf{Cat}^{\mathrm{nu}}}(\mathrm{incl}(\mathbf{C}), \mathbf{D}) \ .$$

We must check that this map is a bijection. Let $\phi, \phi' : \mathbf{C} \to U(\mathbf{D})$ be two functors such that $\alpha(\phi) = \alpha(\phi')$. Then we have $\alpha(\phi(C)) = \alpha(\phi'(C))$ for all objects C' in $\mathbf{C}$ and also $\alpha(\phi(1_C)) = \alpha(\phi'(1_C))$. This implies $\phi(C) = \phi'(C)$ in $U(\mathbf{D})$. It is clear that α is injective on morphisms. We conclude that α is injective.

Let now $\psi : \mathrm{incl}(\mathbf{C}) \to \mathbf{D}$ be given. Then we define $\phi : \mathbf{C} \to U(\mathbf{D})$ by $\phi(C) := (\psi(C), \psi(1_C))$ and $\phi(f) := \psi(f)$. Then $\alpha(\phi) = \psi$. This shows surjectivity. □

Example 9.7. If A is in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$, then $U(A)$ in $\mathbf{Cat}_{\mathbb{C}}$ is in general a C^*-category with many objects, namely the set of projections in A. □

Example 9.8. For any $\mathbf{C}$ in $\mathbf{Cat}_{\mathbb{C}}$ we can consider the category $\mathbf{Fun}(\mathbf{C}, \mathbf{Vect}_{\mathbb{C}})$ of unital functors in $\mathbf{Cat}_{\mathbb{C}}$ and natural transformations. It is again an object of $\mathbf{Cat}_{\mathbb{C}}$.

We have a Yoneda embedding

$$Y : \mathbf{C} \to \mathbf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Vect}_{\mathbb{C}})$$

given as follows:

1. objects: An object C in $\mathbf{C}$ is sent to the functor $Y(C) := \mathtt{Hom}_{\mathbf{C}}(-, C)$

2. morphisms: A morphism $f : C \to C'$ in $\mathbf{C}$ is sent to the natural transformation $f_* : \mathtt{Hom}_{\mathbf{C}}(-, C) \to \mathtt{Hom}_{\mathbf{C}}(-, C')$.

We have the Yoneda Lemma

$$\mathtt{Hom}_{\mathbf{C}}(C, C') \cong \mathtt{Hom}_{\mathbf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Vect}_{\mathbb{C}})}(Y(C), Y(C'))$$

with the usual proof.

All this except the Yoneda Lemma extends to the non-unital case.

In the case of $*$-categories below the analogue of the Yoneda embedding turns out to be considerably more involved.

□

An involution on $\mathbf{C}$ in $\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ is a functor $* : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ which fixes objects, is anti-linear on morphism spaces, and satisfies $*^{\mathrm{op}} \circ * = \mathtt{id}$. In the unital case we require that $*$ is unital.

A $\mathbb{C}$-linear $*$-category as a pair of an object $\mathbf{C}$ in $\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ or $\mathbf{Cat}_{\mathbb{C}}$ together with an involution. We get the categories ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ and ${}^*\mathbf{Cat}_{\mathbb{C}}$ of $\mathbb{C}$-linear $*$-categories and $*$-preserving functors and the subcategory of unital $\mathbb{C}$-linear $*$-categories and $*$-preserving unital functors.

Example 9.9. The categories ${}^*\mathbf{Alg}_{\mathbb{C}}$ and ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ are full subcategories of $\mathbf{Cat}_{\mathbb{C}}$ and $\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ consisting of the categories with a single object. □

Example 9.10. The category $\mathbf{Hilb}(\mathbb{C})$ of Hilbert spaces and bounded operators belongs to ${}^*\mathbf{Cat}_{\mathbb{C}}$. The involution is given by taking adjoints. Its wide subcategory of Hilbert spaces and compact operators $\mathbf{Hilb}_c(\mathbb{C})$ belongs to ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$

We have a unitalization adjunction

$$(-)^u : {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}} : \mathrm{incl}\ .$$

Again the inclusion is also a left-adjoint.

Lemma 9.11. *We have an adjunction* $\mathrm{incl} : {}^*\mathbf{Cat}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} : U$.

Proof. The argument is similar as for Lemma 9.6. The only difference is that $U(\mathbf{D})$ consists of the pairs (D, p) with $p^* = p$. □

We have a functor

$$\mathrm{Ob} : C^*\mathbf{Cat}^{\mathrm{nu}} \to \mathbf{Set}$$

which takes the set of objects. We have a functor

$$\mathbb{C}[-] : \mathbf{Set} \to {}^*\mathbf{Cat}_{\mathbb{C}}$$

which sends a set to the category obtained by linearization of the set considered as a discrete category. There is a unique involution on this category. We furthermore have a functor

$$0[-] : \mathbf{Set} \to {}^*\mathbf{Cat}_{\mathbb{C}}$$

which sends a set to the zero-category with the given set of objects (and with zero morphism spaces).

Lemma 9.12. *We have the following adjunctions:*

$$\mathrm{Ob} : {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} \leftrightarrows \mathbf{Set} : 0[-] \tag{9.1}$$
$$0[-] : \mathbf{Set} \leftrightarrows {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} : \mathrm{Ob} \tag{9.2}$$
$$\mathrm{Ob} : {}^*\mathbf{Cat}_{\mathbb{C}} \leftrightarrows \mathbf{Set} : 0[-] \tag{9.3}$$
$$\mathbb{C}[-] : \mathbf{Set} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}} : \mathrm{Ob} \tag{9.4}$$

Proof. We just give the units or counits determining the structure. The counit of (9.1) at a set X given by the identity $\mathrm{Ob}(0[X]) \to X$.

The counit of (9.2) at $\mathbf{C}$ is given by the canonical functor $0[\mathrm{Ob}(\mathbf{C})] \to \mathbf{C}$. Note that this is not unital in general.

The counit of (9.3) at a set X is again given by the identity $\mathrm{Ob}(0[X]) \to X$.

The counit of (9.4) at $\mathbf{C}$ is given by the canonical functor $\mathbb{C}[\mathrm{Ob}(\mathbf{C})] \to \mathbf{C}$, which sends the unit of C considered as an object of $\mathbb{C}[\mathrm{Ob}(\mathbf{C})]$ to 1_C. This functor is unital. □

A representation of $\mathbf{C}$ in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ is a functor $\rho : \mathbf{C} \to A$ where A is in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$. Note that $C^*\mathbf{Alg}^{\mathrm{nu}}$ is a full subcategory of $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and hence of ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$.

A norm on $\mathbf{C}$ in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ is a function $\mathrm{Mor}(\mathbf{C}) \to [-\infty, \infty]$.

Definition 9.13. *We define the maximal norm* $\| - \|_{\max}$ *on* $\mathbf{C}$ *by*

$$\|f\| := \sup_{\rho:\mathbf{C}\to A} \|\rho(f)\|_A$$

where the supremum is taken over all representations of $\mathbf{C}$ *into some* A *in* $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Since we alway have the zero representation we have $\|f\| \in [0, \infty]$.

Lemma 9.14. *The maximal norm on* $\mathbf{C}$ *has the following properties:*

1. *For all* f *in* $\mathbf{C}$ *and* λ *in* $\mathbb{C}$ *we have* $\|\lambda f\|_{\max} = |\lambda|\|f\|_{\max}$.
2. *For all* f *in* $\mathbf{C}$ *we have* $\|f^*\|_{\max} = \|f\|_{\max}$.
3. *For all composeable* f, f' *in* $\mathbf{C}$ *we have* $\|ff'\|_{\max} \le \|f\|_{\max}\|f'\|_{\max}$.
4. *For all parallel* f, f' *in* $\mathbf{C}$ *we have* $\|f + f'\|_{\max} \le \|f\|_{\max} + \|f'\|_{\max}$.
5. *For all* f *in* $\mathbf{C}$ *we have* $\|f^*f\|_{\max} = \|f\|^2_{\max}$ *(*C^**-property).*
6. *For every functor* $\phi : \mathbf{C} \to \mathbf{C}'$ *in* ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ *and* f *in* $\mathbf{C}$ *we have* $\|\phi(f)\|_{\max} \le \|f\|_{\max}$.
7. *For every pair of morphisms* f, g *with the same domain we have the* C^**-inequality* $\|ff^*\|_{max} \le \|ff^* + gg^*\|_{max}$.

Proof. The argument is the same as for Lemma 7.8. □

Let $\mathbf{C}$ be in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$.

Definition 9.15. *We call* $\mathbf{C}$ *a pre-*C^**-category if* $\|f\| < \infty$ *for every morphism* f *in* $\mathbf{C}$*.*

We get the full subcategories ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}$ and ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$ of ${}^*\mathbf{Cat}_{\mathbb{C}}$ and ${}^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$, respectively. The categories ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$ and ${}_{\mathrm{pre}}C^*\mathbf{Alg}$ are full subcategories of ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$ and ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}$, respectively.

Lemma 9.16. *We have colocalizations*

$$\mathrm{incl} : {}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}} : \mathrm{Bd}^{\infty} \ , \quad \mathrm{incl} : {}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} : \mathrm{Bd}^{\infty} \ .$$

Proof. This is completely analoguous to the one of Lemma 7.14. □

These colocalizations restrict to the colocalizations from Lemma 7.14.

Let $\mathbf{C}$ be in ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$.

Definition 9.17. $\mathbf{C}$ *is called a* C^**-category if its morphism spaces are complete with respect to the maximal norm.*

Lemma 9.18. *We have adjunctions*

$$\mathrm{compl} : {}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}} \leftrightarrows C^*\mathbf{Cat} : \mathrm{incl} \ , \quad \mathrm{compl} : {}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Cat}^{\mathrm{nu}} : \mathrm{incl} \ .$$

Proof. The proof is completely analoguous to the proof of Lemma 7.17. □

We now show that all the categories above are complete and cocomplete.

To this end we consider the adjunction

$$\mathbb{C}[-] : {}^*\mathbf{Cat} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}} : \mathrm{incl} \ , \quad \mathbb{C}[-] : {}^*\mathbf{Cat}^{\mathrm{nu}} \leftrightarrows {}^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}} : \mathrm{incl}$$

where ${}^*\mathbf{Cat}$ and ${}^*\mathbf{Cat}^{\mathrm{nu}}$ are categories with involution (fixing objects) and the left adjoint linearizes the morphism spaces and extends the involution antilinearly. Note that this is not a localization since the right-adjoint is not fully faithful.

Our starting point is that $\mathbf{Cat}$ is complete and cocomplete. By the localization $(-)^u : \mathbf{Cat}^{\mathrm{nu}} \leftrightarrow \mathbf{Cat} : \mathrm{incl}$ also $\mathbf{Cat}^{\mathrm{nu}}$ is complete and cocomplete.

Lemma 9.19. *The categories* ${}^*\mathbf{Cat}$ *and* ${}^*\mathbf{Cat}^{\mathrm{nu}}$ *are complete and cocomplete.*

Proof. One calculates the limits and colimits on the level of underlying categories and then implements the involution by functoriality. Then one checks the universal properties. □

Lemma 9.20. *The categories* ${}^*\mathbf{Cat}_{\mathbb{C}}$ *and* ${}^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$ *are complete and cocomplete.*

Proof. One calculates the limits and colimits on the level of underlying $*$-categories. Then one introduces the $\mathbb{C}$-enrichment in the natural way and checks universal properties. $\square$

Corollary 9.21. *The categories* ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}$*,* ${}_{\mathrm{pre}}C^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}}$*,* $C^*\mathbf{Cat}$ *and* $C^*\mathbf{Cat}^{\mathrm{nu}}$ *are complete and cocomplete.*

Proof. Use the adjunctions. $\square$

Lemma 9.22 (M. Joachim)**.** *We have adjunctions*

$$A^f : C^*\mathbf{Cat} \leftrightarrows C^*\mathbf{Alg} : \mathrm{incl}\ , \quad A^f : C^*\mathbf{Cat}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl}\ .$$

Proof. We disuss the non-unital case. The unital case is analoguous.

We construct the C^*-algebra $A^f(\mathbf{C})$. We first form the free non-unital $*$-algebra $\mathrm{Free}^{*,\mathrm{nu}}(\mathrm{Mor}(C))$ on the C_2-set of morphisms if $\mathbf{C}$. In $\mathrm{Free}^{*,\mathrm{nu}}(\mathrm{Mor}(C))$ we form the ideal $I(\mathbf{C})$ generated by

1. $[f]+\lambda[g]-[f+\lambda g]$ if $f,g:C\to C'$ and λ in $\mathbb{C}$.
2. $[f][g]-[f\circ g]$ if $f\circ g$ is defined.

Then we set

$$A^{f,\mathrm{alg}}(\mathbf{C}) := \mathrm{Free}^{*,\mathrm{nu}}(\mathrm{Mor}(C))/I(\mathbf{C})\ .$$

We have a canonical functor

$$\mathbf{C}\to A^{f,\mathrm{alg}}(\mathbf{C})\ , \quad f\mapsto [f]\ .$$

The relations are minimal such that this functor is well-defined. We observe that this is a pre-C^*-algebra. To this end we show that $\|[f]\|_{max}\le \|f\|$. Let $\rho : A^{f,\mathrm{alg}}(\mathbf{C})\to B$ be a $*$-homomorphism. Let $f:C\to C'$. Then we get a $*$-homomorphism $\mathtt{End}_{\mathbf{C}}(C)\to A^f(\mathbf{C})\to B$. Since $\mathtt{End}_{\mathbf{C}}(C)$ is a C^*-algebra we see that

$$\|\rho([f])\|_B^2 = \|\rho([f^*f])\|_B \le \|f^*f\|_{\mathtt{End}_{\mathbf{C}}(C)} = \|f\|^2_{\mathbf{C}}\ .$$

We define

$$A^f(\mathbf{C}) := \mathrm{compl}(A^{f,\mathrm{alg}}(\mathbf{C}))\ .$$

We have a canonical functor

$$\mathbf{C}\to A^f(\mathbf{C})\ , \quad f\mapsto [f]\ .$$

This functor is natural in $\mathbf{C}$ and the unit of the adjunction.

We check that

$$\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A^f(\mathbf{C}),B) \stackrel{incl}{\to} \mathtt{Hom}_{C^*\mathbf{Cat}^{\mathrm{nu}}}(A^f(\mathbf{C}),B) \stackrel{unit^*}{\to} \mathtt{Hom}_{C^*\mathbf{Cat}^{\mathrm{nu}}}(\mathbf{C},B)$$

is a bijection.

We first show surjectivity. Let $\phi : \mathbf{C} \to B$ be given. Then we define $A^f(\mathbf{C}) \to B$ such that $[f] \mapsto \phi(f)$. A priori this defines a homomorphism $\mathrm{Free}^{*,\mathrm{nu}}(\mathrm{Mor}(C)) \to B$, but it annihilates the ideal $I(\mathbf{C})$ and therefore factorizes over $A^f(\mathbf{C})$. This homomorphism is a pre-image of ϕ.

In order to show injectivity, consider $\psi, \psi' : A^f(\mathbf{C}) \to B$ and assume that they go to the same functor $\mathbf{C} \to B$, then $\psi([f]) = \psi'([f])$ for all morphisms and hence $\psi = \psi'$. □

Some facts about C^*-categories can be deduced from the corresponding facts for C^*-algebras. It turns out that the functor A^f is not so appropriate for this purpose. We therefore associate to every C^*-category $\mathbf{C}$ another C^*-algebra $A(\mathbf{C})$.

We first define the algebra

$$A^{\mathrm{alg}}(\mathbf{C}) := \bigoplus_{C,C' \in \mathrm{Ob}(\mathbf{C})} \mathtt{Hom}_{\mathbf{C}}(C, C') \ .$$

We consider elements of $A^{\mathrm{alg}}(\mathbf{C})$ as matrices $M = (M_{C',C})$ indexed by the set $\mathrm{Ob}(\mathbf{C})$ with finitely many non-zero entries, where $M_{C',C} \in \mathtt{Hom}_{\mathbf{C}}(C, C')$. The composition is induced by matrix multiplication

$$(M'M)_{C'',C} := \sum_{C' \in \mathrm{Ob}(\mathbf{C})} M'_{C'',C'} \circ M_{C',C}$$

and the involution is given by

$$(M^*)_{C',C} := M^*_{C,C'} \ .$$

If $f : C \to C'$ is a morphism in $\mathbf{C}$, then we let $[f]$ denote the corresponding matrix with a single non-trivial entry. We have a canonical functor

$$\mathbf{C} \to A^{\mathrm{alg}}(\mathbf{C}) \ , \quad f \mapsto [f] \ .$$

It is initial for functors $\rho : \mathbf{C} \to B$ to $*$-algebras such that

$$\rho(f')\rho(f) = \begin{cases} \rho(f' \circ f) & f' \circ f \text{ is defined} \\ 0 & \textit{else} \end{cases}$$

Lemma 9.23. *$A^{\mathrm{alg}}(\mathbf{C})$ is in ${}_{\mathrm{pre}}C^*\mathbf{Alg}^{\mathrm{nu}}$.*

Proof. Let $\rho : A^{\mathrm{alg}}(\mathbf{C}) \to B$ be a homomorphism to a C^*-algebra. Let $f : C \to C'$ be a morphism in $\mathbf{C}$. Then we have

$$\|\rho(f)\|^2 = \|\rho(f)^*\rho(f)\|_B = \|\rho(f^*f)\|_B \le \|f^*f\|^2_{\mathtt{End}_{\mathbf{C}}(C)} = \|f\|_{\mathbf{C}} \ .$$

Since B is arbitrary this implies that $\|[f]\|_{\max} \le \|f\|_{\mathbf{C}}$. Since $A^{\mathrm{alg}}(\mathbf{C})$ is algebraically generated by elements of finite maximal norm we conclude that $A^{\mathrm{alg}}(\mathbf{C})$ is a pre-C^*-algebra. □

Definition 9.24. *We define the C^*-algebra $A(\mathbf{C}) := \mathrm{compl}(A^{\mathrm{alg}}(\mathbf{C}))$.*

The construction $\mathbf{C} \mapsto A(\mathbf{C})$ is functorial for functors which are injective on objects.

Our next goal is to show that the functor $\mathbf{C} \to A(\mathbf{C})$ is isometric. In order to do this we must show that $\|-\|_{\max}$ on $A^{\mathrm{alg}}(\mathbf{C})$ is sufficiently large. To this end we must construct representations of this algebra. These will be representations by endomorphisms on some Hilbert C^*-modules.

10 Hilbert C^*-modules and Yoneda

In this section we associate to any C^*-algebra A the C^*-category $\mathbf{Hilb}(A)$ of Hilbert C^*-modules over A. The notion of a Hilbert C^*-module generalizes the notion of a Hilbert space in the case of $A = \mathbb{C}$.

Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Let M be a right A-module.

Definition 10.1. *An A-valued scalar product on M is a map*

$$\langle -, - \rangle : M \times M \to A$$

with the following properties:

1. *$\langle -, - \rangle$ is $\mathbb{C}$-antilinear in the first and $\mathbb{C}$-linear in the second argument.*
2. *$\langle m, m' \rangle = \langle m', m \rangle^*$ for all m, m' in M*
3. *$\langle m, m'a \rangle = \langle m, m' \rangle a$ for all m, m' in M and a in A*
4. *$\langle m, m \rangle \geq 0$ for all m in M and $\langle m, m \rangle = 0$ if and only if $m = 0$.*

We define

$$\|m\| := \sqrt{\|\langle m, m \rangle\|} \ .$$

Lemma 10.2 (Cauchy-Schwarz)**.** *For all m, n in M we have the inequalities*

$$\langle m, n \rangle \langle m, n \rangle^* \leq \|n\|^2 \|\langle m, m \rangle \ , \quad \|\langle m, n \rangle\| \leq \|n\| \|m\| \ .$$

Proof. The second inequality follows from the first by applying $\|-\|$ and taking the root. In order to show the first we calculate

$$\begin{aligned} 0 &\leq \langle m - n\frac{\langle n, m \rangle}{\|n\|^2}, m - n\frac{\langle n, m \rangle}{\|n\|^2} \rangle \\ &= \langle m, m \rangle - \frac{2}{\|n\|^2} \langle m, n \rangle \langle n, m \rangle + \frac{1}{\|n\|^4} \langle m, n \rangle \langle n, n \rangle \langle n, m \rangle \end{aligned}$$

The inequality

$$\frac{\langle n,n\rangle}{\|n\|^2} \le 1$$

implies by Lemma 6.15 that

$$\frac{1}{\|n\|^4}\langle m,n\rangle\langle n,n\rangle\langle n,m\rangle \le \frac{1}{\|n\|^2}\langle m,n\rangle\langle n,m\rangle$$

We conclude that

$$0 \le \langle m,m\rangle - \frac{1}{\|n\|^2}\langle m,n\rangle\langle n,m\rangle$$

which implies the desired inequality. □

We define a norm in M by

$$\|m\| := \sqrt{\|\langle m,m\rangle\|}\ .$$

To see the triangle inequality note that

$$\|m+n\|^2 = \|\langle m+n,m+n\rangle\| = \|\langle m,m\rangle+\langle m,n\rangle+\langle n,m\rangle+\langle n,n\rangle\| \le \|m\|^2+\|n\|^2+2\|\langle m,n\rangle\|\ .$$

Using the Cauchy-Schwartz inequality we get $2\|\langle m,n\rangle\| \le 2\|m\|\|n\|$ and hence

$$\|m+n\|^2 \le (\|m\|+\|n\|)^2\ .$$

Definition 10.3. *$(M,\langle -,-\rangle)$ is a Hilbert A-module if M is complete w.r.t the norm $\|-\|$.*

Example 10.4. If $(M,\langle -,-\rangle)$ is a right A-module with a scalar product. Then we can form the closure $\bar{M}$ with respect to the norm $\|-\|$. The A-action and the scalar product extend by continuity and $(\bar{M},\langle -,-\rangle)$ is a Hilbert A-module. In order to see that the right multiplication extends note that

$$\|ma\| = \sqrt{\|\langle ma,ma\rangle\|_A} = \sqrt{\|a^*\langle m,m\rangle a\|_A} \le \|a\|_A\|m\|\ .$$

□

Example 10.5. The algebra A itself is a Hilbert A-C^*-module. Thereby we consider A as a right-module by right multiplication. The A-valued scalar product is defined by $\langle a,a'\rangle = a^*a'$. The norm on A as Hilbert module is the original norm of A:

$$\|a\| = \sqrt{\|\langle a,a\rangle\|_A} = \sqrt{\|a^*a\|_A} = \|a\|_A\ .$$

□

Example 10.6. Let $(M_i,\langle -,-\rangle_i)$ be a family of Hilbert A-modules. Then we form the algebraic sum

$$M := \bigoplus_{i\in I}^{\mathrm{alg}} M_i$$

and define the scalar product

$$\langle \oplus_i m_i, \oplus_i m'_i \rangle_M := \sum_{i\in I} \langle m_i, m'_i \rangle_i \ .$$

We then form the closure which will be denoted by $\bigoplus_{i\in I} M_i$. In particular

$$H_A := \bigoplus_{\mathbb{N}} A$$

is called the standard Hilbert A-module.

We consider a Hilbert A-module $(M, \langle -,- \rangle)$. Let $T, T' : M \to M$ be maps.

Definition 10.7. *We say that T' is adjoint to T if*

$$\langle Tm, n \rangle = \langle m, T'n \rangle$$

for all m, n in M.

Lemma 10.8. *If T admits an adjoint T', then*

1. *T is bounded.*
2. *T is an A-module map.*
3. *T' is uniquely determined.*
4. *T is the adjoint of T'.*

Proof. In order to show that T is bounded we use the closed graph theorem. Assume that $(m_i)_{i\in\mathbb{N}}$ is a sequence such that $m_i \to m$ and $Tm_i \to n$. Then we have for any k in M that

$$\begin{aligned} \langle Tm, k \rangle &= \langle m, T'k \rangle \\ &= \langle \lim_{i\in\mathbb{N}} m_i, T'k \rangle \\ &= \lim_{i\in\mathbb{N}} \langle m_i, T'k \rangle \\ &= \lim_{i\in\mathbb{N}} \langle Tm_i, k \rangle \\ &= \langle \lim_{i\in\mathbb{N}} Tm_i, k \rangle \\ &= \langle n, k \rangle \end{aligned}$$

This implies that $Tm = n$. Consequently T has a closed graph and is therefore bounded.

We have for any m, n in M and a in A that

$$\begin{aligned} \langle T(ma), n \rangle &= \langle ma, T'n \rangle \\ &= a^* \langle m, T'n \rangle \\ &= a^* \langle Tm, n \rangle \\ &= \langle (Tm)a, n \rangle \ . \end{aligned}$$

This implies that T is A-linear.

Assume that T'' is also adjoint to T. Then

$$0 = \langle m, (T' - T'')n \rangle$$

for all $m, n \in M$. This implies $T' = T''$.

Finally we have

$$\begin{aligned} \langle T'm, n \rangle &= \langle n, T'm \rangle^* \\ &= \langle Tn, m \rangle^* \\ &= \langle m, Tn \rangle \end{aligned}$$

for all m, n in M showing that T is the adjoint of T'. □

We consider a Hilbert A-module $(M, \langle, \rangle_M)$.

Definition 10.9. *We define $B(M)$ as the set of maps $M \to M$ which admit an adjoint.*

Remark 10.10. Let $M^* := \mathtt{Hom}_A(M, A)$ be the space of bounded A-linear operators. The scalar product of M induces antilinear map

$$i : M \to M^* \ , \quad m \mapsto \langle m, - \rangle \ .$$

In we do not have the analogue of the Riez representation theorem. In particular, this map is not an isomorphism in general.

If $T : M \to M$ is bounded A-linear, then we have a continuous adjoint $T^* : M \to M^*$. The existence of an adjoint T' of T in the sense of Hilbert modules is equivalent to the existence of a factorization T' as in

$$\begin{array}{ccc} M & \overset{T'}{\dashrightarrow} & M \\ \downarrow i & & \downarrow i \\ M^* & \xrightarrow{\circ T} & M^* \end{array} \ .$$

We refer to Example 10.21 for an operator without adjoint. □

We write T^* for the adjoint.

Lemma 10.11. *$B(M)$ is a C^*-algebra with the operator norm*

$$\|T\|_{B(M)} := \sup_{m \in M \ \|m\|=1} \|Tm\|_M \ .$$

Proof. It is clear that $B(M)$ is closed under sum and composition. Thereby $(T+\lambda T')^* = T^* + \bar{\lambda} T^*$ and $(TT')^* = T'^{,*} T^*$. The operator norm is submultiplicative and $*$ is an isometry. The last property implies that $B(M)$ is complete with respect to $\| - \|_{B(M)}$.

We now show the C^*-equality. It is clear from the properties already seen that we have

$$\|T^*T\|_{B(M)} \le \|T\|^2_{B(M)} \ .$$

If $\|m\|_M = 1$, then we have by Cauchy-Schwarz

$$\|Tm\|^2 = \|\langle Tm, Tm\rangle\|_A = \|\langle m, T^*Tm\rangle\|_A \le \|T^*T\|_{B(M)} \ .$$

This implies

$$\|T\|^2_{B(M)} \le \|T^*T\|_{B(M)} \ .$$

We conclude the C^*-equality. □

Example 10.12. We consider A as a Hilbert A-module. Then every a in A acting by left multiplication belongs to $B(A)$. Its adjoint is given by a^*. Indeed

$$\langle ab, b'\rangle = b^*a^*b' = \langle b, a^*b'\rangle$$

for all b, b' in A.

If A is unital, then $A \to B(A)$ is an isomorphism with inverse $T \mapsto T1_A$.

In general $A \to B(A)$ is the inclusion of A into the multiplier algebra $\mathcal{M}(A)$ of A.

Let A be a C^*-algebra. Then the multiplier algebra $\mathcal{M}(A)$ of A is defined as follows. The set $\mathcal{M}(A)$ is the set of pair (l, r) of maps $A \to A$ such that $r(a)b = al(b)$ for all a, b in A. Linear combinations are defined in the obvious manner. The composition is defined by $(l', r') \circ (l, r) = (l'l, rr')$, and the adjoint is defined by $(l, r)^* := (r(-^*)^*, l(-^*)^*)$.

We get $\mathcal{M}(A)$ in $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ and a homomorphism $A \to \mathcal{M}(A)$ which sends c to (l, r) given by $l(b) := cb$, $r(b) := bc$. We check that $al(b) = a(cb) = (ac)b = r(a)b$.

We now observe that

$$\langle l(a), b\rangle = l(a)^*b = (b^*l(a))^* = (r(b^*)a)^* = a^*r(b^*)^* = \langle a, r(b^*)^*\rangle \ .$$

This shows that l has an adjoint, namly $b \mapsto r(b^*)^*$. This we have a homomorphism $\mathcal{M}(A) \to B(A)$ given by $(l, r) \mapsto l$. On the other and, if T is in $B(A)$, then we define (l, r) in $\mathcal{M}(A)$ by $l(a) = T(a)$ and $r(a) := T^*(a^*)^*$. We check that

$$al(b) = aT(b) = \langle a^*, T(b)\rangle = \langle T^*(a^*), b\rangle = T^*(a^*)^*b = r(a)b \ .$$

This gives a map $B(A) \to \mathcal{M}(A)$. These two maps are bijections. Consequently, $\mathcal{M}(A) = B(A)$. In particular $\mathcal{M}(A)$ is a C^*-algebra.

Note that we gave a constructive definition of the multiplier algebra. It can alternatively be characterized by the universal property being the maximal unital extension of A which contains A as an essential ideal (intersects A intersects every non-zero ideal non-trivially). □

Example 10.13. Let $(M, \langle -, - \rangle)$ be a Hilbert A-module. For k, n we can define the operator $\theta_{k,n}$ in $B(M)$ by

$$\theta_{k,n}(m) = k\langle n, m \rangle \ .$$

The adjoint of $\theta_{k,n}$ is $\theta_{n,k}$. Indeed, for all m, l in M we have

$$\begin{aligned} \langle \theta_{k,n}(m), l \rangle &= \langle k\langle n, m \rangle, l \rangle \\ &= \langle n, m \rangle^* \langle k, l \rangle \\ &= \langle m, n \rangle \langle k, l \rangle \\ &= \langle m, n\langle k, l \rangle \rangle \\ &= \langle m, \theta_{n,k}(l) \rangle \ . \end{aligned}$$

Let $(M, \langle -, - \rangle)$ be a Hilbert A-module.

Definition 10.14. *We let $K(M)$ be the subalgebra of $B(H)$ generated by the operators $\theta_{n,k}$ for all n, k in A.*

The C^*-algebra $K(M)$ is the algebra of compact operators (in the sense of Hilbert A-modules) on M.

Example 10.15. If $A = \mathbb{C}$ and M is a Hilbert space, then $K(M)$ is the usual algebra of compact operators on M. The operators $\theta_{k,n}$ are exactly the one-dimensional operators. □

Lemma 10.16. *$K(M)$ is a closed $*$-ideal in $B(M)$.*

Proof. This immediately follows from

$$T\theta_{n,k} = \theta_{Tn,k} \ , \quad \theta_{n,k}T = \theta_{n,T^*k}$$

for all n, k in M and T in $B(M)$. □

Example 10.17. We consider A as a Hilbert C^*-module. Then we have $K(A) = A$. First of all, for all n, k in A the operator $\theta_{n,k}$ is given by left-multiplication by nk^* and hence belongs to A. This implies that $K(A) \subseteq A \subseteq B(A)$.

In order to see the converse, let (u_ν) be an approximative unit. We then have $\theta_{u_\nu,a}(b) = u_\nu a^* b$. Since $\mathtt{lim}_\nu\, u_\nu a^* = a^*$ it follows that

$$\theta_{u_\nu,a} \to a^* \cdot \ .$$

This shows that $K(A)$ is dense in A. We conclude equality. □

Example 10.18. Let (V, h) be an euclidean vector bundle on a locally compact space X. We consider the right $C_0(X)$-module $C_c(X, V)$ with scalar product $\langle \phi, \psi \rangle := (x \mapsto h(\phi(x), \psi(x))$. We define the Hilbert $C_0(X)$-module $C_0(X, V) := \overline{C_c(X, V)}$ as the closure.

The identity of $C_0(X, V)$ is compact if and only if X is compact. If X is not second countable, then $C_0(X, V)$ might not be full in the sense that the values of the scalar product generate the algebra. □

We can extend the definitions to operators between different Hilbert H-modules M and M'. Formally we define $B(M, M')$ such that

$$B(M \oplus M') = \begin{pmatrix} B(M) & B(M', M) \\ B(M, M') & B(M') \end{pmatrix} .$$

Definition 10.19. *We define the* $\mathbf{Hilb}(A)$ *in* ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ *such that:*

1. *objects: Hilbert A-modules.*
2. *morphisms: $B(M, M')$*
3. *composition and $*$: composition and adjoint*

Lemma 10.20. $\mathbf{Hilb}(A)$ *is a C^*-category.*

Proof. The operator norm on the spaces $B(M, M')$ exhibits $\mathbf{Hilb}(A)$ as a C^*-category. □

Example 10.21. We let $B = B(\ell^2)$ and $K = K(\ell^2)$. We consider K and B as Hilbert B-modules. Then the inclusion $K \to B$ is bounded B-linear, but does not have an adjoint. To see this assume by contradiction $T' : B \to K$ is an adjoint. Then we have $k^*b = k^*T'(b)$ for all k in K and B in b. This implies $T'(b) = b$ but this is impossible.

Note that K is a closed submodule of B. But $K^\perp$ (with the obvious definition) vanishes, and hence $(K^\perp)^\perp = B$ is strictly larger than K. There is no orthogonal projection $B \to K$.

We can consider the inclusion $K \to B$ as an operator on the B-Hilbert module $K \oplus B$. In this way we get an example of a bounded B-linear map $K \oplus B \to K \oplus B$ which does not belong to $B(K \oplus B)$. □

Example 10.22. We consider the C^*-algebra $C([-1, 1])$ and the submodule

$$N := \{f \in C([-1, 1]) \mid f(0) = 0\} .$$

The inclusion $i : N \to C([-1, 1])$ does not have an adjoint. If T would be such an adjoint, then we set $T(1) =: f$ in N. Then $\bar{h}g = \bar{h}fg$ for all g in $C([-1, 1])$ and h in N. Setting $h = \bar{f}$ and $g = 1$ we get $f = f^2$. Hence f takes values in $\{0, 1\}$. Since $f(0) = 0$ we have $f \equiv 0$. But the embedding is non-zero so that we get a contradiction. □

Example 10.23. Let A be a C^*-algebra. Let $(M_i)_{i\in I}$ be a famliy of Hilbert C^*-modules. Then we have inclusions $e_i : M_i \to \bigoplus_{i\in I} M_i$. We have $e_i \in B(M_i, \bigoplus_{i\in I} M_i)$ and $e_i^* e_i = \mathtt{id}_{M_i}$. Indeed, one checks that $e_i^*(\oplus_{i\in I} m_i) = m_i$. □

Let $\mathbf{C}$ be a C^*-category.

Theorem 10.24 (M. Joachim)**.** *The canonical map $\mathbf{C} \to A(\mathbf{C})$ is isometric.*

Proof. Let $\mathbf{C}$ be a C^*-category and C, C' in $\mathbf{C}$. Then $\mathtt{Hom}_{\mathbf{C}}(C, C')$ is an $\mathtt{End}_{\mathbf{C}}(C)$-Hilbert module with scalar product $\langle f, g\rangle := f^* g$. We form the sum

$$M_C := \bigoplus_{C' \in \mathbf{C}} \mathtt{Hom}_{C,C'}$$

in $\mathbf{Hilb}(\mathtt{End}_{\mathbf{C}}(C))$. We have a canonical homomorphism

$$A^{\mathrm{alg}}(\mathbf{C}) \to B(M_C)$$

which sends $f : C' \to C''$ to the one-entry matrix $[f]$ acting by left multiplication. In detail

$$([f] \oplus_{C'} g_{C'})_{C'''} = \left\{ \begin{array}{cc} 0 & C''' \neq C'' \\ f \circ g_{C'} & C''' = C'' \end{array} \right. .$$

By the universal property of the maximal norm involved in the construction of $A(\mathbf{C})$ this homomorphism uniquely extends to a homomorphism of C^*-algebras

$$A(\mathbf{C}) \to B(M_C) .$$

We now consider the composition

$$\mathtt{End}_{\mathbf{C}}(C) \to \mathbf{C} \to A(\mathbf{C}) \to B(M_C) .$$

This map is the canonical identification of $\mathtt{End}_{\mathbf{C}}(C)$ with one-entry diagonal matrices located at the position with index C. In particular it is an isometric embedding. We conclude that $\mathtt{End}_{\mathbf{C}}(C) \to \mathbf{C} \to A(\mathbf{C})$ is isometric for every C in $\mathbf{C}$.

Since C can be taken arbitrary, using the C^*-equality we see that $\mathbf{C} \to A(\mathbf{C})$ is isometric. □

We now construct the Yoneda embedding

$$Y : \mathbf{C} \to \mathbf{Hilb}(A(\mathbf{C})) .$$

Let C be in $\mathbf{C}$. Then we consider the right A^{alg}-module

$$Y^{\mathrm{alg}}(C) := \bigoplus_{C' \in \mathrm{Ob}(\mathbf{C})} \mathtt{Hom}_{\mathbf{C}}(C', C) .$$

We define the $A^{\mathrm{alg}}(\mathbf{C})$ (and therefore also $A(\mathbf{C})$)-valued scalar product by

$$\langle \oplus_{C'} f_{C'}, \oplus_{C''} g_{C''} \rangle := \sum_{C',C''} f^*_{C'} g_{C''} \ .$$

We then form the closure and get $Y(\mathbf{C})$ in $\mathbf{Hilb}(A(\mathbf{C}))$.

If $f : C_0 \to C_1$ is a morphism, then we get a morphism $Y(f)$ in $B(Y(C_0), Y(C_1))$. It is given by

$$Y(f)(\oplus_{C'} g_{C'}) := \oplus_{C'} f \circ g \ .$$

Its adjoint is f^*.

Lemma 10.25.

1. *The Yoneda embedding is isometric.*
2. *If $\mathbf{C}$ is unital, then the Yoneda embedding is full.*

Proof. Let $f : C \to C'$. We must show that $\|Y(f^*f)\| = \|f\|$. Note that $Y(f^*f)$ is the endomorphism of $Y(\mathbf{C})$ given by

$$Y(f^*f)(\oplus_{C'} g_{C'}) := \oplus_{C'} f^*fg \ .$$

We restrict this to g in $\mathtt{Hom}_{\mathbf{C}}(C, C)$. Inserting for g the members of an approximative unit $(u_\nu)_\nu$ we get

$$\mathtt{lim}_{\nu} Y(f^*f)(u_\nu)_C := (f^*f)_C \ .$$

This shows that

$$\|Y(f^*f)\| \geq \|f^*f\| \ .$$

If $\mathbf{C}$ is unital and $a : Y(C) \to Y(C')$ is a homomorphism, then $f := a(\mathtt{id}_C)$ in $\mathtt{Hom}_{\mathbf{C}}(C, C')$ is such that $Y(f) = a$. □

Example 10.26. Let B be a unital C^*-algebra. Then $B \cong A(B)$. Hence $Y : B \to \mathbf{Hilb}(A(B))$ is the usual embedding which considers B as a Hilbert B-module.

11 Constructions

Let $\mathbf{C}$ be in $C^*\mathbf{Cat}$.

Let $(C_i)_{i\in I}$ be a finite family of objects.

Definition 11.1. *A direct sum of this family is a pair $(C, (e_i)_{i\in I})$ such that*

1. *C in $\mathbf{C}$*

2. *$e_i : C_i \to C$ is an isometry for every i.*

3. *$\sum_{i\in I} e_i e_i^* = \mathtt{id}_C$.*

We call the morphisms e_i the structure morphisms of the sum.-

Example 11.2. If $(C'_i, (e'_i)_{i\in I})$ is a second sum of the family, then we define $u : C \to C'$ by $u := \sum_{i\in I} e'_i e_i^*$ and $v : C' \to C$ by $v = \sum_{i\in I} e_i e_i'^{,*}$. Then $v = u^*$ and $vu = \mathtt{id}_C$ and $uv = \mathtt{id}_{C'}$ and $ue_i = e'_i$ and $ve'_i = e_i$ for all i. The verification is by direct calculations.

If $u' : C \to C'$ is another morphism such that $e'_i = ue_i$ for all i in I, then $u = u'$. We conclude that a sum is unique up to unique isomorphism which is compatible with the structure maps.

Definition 11.3. *We call $\mathbf{C}$ additive if it admits sums for all finite families of objects.*

Example 11.4. $\mathbf{Hilb}(A)$ is additive. The orthogonal sum is represented by the orthogonal sum in the sense of Hilbert modules. □

Example 11.5. The algebra $B(\ell^2)$ is additive. Let I be a finite set We choose a bijection $\mathbb{N} \to \bigoplus_I \mathbb{N}$. Then we get a unitary isomorphism $u : \ell^2 \to \bigoplus_{i\in I} \ell^2$. We let p_i be the orthogonal projection onto the i'th summand. We let $e_i := u^* p_i u$. Then $(e_i)_{i\in I}$ is a family of isometries in $B(\ell^2)$ and $\sum_{i\in I} e_i e_i^* = 1$. The object $(B(\ell^2), (e_i)_{i\in I})$ represents the sum of the family $(B(\ell^2)_I$. □

Let $\phi : \mathbf{C} \to \mathbf{C}'$ be a functor.

Lemma 11.6. *ϕ preserves finite orthogonal sums.*

Proof. Let $(C, (e_i)_{i\in I})$ is an orthogonal sum of the family $(C_i)_{i\in I}$ in $\mathbf{C}$. Then $(\phi(C), (\phi(e_i))_{i\in I})$ is an orthogonal sum of the family $(\phi(C_i))_{i\in I}$ in $\mathbf{D}$. Indeed, the identity $\sum_{i\in I} e_i e_i^* = \mathtt{id}_C$ implies $\sum_{i\in I} \phi(e_i)\phi(e_i)^* = \mathtt{id}_{\phi(C)}$. □

Remark 11.7. Let $\mathbf{C}$ be in $C^*\mathbf{Cat}$. Then we consider the Yoneda embedding $Y : \mathbf{C} \to \mathbf{Hilb}(A(\mathbf{C}))$. We let $\mathbf{C}_\oplus$ be the full subcategory of $\mathbf{Hilb}(A(\mathbf{C}))$ of objects which are isomorphic to finite direct sums of objects of $\mathbf{C}$. The Yoneda embedding factorizes through a functor

$$(-)_\oplus : \mathbf{C} \to \mathbf{C}_\oplus \ .$$

This functor has the following universal property: For any unital functor $\psi : \mathbf{C} \to \mathbf{D}$ with $\mathbf{D}$ additive we have an extension

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\psi} & \mathbf{D} \\ & \searrow & \nearrow \\ & \mathbf{C}_\oplus & \end{array}$$

as indicated by the dotted arrow. This extension is unique up to unitary isomorphism. We say that $\mathbf{C} \to \mathbf{C}_\oplus$ is a finite additive completion. Note that a finite additive completion is unique up to unitary equivalence.

If we consider an algebra A as a C^*-category, then $A_\oplus$ is the category $\mathbf{Mod}(A)^{\mathrm{fg,free}}$ of finitely generated free A-modules. $\square$

We now discuss orthogonal sums of infinite families.

Remark 11.8. We consider the C^*-category $\mathbf{Hilb}(A)$. If $(M_i)_{i\in I}$ is a family of objects, then we can define $(M,(e_i)_{i\in I})$ with $M = \bigoplus_{i\in I}$. If I is infinite and infinitely many of the M_i are non-zero, then $\|1_M - \sum_{i\in J} e_i^* e_i\| = 1$ for all finite subset J of I. Hence the Definition 11.1 does not generalize to capture the case of infinite sums of Hilbert modules.

In order to define infinite orthgonal sums we consider $\mathbf{Hilb}(A)$ as a model case. Let $(C_i)_{i\in I}$ be a family of objects.

Definition 11.9. *A pair $(C,(e_i)_{i\in I})$ represents the sum of the family if there exists a unitary isomorphism $u : Y(C) \to \bigoplus_{i\in I} Y(C_i)$ such that $u \circ Y(e_i) : Y(C_i) \to \bigoplus_{i\in I} Y(C_i)$ is the canonical embedding.*

Example 11.10. In this construction we again show that also infinite sums are unique up to unique isomorphism provided they exists.

We first show that the isomorphism u in the definition above is unique. Indeed, if u' is a second one, then

$$(Y(e_j)^* u^{*,\prime}) u' u^* (u Y(e_i)) = \begin{cases} \mathtt{id}_{C_i} & i = j \\ 0 & else \end{cases}$$

This implies $u'u^* = \mathtt{id}_{Y(C)}$ and hence $u = u'$.

If $(C_i', (e_i')_{i\in I})$ is a second sum of the family. Then we get a unitary isomorphism

$$\hat{v} : Y(C) \stackrel{u}{\to} \bigoplus_{i\in I} Y(C_i) \stackrel{u',*}{\to} Y(C') \ .$$

Using that Y is fully faithul we get $v : C \to C'$ such that $Y(v) = \hat{v}$. One then checks that v is unitary and $ve_i = e_i'$.

Furthermore, v is unique with these properties. $\square$

Example 11.11. The algebra $B(\ell^2)$ is countably additive.

Let I be a countable set We choose a bijection $\mathbb{N} \to \bigoplus_I \mathbb{N}$. Then we get a unitary isomorphism $u : \ell^2 \to \bigoplus_{i\in I} \ell^2$. We let p_i be the orthogonal projection onto the i'th summand. We let $e_i := u^* p_i u$. Then $(e_i)_{i\in I}$ is a family of isometries in $B(\ell^2)$. We want to show that $(B(\ell^2),(e_i)_i)$ represents the sum of the family $(B(\ell^2))_I$.

We use that in $\mathbf{Hilb}(B(\ell))$ we have an isomorphism of C^*-algebras

$$B(\bigoplus_I B(\ell^2)) \cong B(\bigoplus_I \ell^2) \cong B(\ell^2) .$$

□

Let $\phi : \mathbf{C} \to \mathbf{D}$ be a functor in $C^*\mathbf{Cat}$. We have seen that ϕ preserves finite orthogonal sums. For infinite sums the situation is more complicate.

Proposition 11.12. *If ϕ is fully faithful, then it preserves sums.*

Proof. This is shown in [BE, Sec. 6]. □ □

Let A be a C^*-algebra.

Lemma 11.13. *Then the notions of an orthogonal sum of Hilbert A-modules and the orthogonal sum in $\mathbf{Hilb}(A)$ coincide.*

Proof. (sketch) Let $(M_i)_{i\in I}$ be a family in $\mathbf{Hilb}(A)$. We must show that there exists an isomorphism

$$u : Y(\bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} Y(M_i)$$

which is compatible with the structure maps. It is given

$$\sum_{i\in I} u_i Y(e_i^*) ,$$

where $e_i^* : \bigoplus_{i\in I} M_i \to M_i$ is the projection and $u_i : Y(M_i) \to \bigoplus_{i\in I} Y(M_i)$ is the embedding. The problem is to show that this sum converges. It does not converge in norm, but strongly. The inverse of u is given by a similar formula

$$\sum_{i\in I} Y(e_i)u_i^*) : \bigoplus_{i\in I} Y(M_i) \to \bigoplus_{i\in I} M_i) .$$

□

Example 11.14. In this example we exhibit a functor which does not preserve infinite orthogonal sums. Let $\mathbf{C}$ be in $C^*\mathbf{Cat}$. Note that objects of $\prod_{\mathbb{N}} \mathbf{C}$ are families $(C_k)_{k\in\mathbb{N}}$ of objects in $\mathbf{C}$, and morphisms are families $(f_k : C_k \to C'_k)_{k\in\mathbb{N}}$ such that $\sup_{k\in\mathbb{N}} \|f_k\| < \infty$.

The product category $\prod_{\mathbb{N}} \mathbf{C}$ contains the ideal $\mathbf{I}$ consisting of all families of morphisms $(f_k)_{k\in\mathbb{N}}$ such that $\mathtt{lim}_{k\to\infty} \|f_k\| = 0$.

We let $\mathbf{D} := (\prod_{\mathbb{N}} \mathbf{C})/\mathbf{I}$ and consider the functor

$$h\colon \mathbf{C} \xrightarrow{\mathtt{diag}} \prod_{\mathbb{N}} \mathbf{C} \xrightarrow{\mathtt{pr}} \mathbf{D}\,. \tag{11.1}$$

We note that h is an isometric inclusion.

We consider a family of non-zero objects $(C_i)_{i\in\mathbb{N}}$ in $\mathbf{C}$ which admits an orthogonal sum $(C,(e_i)_{i\in\mathbb{N}})$. We claim that then $(h(C),h(e_i)_{i\in\mathbb{N}})$ in $\mathbf{D}$ does not represent the sum of the family $(h(C_i))_{i\in\mathbb{N}}$.

To show the claim we consider the morphism $f := (\sum_{i>k} e_i e_i^*)_{k\in\mathbb{N}}$ in $\mathtt{End}_{\prod_{\mathbb{N}}\mathbf{C}}(\mathtt{diag}(C))$ and let

$$\tilde{f} := \mathtt{pr}(f) \tag{11.2}$$

in $\mathtt{End}_{\mathbf{D}}(h(C))$ be the image of f in $\mathbf{D}$. We note that $\|\tilde{f}\| = 1$. We further observe that for every i in I we have $f \circ \mathtt{diag}(e_i) \in \mathbf{I}$, and consequently $\tilde{f} \circ h(e_i) = 0$. The relations $\tilde{f} \neq 0$ and $\tilde{f} \circ h(e_i)$ contradict each other $(h(C),h(e_i)_{i\in\mathbb{N}})$ would have been the orthogonal sum. □

Let $\mathbf{C}$ be in $C^*\mathbf{Cat}$, C be in $\mathbf{C}$ and p be in $\mathtt{End}_{\mathbf{C}}(C)$ such that $p^2 = p = p^*$, i.e, p is an orthogonal projection.

Definition 11.15. *An image of p is an isometry $u : D \to C$ such that $u^*u = p$.*

Remark 11.16. If $u' : D' \to C$ is a second image, then there exists a unique unitary $v : D \to D'$ such that $u'v = u$. Indeed, we can take $v = u'^{,*}u$. Hence an image of a projection is unique up to unique unitary isomorphism.

Definition 11.17. *We say that $\mathbf{C}$ is idempotent complete if it is finitelöy adittively complete and every projection in $\mathbf{C}$ admits an image.*

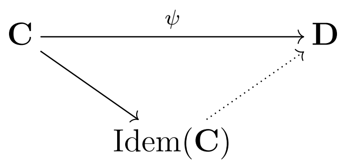
Example 11.18. If A is a C^*-algebra, then $\mathbf{Hilb}(A)$ is idempotent complete. Indeed, if p is a projection on some M in $\mathbf{Hilb}(A)$, then the submodule pM is in $\mathbf{Hilb}(A)$ and the inclusion $u : pM \to M$ is an image. Indeed, the adjoint $u* : M \to pM$ is given by p.

Example 11.19. We consider the C^*-algebra $A := \mathbb{C} \oplus \mathbb{C}$. Then $\mathbf{Mod}(A)^{\mathrm{free}}$ is not idempotent complete. E.g. the submodule $\mathbb{C} \oplus 0$ of A does not belong to $\mathbf{Mod}(A)^{\mathrm{free}}$.

We consider the full subcategory $\mathrm{Idem}(\mathbf{C})$ of $\mathbf{Hilb}(A(\mathbf{C}))$ of objects which direct summands of modules which are isomorphic to finite sums of objects in the image of the Yoneda embedding. The Yoneda embedding factorizes over a functor

$$\mathbf{C} \to \mathrm{Idem}(\mathbf{C})\ .$$

It has following universal property: Any unital functor $\psi : \mathbf{C} \to \mathbf{D}$ with $\mathbf{D}$ idempotent complete has a factorization



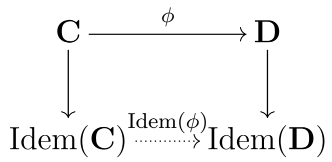
indicated by the dotted arrow. This factorization is unique up to unitary isomorphism.

Definition 11.20. *We call* $\mathbf{C} \to \mathrm{Idem}(\mathbf{C})$ *and idempotent completion functor.*

An idempotent completion functor is uniquely determined up to unitary equivalence.

Example 11.21. Let A be in $C^*\mathbf{Alg}$. Then $\mathrm{Idem}(A) \simeq \mathbf{Mod}(A)^{\mathrm{fg,Proj}}$. □

Let $\mathbf{C} \to \mathbf{D}$ be a functor. By the universal property of the idempotent completion functor we get a factorization



The functor $\mathrm{Idem}(\phi)$ is unique up to unitary isomorphism.

Definition 11.22. *We call* $\mathbf{C} \to \mathbf{D}$ *a Morita equivalence of* $\mathrm{Idem}(\mathbf{C}) \to \mathrm{Idem}(\mathbf{D})$ *is a unitary equivalence.*

Example 11.23. The embedding $A \to \mathbf{Mod}(A)^{\mathrm{fg,Proj}}$ is a Morita equivalence. Also $\mathbf{Mod}(A)^{\mathrm{fg,free}} \to \mathbf{Mod}(A)^{\mathrm{fg,Proj}}$ is a Morita equivalence.

We discuss tensor products of C^*-categories. This includes a discussion of tensor products of C^*-algebras.

Definition 11.24. *For* $\mathbf{C}$ *and* $\mathbf{D}$ *in* ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ *the algebraic tensor product* $\mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D}$ *is characterized by the property that the morphism*

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D}$$

in ${}^*\mathbf{Cat}^{\mathrm{nu}}$ *(possibly non-unital categories with involution) is universal for morphisms from* $\mathbf{C} \times \mathbf{D}$ *to objects from* ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ *which are bilinear on morphism spaces.*

Here is an description of the algebraic tensor product of $\mathbf{C}$ and $\mathbf{D}$ in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$:

1. objects: We have $\mathrm{Ob}(\mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D}) \cong \mathrm{Ob}(\mathbf{C}) \times \mathrm{Ob}(\mathbf{D})$.

2. morphisms: For objects (C,D) and (C',D') in $\mathbf{C}\otimes^{\mathrm{alg}}\mathbf{D}$ we have

$$\mathtt{Hom}_{\mathbf{C}\otimes^{\mathrm{alg}}\mathbf{D}}((C,D),(C',D')) \cong \mathtt{Hom}_{\mathbf{C}}(C,C')\otimes \mathtt{Hom}_{\mathbf{D}}(D,D')\,.$$

3. composition and involution: These structures are defined in the obvious manner.

We can thus form the algebraic tensor product of C^*-algebras $A\otimes^{\mathrm{alg}} B$. Assume that C is any C^*-algebra and $\rho : A\otimes^{\mathrm{alg}} B \to C$ is a homomorphism. Let $\mathcal{M}(C)$ denote the multiplier algebra of C.

Proposition 11.25. *There exists representations $\alpha : A \to \mathcal{M}(C)$ and $\beta : B \to \mathcal{M}(C)$ such that $\rho(a\otimes b) = \alpha(a)\beta(b)$ for all a in A and B in B.*

Proof. See [Tak97, Lemma 4.1] or [Mur90, Cor. 6.3.6]. □

Corollary 11.26. *For all a in A and B in B we have $\|\rho(a\otimes b)\|_C \le \|a\|_A\|b\|_B$.*

We now introduce the maximal tensor product in $C^*\mathbf{Cat}$ or $C^*\mathbf{Cat}^{\mathrm{nu}}$ has a similar description by a universal property:

Definition 11.27. *For $\mathbf{C}$ and $\mathbf{D}$ in $C^*\mathbf{Cat}^{\mathrm{nu}}$ the maximal tensor product is characterized by the property that the morphism*

$$\mathbf{C}\times\mathbf{D}\to\mathbf{C}\otimes_{\max}\mathbf{D}$$

in ${}^*\mathbf{Cat}^{\mathrm{nu}}$ is universal for morphisms from $\mathbf{C}\times\mathbf{D}$ to objects from $C^*\mathbf{Cat}^{\mathrm{nu}}$ which are bilinear on morphism spaces.

One must check that the maximal tensor product exists. The first step in the verification is the following lemma. Assume that $\mathbf{C}$ and $\mathbf{D}$ are in $C^*\mathbf{Cat}^{\mathrm{nu}}$.

Lemma 11.28. *The algebraic tensor product $\mathbf{C}\otimes^{\mathrm{alg}}\mathbf{D}$ is a pre-C^*-category.*

Proof. It suffices to check that $f\otimes g$ has a finite maximal norm for every pair of morphisms f in $\mathbf{C}$ and g in $\mathbf{D}$. We will show that

$$\|f\otimes g\|_{\max} \le \|f\|_{\mathbf{C}}\|g\|_{\mathbf{D}}\,. \tag{11.3}$$

Let $\rho\colon \mathbf{C}\otimes^{\mathrm{alg}}\mathbf{D}\to A$ be a functor into to a C^*-algebra (considered as a morphism in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$). Then we will show that $\|\rho(f\otimes g)\|_A \le \|f\|_{\mathbf{C}}\|g\|_{\mathbf{D}}$. Using the C^*-equality for the norm on C^*-categories, the case of C^*-categories can be reduced to the case of C^*-algebras as follows. We have

$$\|\rho(f\otimes g)\|_A^2 = \|\rho(f^*\otimes g^*)\rho(f\otimes g)\|_A = \|\rho(f^*f\otimes g^*g)\|_A \le \|f^*f\|_{\mathbf{C}}\|g^*g\|_{\mathbf{D}} = \|f\|^2_{\mathbf{C}}\|g\|^2_{\mathbf{D}}\,,$$

where for the inequality we use that ρ induces a representation of the algebraic tensor product of C^*-algebras $\mathtt{End}_{\mathbf{C}}(C)\otimes^{\mathrm{alg}}\mathtt{End}_{\mathbf{D}}(D)$ to A. Since ρ is arbitrary the inequality (11.3) follows. □

Proposition 11.29. *The maximal tensor product $\otimes_{\max}$ on $C^*\mathbf{Cat}^{\mathrm{nu}}$ exists and equips this category with a symmetric monoidal structure.*

Proof. In view of Lemma 11.28 the algebraic tensor product induces a symmetric monoidal functor $C^*\mathbf{Cat}^{\mathrm{nu}} \to {}_{\mathrm{pre}}\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$. Using the completion functor we define

$$\mathbf{C} \otimes_{\max} \mathbf{D} := \mathrm{compl}(\mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D})\,.$$

It remains to define the unit, associativity and symmetry constraints. Thereby only the associativity is not completely straightforward. In order to construct it we consider the bold part of the commutative diagram

$$\begin{array}{ccc}
(\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B}) \otimes^{\mathrm{alg}} \mathbf{C} & \xrightarrow{\cong} & \mathbf{A} \otimes^{\mathrm{alg}} (\mathbf{B} \otimes^{\mathrm{alg}} \mathbf{C}) \\
\downarrow & & \downarrow \\
(\mathbf{A} \otimes_{\max} \mathbf{B}) \otimes^{\mathrm{alg}} \mathbf{C} & \searrow & \mathbf{A} \otimes^{\mathrm{alg}} (\mathbf{B} \otimes_{\max} \mathbf{C}) \\
\downarrow & & \downarrow \\
(\mathbf{A} \otimes_{\max} \mathbf{B}) \otimes_{\max} \mathbf{C} & \dashrightarrow & \mathbf{A} \otimes_{\max} (\mathbf{B} \otimes_{\max} \mathbf{C})
\end{array}$$

whose vertical morphisms are all given by canonical morphisms into completions and the functoriality of the algebraic tensor product. The upper horizontal functor is the associativity constraint of the algebraic tensor product. We obtain the dotted arrow from the universal property of the algebraic tensor product: To this end we must show that the bilinear functor

$$(\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B}) \times^{\mathrm{alg}} \mathbf{C} \to \mathbf{A} \otimes_{\max} (\mathbf{B} \otimes_{\max} \mathbf{C})$$

induced by the right-down composition extends by continuity to a bilinear functor

$$(\mathbf{A} \otimes_{\max} \mathbf{B}) \times^{\mathrm{alg}} \mathbf{C} \to \mathbf{A} \otimes_{\max} (\mathbf{B} \otimes_{\max} \mathbf{C})\,.$$

For a morphism ϕ in $\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B}$ and h in $\mathbf{C}$ we have by (11.3) that

$$\|\phi \otimes h\|_{\mathbf{A} \otimes_{\max} (\mathbf{B} \otimes_{\max} \mathbf{C})} \le \|\phi\|_{\max} \|h\|_{\mathbf{C}}\,.$$

This estimate implies that the bilinear functor extends as desired, and the existence of the dotted arrow follows.

We finally get the dashed arrow from the universal property of the lower left vertical arrow applied to the dotted arrow. In order to show that it is an isomorphism we construct an inverse by a similar argument starting from the inverse of the upper horizontal arrow. □

Restricting to $C^*\mathbf{Alg}^{\mathrm{nu}}$ we obtain the maximal tensor product for C^*-algebras.

We now turn to the minimal tensor product on $C^*\mathbf{Cat}^{\mathrm{nu}}$.

The category **Hilb** of small Hilbert spaces is a commutative algebra in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ such that the structure morphism

$$\mathbf{Hilb} \otimes^{\mathrm{alg}} \mathbf{Hilb} \to \mathbf{Hilb}$$

is induced by the universal property of $\otimes^{\mathrm{alg}}$ by the functor $\mathbf{Hilb} \times \mathbf{Hilb} \to \mathbf{Hilb}$ given as follows:

1. objects: A pair (H, H') of Hilbert spaces is sent to $H \otimes H'$ (tensor product in the sense of Hilbert spaces).

2. morphisms: A pair of morphism $(f, g)\colon (H_0, H'_0) \to (H_1, H'_1)$ is sent to the morphism $f \otimes g\colon H_0 \otimes H'_0 \to H_1 \otimes H'_1$.

The unit of this algebra is the inclusion functor $\mathbb{C} \to \mathbf{Hilb}$.

Let $\mathbf{C}, \mathbf{D}$ be in $C^*\mathbf{Cat}^{\mathrm{nu}}$ and $c\colon \mathbf{C} \to \mathbf{Hilb}$ and $d\colon \mathbf{D} \to \mathbf{Hilb}$ be functors. Then we can define a functor

$$c \otimes d\colon \mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D} \to \mathbf{Hilb} \otimes^{\mathrm{alg}} \mathbf{Hilb} \to \mathbf{Hilb}\,.$$

Definition 11.30. *The minimal tensor product $\mathbf{C} \otimes_{\mathrm{min}} \mathbf{D}$ is defined as the completion of the algebraic tensor product such that for every c, d as above we have a factorization*

$$\begin{array}{ccc} \mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D} & \xrightarrow{\quad c\otimes d \quad} & \mathbf{Hilb}\,. \\ & \searrow \qquad \nearrow & \\ & \mathbf{C} \otimes_{\mathrm{min}} \mathbf{D} & \end{array}$$

In other words, the minimal norm of a morphism ϕ in $\mathbf{C} \otimes^{\mathrm{alg}} \mathbf{D}$ is given by

$$\|\phi\|_{\mathrm{min}} \coloneqq \sup\nolimits_{c,d} \|(c \otimes d)(\phi)\|_{\mathbf{Hilb}}\,. \tag{11.4}$$

Proposition 11.31. *The minimal tensor product $\otimes_{\mathrm{min}}$ equips $C^*\mathbf{Cat}^{\mathrm{nu}}$ with a symmetric monoidal structure.*

Proof. We must provide the unit, associativity, and symmetry constraints. As in the case of the maximal tensor product only the associativity constraint is non-straightforward. In order to construct it we consider the bold part of the commutative diagram

$$\begin{array}{ccc} (\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B}) \otimes^{\mathrm{alg}} \mathbf{C} & \xrightarrow{\cong} & \mathbf{A} \otimes^{\mathrm{alg}} (\mathbf{B} \otimes^{\mathrm{alg}} \mathbf{C}) \\ \downarrow & & \downarrow \\ (\mathbf{A} \otimes_{\mathrm{min}} \mathbf{B}) \otimes^{\mathrm{alg}} \mathbf{C} & \searrow & \mathbf{A} \otimes^{\mathrm{alg}} (\mathbf{B} \otimes_{\mathrm{min}} \mathbf{C}) \\ \downarrow & & \downarrow \\ (\mathbf{A} \otimes_{\mathrm{min}} \mathbf{B}) \otimes_{\mathrm{min}} \mathbf{C} & \dashrightarrow & \mathbf{A} \otimes_{\mathrm{min}} (\mathbf{B} \otimes_{\mathrm{min}} \mathbf{C}) \end{array}$$

where the vertical maps are given by the canonical maps from the algebraic tensor products to the respective completions.

As in the case of the maximal tensor product, in order to show the existence of the dotted arrow we must show that the bilinear functor

$$(\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B}) \times \mathbf{C} \to \mathbf{A} \otimes_{\min} (\mathbf{B} \otimes_{\min} \mathbf{C})$$

induced by the right-down composition extends by continuity to a bilinear functor

$$(\mathbf{A} \otimes_{\min} \mathbf{B}) \times \mathbf{C} \to \mathbf{A} \otimes_{\min} (\mathbf{B} \otimes_{\min} \mathbf{C})\,.$$

Let $a\colon \mathbf{A} \to \mathbf{Hilb}$, $b\colon \mathbf{B} \to \mathbf{Hilb}$ and $c\colon \mathbf{C} \to \mathbf{Hilb}$ be representations. Let ϕ be in $(\mathbf{A} \otimes^{\mathrm{alg}} \mathbf{B})$ and h be in $\mathbf{C}$. Then we have the inequalities

$$\|(a \otimes b \otimes c)(\phi \otimes h)\|_{\mathbf{Hilb}} \le \|(a \otimes b)(\phi)\|_{\mathbf{Hilb}} \|c(h)\|_{\mathbf{Hilb}} \le \|\phi\|_{\min} \|h\|_{\mathbf{C}}\,.$$

Since a, b, c are arbitrary we conclude that $\|\phi \otimes h\|_{\mathbf{A} \otimes_{\min} (\mathbf{B} \otimes_{\min} \mathbf{C})} \le \|\phi\|_{\min} \|h\|_{\mathbf{C}}$. This estimate implies that the bilinear functor extends as desired and that the dotted arrow exists.

The first part of the estimate above shows that the dotted arrow further extends by continuity to the dashed arrow. An inverse of the dashed arrow can be constructed in a similar manner starting from the inverse of the upper horizontal arrow. □

It is again clear from the universal property of $\otimes_{\min}$, or alternatively from the construction of the minimal norm in (11.4), that the inclusion functor $\mathrm{incl}\colon C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Cat}^{\mathrm{nu}}$ has a canonical symmetric monoidal refinement for the minimal tensor structures on the domain and the target.

Let us now collect some facts about the minimal tensor product which we will use at various places in the present section.

If A is in $C^*\mathbf{Alg}^{\mathrm{nu}}$, then a representation $\alpha\colon A \to \mathbf{Hilb}$ of A is the same datum as a homomorphism $\alpha\colon A \to B(H)$ for some Hilbert space H. If $\beta\colon B \to B(H')$ is a second homomorphism, then their tensor product in the sense of representations to $\mathbf{Hilb}$ is simply the tensor product

$$\alpha \otimes \beta\colon A \otimes^{\mathrm{alg}} B \to B(H) \otimes B(H') \cong B(H \otimes H')\,.$$

It is known that if α and β are faithful representations, then

$$\|x\|_{\min} = \|(\alpha \otimes \beta)(x)\|_{B(H \otimes H')} \tag{11.5}$$

for all x in $A \otimes^{\mathrm{alg}} B$. Thus for C^*-algebras the supremum in (11.4) is realized by any pair of faithful representations.

Proposition 11.32. *The functor* $A : C^*\mathbf{Cat}_i^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ *is symmetric monoidal for* $\otimes_{\min}$ *and* $\otimes_{\max}$.

Proof. See [BEL]. □

The following is a deeper property of $\otimes_{\max}$ on $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Theorem 11.33. *For* A *in* $C^*\mathbf{Alg}^{\mathrm{nu}}$ *the functor* $A \otimes_{\max} -$ *preserves exact sequences.*

Proof. See [BO08, Prop. 3.7.1]. □

Corollary 11.34. *For* $\mathbf{C}$ *in* $C^*\mathbf{Cat}^{\mathrm{nu}}$ *the functor* $A \otimes_{\max} -$ *preserves exact sequences.*

Proof. We use that $A(-)$ detects and preserves exact sequences □

We consider a group G. We let C be in $C^*\mathbf{Cat}^{\mathrm{nu}}$ with act strict action of G. We use the notation $(g, C) \mapsto gC$ and $(g, f) \mapsto gf$ for the action on objects and morphisms.

Example 11.35. If $\mathbf{C}$ is in $C^*\mathbf{Cat}^{\mathrm{nu}}$, then we can consider the trivial G-action on $\mathbf{C}$.

Example 11.36. Let X be a bornological coarse space and $\mathbf{V}_{lf}(X)$ be the C^*-category of locally small X-controlled objects. If G acts on X by automorphisms, then it acts on $\mathbf{V}_{lf}(X)$ by functoriality. We have $g(H, \mu) = (H, g_*\mu)$ and $gA = A$. □

Example 11.37. Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and consider the C^*-category $\mathbf{Hilb}(A)^G$ consisting of pairs (H, ρ) if Hilbert A-modules with unitary G-action ρ and equivariant operators. We define the G-action on $\mathbf{Hilb}(A)^G$ by $g(H, \rho) := (H, \rho(g^{-1} - g))$ and $gA := A$. □

Example 11.38. If $\mathbf{C}$ is a C^*-category with G-action, then we can form a new category $\mathbf{C}^{(G)}$ with G-action defined as follows.

1. objects: (C, ρ), where C in $\mathrm{Ob}(\mathbf{C})$ and $\rho = (\rho_g)_{g \in G}$ is a family of unitary maps $\rho_g : C \to gC$ such that $g\rho_h\rho_g = \rho_{gh}$ for all g, h.

2. morphisms, composition, involution: inheritied from $\mathbf{C}$

3. G-action: G fixes the objects of $\mathbf{C}$ and acts on morphisms by $g \cdot A := \rho_{g^{-1}} \circ gA \circ \rho_g$.

□

We consider $\mathbf{C}$ in $\mathbf{Fun}(BG, {}^*\mathbf{Cat}_{\mathbb{C}}^{\mathrm{nu}})$. We use the notation $(g, C) \mapsto gC$ and $(g, f) \mapsto gf$ for the G-action on objects and morphisms of $\mathbf{C}$.

Definition 11.39. *We define the crossed product* $\mathbf{C} \rtimes^{\mathrm{alg}} G$ *of* $\mathbf{C}$ *with* G *as an object of* ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ *as follows:*

1. *objects: The set of objects of* $\mathbf{C} \rtimes^{\mathrm{alg}} G$ *is the set of objects of* $\mathbf{C}$.

2. *morphisms: For any two objects* C, C' *of* $\mathbf{C}$ *we define the* $\mathbb{C}$*-vector space*
$$\mathtt{Hom}_{\mathbf{C} \rtimes^{\mathrm{alg}} G}(C, C') := \bigoplus_{g \in G} \mathtt{Hom}_{\mathbf{C}}(C, g^{-1}C') \ .$$
Elements f *in the summand* $\mathtt{Hom}_{\mathbf{C}}(C, g^{-1}C')$ *will be denoted by* (f, g).

3. *composition: For* (f, g) *in* $\mathtt{Hom}_{\mathbf{C} \rtimes^{\mathrm{alg}} G}(C, C')$ *and* (f', g') *in* $\mathtt{Hom}_{\mathbf{C} \rtimes^{\mathrm{alg}} G}(C', C'')$ *we set*
$$(f', g') \circ (f, g) := (g^{-1}f' \circ f, g'g) \ .$$
For general elements the composition is defined by linear extension.

4. $*$*-operation: We define* $(f, g)^* := (gf^*, g^{-1})$.

Note that if $f : C \to C'$ is a morphism in $\mathbf{C}$ and g is in G, then we get a morphism $(f, g) : C \to gC'$ in $\mathbf{C} \rtimes^{\mathrm{alg}} G$.

The construction of the crossed product is functorial in $\mathbf{C}$ in an obvious manner. Let $\phi : \mathbf{C} \to \mathbf{C}'$ be a morphism in $\mathbf{Fun}(BG, {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}})$. Then we get a morphism
$$\phi \rtimes^{\mathrm{alg}} G : \mathbf{C} \rtimes^{\mathrm{alg}} G \to \mathbf{C}' \rtimes^{\mathrm{alg}} G$$
defined as follows:

1. objects: The action of $\phi \rtimes^{\mathrm{alg}} G$ on objects is given by the action of ϕ on objects.

2. morphisms: For a morphism f in $\mathbf{C}$ and g in G we set $(\phi \rtimes^{\mathrm{alg}} G)(f, g) := (\phi(f), g)$.

We have thus defined a functor
$$- \rtimes^{\mathrm{alg}} G : \mathbf{Fun}(BG, {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}) \to {}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} \ . \tag{11.6}$$

The crossed product functor preserves unitality of objects and morphisms and therefore restricts to a functor
$$- \rtimes^{\mathrm{alg}} G : \mathbf{Fun}(BG, {}^*\mathbf{Cat}_{\mathbb{C}}) \to {}^*\mathbf{Cat}_{\mathbb{C}} \ . \tag{11.7}$$

Remark 11.40. The crossed product functor $- \rtimes^{\mathrm{alg}} G$ preserves the full subcategories of algebras $\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ (in the possibly non-unital case) and $\mathbf{Alg}_{\mathbb{C}}$ of ${}^*\mathbf{Cat}_{\mathbb{C}}$ (in the unital case). The restrictions of the crossed product to these subcategories recovers the classical definitions. □

We have a canonical morphism

$$\iota^{\mathrm{alg}}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \rtimes^{\mathrm{alg}} G \tag{11.8}$$

in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$ which is the identity on objects and sends a morphism f in $\mathbf{C}$ to the morphism (f, e) in $\mathbf{C} \rtimes^{\mathrm{alg}} G$. If $\mathbf{C}$ is unital, then $\iota^{\mathrm{alg}}_{\mathbf{C}}$ is unital.

Remark 11.41. Note that in the domain of $\iota^{\mathrm{alg}}_{\mathbf{C}}$ we omitted to write the functor which forgets the G-action. Below we will also omit the various inclusion functors from the notation. □

Lemma 11.42. *If $\mathbf{C}$ is in $\mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, then $\mathbf{C} \rtimes^{\mathrm{alg}} G$ is a pre-C^*-category.*

Proof. We first show that for every morphism f in $\mathbf{C}$ and g in G we have

$$\|(f, g)\|_{\max} \le \|f\|_{\mathbf{C}} \ . \tag{11.9}$$

Let A be a C^*-algebra and $\lambda : \mathbf{C} \rtimes^{\mathrm{alg}} G \to A$ be a morphism in ${}^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}}$. Then the composition $\lambda \circ \iota^{\mathrm{alg}}_{\mathbf{C}} : \mathbf{C} \to A$ is a functor between C^*-categories. This implies

$$\|\lambda(f, e)\|_A = \|\lambda(\iota^{\mathrm{alg}}_{\mathbf{C}}(f))\|_A \le \|f\|_{\mathbf{C}} \ .$$

We now have

$$\begin{aligned} &\|\lambda(f, g)\|^2_A \\ &= \|\lambda(f, g)^* \lambda(f, g)\|_A = \|\lambda(gf^*, g^{-1})\lambda(f, g)\|_A = \|\lambda((gf^*, g^{-1})(f, g))\|_A = \|\lambda(f^*f, e)\|_A \\ &\le \|f^*f\|_{\mathbf{C}} = \|f\|^2_{\mathbf{C}} \ . \end{aligned}$$

Since λ is arbitrary this implies that $\|(f, g)\|_{\max} \le \|f\|_{\mathbf{C}}$.

Since every morphism of $\mathbf{C} \rtimes^{\mathrm{alg}} G$ is a finite linear combination of elements of the form (f, g) this implies that $\| - \|_{\max}$ is finite. Hence $\mathbf{C} \rtimes^{\mathrm{alg}} G$ is a pre-C^*-category. □

In view of Lemma 11.42 we can restrict the crossed product functor to a functor

$$- \rtimes^{\mathrm{alg}} : \mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to {}_{\mathrm{pre}}C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathbb{C}} \ .$$

Definition 11.43. *We define the crossed product for C^*-categories by*

$$\mathbf{C} \rtimes G := \mathrm{compl}(\mathbf{C} \rtimes^{\mathrm{alg}} G) \ .$$

Since the crossed-product for C^*-categories is obtained by composing the algebraic crossed product functor (11.6) and the completion functor it is clear that we have defined a functor

$$- \rtimes G : \mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to C^*\mathbf{Cat}^{\mathrm{nu}} \ . \tag{11.10}$$

It again restricts to a functor

$$- \rtimes G : \mathbf{Fun}(BG, C^*\mathbf{Cat}) \to C^*\mathbf{Cat} \ . \tag{11.11}$$

We define the natural morphism

$$\iota_{\mathbf{C}} : \mathbf{C} \stackrel{\iota_{\mathbf{C}}^{\mathrm{alg}}}{\to} \mathbf{C} \rtimes^{\mathrm{alg}} G \to \mathbf{C} \rtimes G \tag{11.12}$$

in $C^*\mathbf{Cat}^{\mathrm{nu}}$.

Proposition 11.44. *$\iota_{\mathbf{C}}$ is isometric.*

Proof. [Bun] □

In the following we explain the universal property of the crossed product in the case of unital categories.

We consider $C^*\mathbf{Cat}$ and the class of unitary equivalences. Then we can form the ∞-category $C^*\mathbf{Cat}_\infty := C^*\mathbf{Cat}[W^{-1}]$ by Dwyer-Kan localization. Let $\ell : C^*\mathbf{Cat} \to C^*\mathbf{Cat}_\infty$ be the canonical morphism. We furthermore let $\ell_{BG} : \mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to \mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_\infty)$ be the induced functor.

Theorem 11.45. *For $\mathbf{C}$ in $\mathbf{Fun}(BG, C^*\mathbf{Cat})$ we have a canonical equivalence*

$$\ell(\mathbf{C} \rtimes G) \simeq \operatorname*{\mathtt{colim}}_{BG} \ell_{BG}(\mathbf{C}) \ .$$

Proof. See [Bun]. □

Example 11.46. Let $\mathbb{C}$ be considered as a C^*-category with the trivial action. Then $C^*(G) := \mathbb{C} \rtimes G$ is also called the maximal group C^*-algebra.

Proposition 11.47.

1. *We have $A(\mathbf{C} \rtimes G) \cong A(\mathbf{C}) \rtimes G$.*

2. *The functor $- \rtimes G : \mathbf{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to C^*\mathbf{Cat}^{\mathrm{nu}}$ preserves exact sequences.*

Proof. For the first assertion see [Bun]. The second assertion follows from the first using (the special case) that $- \rtimes G$ preserves exact sequences of C^*-algebras and that A preserves exact sequences. □

For G-C^*-algebras we also have the reduced crossed product $A \rtimes_r G$. To this end we define the A-valued scalar product on $A \rtimes^{\mathrm{alg}} G$ by

$$\langle (f,g),(f',g')\rangle = \begin{cases} (g^{-1}f^* \circ f' & g = g' \\ 0 & else \end{cases} .$$

We let $L^2(G,A)$ be the closure and get a homomorphism $A \rtimes^{\mathrm{alg}} G \to B(L^2(G,A))$. It induces a C^*-norm $\|-\|_r$ on $A \rtimes^{\mathrm{alg}} G$.

Definition 11.48. *The reduced crossed product $A \rtimes_r G$ is the closure of $A \rtimes^{\mathrm{alg}} G$ with respect to $\|-\|_r$.*

From the universal property of the maximal crossed product we have a canonical homomorphism

$$A \rtimes G \to A \rtimes_r G .$$

Remark 11.49. The calculation of $K_*(A \rtimes_r G)$ in terms of homotopy theory is the content of the Baum-Connes conjecture for G and A.

Let $\mathbf{C}$ be a C^*-category with G-action.

Then we have a homomorphism

$$\mathbf{C} \rtimes^{\mathrm{alg}} G \to A(\mathbf{C} \rtimes G) \cong A(\mathbf{C}) \rtimes G \to A(\mathbf{C}) \rtimes_r G .$$

It induces the reduced norm $\|-\|_r$ on $\mathbf{C} \rtimes^{\mathrm{alg}} G$.

Definition 11.50. *We define the reduced crossed product $\mathbf{C} \rtimes_r G$ as the closure of $\mathbf{C} \rtimes^{\mathrm{alg}} G$ with respect to the norm $\|-\|_r$.*

Corollary 11.51. *We have an isomorphism*

$$A(\mathbf{C} \rtimes_r G) \cong A(\mathbf{C}) \rtimes_r G .$$