Differential Geometry II

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Contents

1 Riemannian manifolds - further examples

1.1 Generalities

want to explain a couple of constructions of Riemannian manifolds and their basic properties

up to now:

- every manifold has a Riemannian metric

– glue local metrics using a partition of unity

– these metrics do not have interesting special properties

a basic property is completeness

- if g is any metric on M

- can find conformal change $e^f g$ which is in addition complete

often one is interesting in metrics with symmetry

- assume that a Lie group H acts on M

Lemma 1.1. *If H acts properly, then there exists a H-invariant metric on M.*

Proof. idea: take any metric g on M, average over H

- H as locally compact group has right-invariant Haar measure dh

- $R_{l,*}dh = dh$ for all l in H

– dh unique up to normalization

- H is Lie group $\Rightarrow$ dh represented by a H-invariant volume form

idea works immediately if H compact:

- H compact $\Rightarrow$ can normalize volume such that $\int_H dh = 1$

- set $\bar{g} := \int_H h^*g \; dh$

– check: $l^*\bar{g} = \int_H l^*h^*g \; dh = \int_H (hl)^*g \; dh = \int_H h^*g R_{l^{-1},*} \; dh = \int_H h^*g \; dh = \bar{g}$

choose any metric g

if H is non-compact:

- can not normalize dh (H has infinite volume)

- by properness of the action can choose a function χ in $C_c(M)$ with

– $\chi \geq 0$

– $\int_{h\in H} h^*\chi dh = 1$

- define $\bar{g} := \int_{h\in H} h^{-1,*}\chi\ h^*g\ dh$

check:

$$\begin{aligned} l^*\bar{g} &= \int_{h\in G} l^*h^*\chi\ l^*h^*g\ dh \\ &= \int_{h\in G} (hl)^*\chi\ (hl)^*g\ dh \\ &= \int_{h\in G} h^*\chi\ h^*gR_{l^{-1},*}\ dh \\ &= \int_{h\in G} h^*\chi h^*\ g\ dh \\ &= \bar{g} \end{aligned}$$

□

Example 1.2. Exercise?

$\mathbb{R}^\times$ acts on $\mathbb{R}$ by multiplication

$\mathbb{R}$ has no $\mathbb{R}^\times$-invariant metric

- assume that g is such a metric

- $g = f(x)dx^2$ with $f > 0$

- $t^*g = f(tx)t^2dx$ for all t in $\mathbb{R}^\times$

- at $x = 0$ get $f(0) = t^2f(0)$

- this implies $f(0) = 0$ (consider limit for $t \to 0$)

– contradicts $f > 0$

What goes wrong?

$R^\times$ does not act properly

- it does act properly on $\mathbb{R} \setminus \{0\}$

- then $x^{-2}dx^2$ is invariant metric

□

$M \to \mathbb{R}^n$ - submanifold

- has induced metric

- can describe properties by second fundamental form, Gauss-Codazzi equations

Problem 1.3. *Given (M, g), is there an isometric embedding $M \to \mathbb{R}^n$ for some n?*

- Whitney: there is an embedding as manifolds if $n \geq 2\dim(M)$.

- Nash: There is an isometric embedding for $n >> \dim(M)$

1.2 Warped products

Construction 1.4. (N, g^N) Riemannian manifold

$f : \mathbb{R} \to (0, \infty)$ - warping function

$M := \mathbb{R} \times N$

$g^M := dr^2 + f(r)g^N$

(M, g^M) is called warped product

sometimes one replaces $\mathbb{R}$ by subintervals

□

Example 1.5. $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$

- coordinates (x, x')

- $g^{\mathbb{R}^n} = dx^2 + g^{\mathbb{R}^{n-1}}$

- constant warping □

Example 1.6. cylinder over (N, g)

$M = \mathbb{R} \times N$

- $g^M = dr^2 + g^N$

$\mathbb{R}^n$ is cylinder over $\mathbb{R}^{n-1}$ □

Example 1.7. H^n

- upper half space model $H^n = \{(x, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x > 0\}$
- $g^H = \frac{1}{x^2} g^{\mathbb{R}^n}$

- solve $dr^2 = \frac{dx^2}{x^2}$
- $dr = \frac{dx}{x}$
- $r = \ln(x)$
- $x = e^r$

$x = e^r$

$H^n = \mathbb{R} \times \mathbb{R}^{n-1}$

- $g^H = dr^2 + e^{-2r} g^{\mathbb{R}^{n-1}}$ □

Example 1.8. cusp over (N, g^N)

- $g^M = dr^2 + r^{-2r} g^N$

$H^n \setminus \{0\}$ is cusp over $\mathbb{R}^{n-1}$

□

Example 1.9. euclidean cone

- replace $\mathbb{R}$ by $(0, \infty)$
- $M = (0, \infty) \times N$
- $g^M = dr^2 + r^2 g^N$

– not complete at $t = 0$

$\mathbb{R}^n \setminus \{0\}$ is euclidean cone over S^{n-1} (Polar coordinates)

- in this case can complete at $t = 0$ □

(N, g^N), f given

(M, g^M) warped product

Lemma 1.10. *(M, g^M) is complete if and only if (N, g^N) is complete*

Proof. exercise?

- (t, x) in M

- $B((t, x), r)$ is contained in $[t - r, t + r] \times N$

- $B((t, x), r)$ is contained in $[t - r, t + r] \times B(x, s)$ with $s := \frac{1}{\min_{u \in [t-r, t+r]} f(u)}$

- this is compact by completeness of (N, g^N) □

Example 1.11. volume

$\mathrm{vol}_g = dr + f(r)^{\frac{n-1}{2}} \mathrm{vol}_{g^N}$

Lemma 1.12. *If N is compact, then $\mathrm{vol}(M, g^M)$ is finite if and only if $\int_{\mathbb{R}} f(r)^{\frac{n-1}{2}} dr < \infty$.*

(N, g^N), f given

(M, g^M) warped product

Example 1.13. exercise? When is the fibre $N_t := \{t\} \times N$ totally geodesic?

Answer: If and only of $f'(t) = 0$ □

1.3 Bundles

$\pi : M \to B$ fibre bundle

g^M and g^B Riemannian metrics on M and B

Definition 1.14. *π is called a Riemannian submersion if $D\pi : TM \to \pi^*TB$ is an isometry.*

- get orthogonal decomposition $TM \cong T^v\pi \oplus T^v\pi^\perp$

– set $T^hM := T^v\pi^\perp$ - this is a connection

– $D\pi$ induces isometry $T^v\pi^\perp \cong \pi^*TB$

reverse construction

choose connection $TM = T^v\pi \oplus T^h M$

choose

- g^B -metric on B

- $g^{T^v\pi}$ vertical metric

define: $g^M := g^{T^v\pi} \oplus \pi^* g^B$

- then π is Riemannian submersion

Example 1.15. warped products are examples

$\pi : \mathbb{R} \times N \to \mathbb{R}$

- $g^{\mathbb{R}} = dr^2$

- connection is $TN \subseteq T\mathbb{R} \boxplus TN = T(\mathbb{R} \times N)$

- $f(r) g^N$ is $g^{T^v\pi}$ □

Lemma 1.16. *If π is proper and B is complete, then M is complete.*

Proof. Exercise?

fix m in M

fix r in $(0, \infty)$

- $B(m, r) \subseteq \pi^{-1} B(\pi(m), r)$

- this is compact since π is proper and (B, g^B) is complete

□

Example 1.17. G-principal bundles

$\pi : P \to B$ - G-principal bundle

- $g^{\mathfrak{g}}$ - Ad-invariant metric on $\mathfrak{g}$

- action defines isomorphism $T^v_p \cong \mathfrak{g}$ at every p in P

- define $g^{T^v\pi}$ so that this is isometric

- choose principal bundle connection ω

- choose metric g^B

- get metric $g^P := g^{T^v\pi} \oplus \pi^* g^B$

Lemma 1.18. *g^P is G-invariant.*

Proof. Exercise

use

$$\begin{array}{ccc} T^v_p\pi & \xrightarrow{TR_g} & T^v_{pg}\pi \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g^{-1})} & \mathfrak{g} \end{array}$$

□

□

1.4 Spaces of loops

(W, g^W) Riemannian manifold

$L(W) := C^\infty(S^1, W)$ - loop space

- this is a set for the moment, more structure later

γ in $L(W)$

- $(\gamma_u)_{u\in I}$ smooth family of loops at γ

– this is a map $S^1 \times I \to W$, $(t, u) \mapsto \gamma_u(t)$

– write $(-)'$ for derivative w.r.t. u

- $\gamma_0' \in \Gamma(S^1, \gamma^*TW)$

- interpret $\Gamma(S^1, \gamma^*TW)$ as $T_\gamma L(W)$

- define scalar product for Y, X in $T_\gamma L(W)$

– $\langle X, Y\rangle := \int_{S^1} g^W(X(t), Y(t))dt$

want to interpret this as Riemannian metric $g^{L(W)}$ on $L(W)$

- consider $f : S^1 \times M \to W$ - interpret as map $f : M \to L(W)$

- get a scalar product $\langle -, -\rangle$ on T_mM by:

- $\gamma := f(-, m)$

- $d_X f(-, m)$ is in $T_\gamma L(W)$

– $\langle X, Y \rangle := \langle d_X f, d_Y f \rangle$

this should be $f^* g^{L(W)}$

- problem: $L(W)$ is not a manifold (infinite-dimensional)

use the language of diffeological spaces

- $L(W)$ is diffeological space:

Cart - category of open subsets of euclidean spaces $\mathbb{R}^n$ (for any n) and smooth maps

Definition 1.19. *A cartesian sheaf is a functor $F : \mathbf{Cart}^{\mathrm{op}} \to \mathbf{Set}$ such that for every U and open covering (U_i) we have*

$$F(U) = \mathbf{eq}(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)) \ .$$

A morphism between cartesian sheaves is a natural transformation.

get category $\mathbf{Sh}(\mathbf{Cart})$ of cartesian sheaves

Example 1.20. example: X a set

$X(U) := \mathrm{Hom}_{\mathbf{Set}}(U, X)$ is a sheaf

□

Definition 1.21. *A concrete cartesian sheaf is a subsheaf of a cartesian sheaf of the form $X(-)$ for some set X.*

Remark 1.22. X - concrete sheaf

- can recover set $X := X(*)$

- u in U is map $u : * \to U$

- interpret ϕ in $X(U)$ as map $U \to X(*)$

– $U \ni u \mapsto u^* \phi \in X(*)$

– $X(U)$ is a subset of $\mathrm{Hom}_{\mathbf{Set}}(U, X(*))$ □

Example 1.23. not every cartesian sheaf is concrete

- consider $\Omega^1 : U \mapsto \Omega^1(U)$ - sheaf of smooth 1-forms

- $\Omega^1(*) = \{0\}$

- $\Omega^1(\mathbb{R})$ is large

□

Definition 1.24. *A diffeological space is a subsheaf of a concrete sheaf.*

get a full subcategory $\mathbf{Mf}_{Diff}$ of $\mathbf{Sh}(\mathbf{Cart})$ of diffeological spaces

- this is category of diffeological spaces

Example 1.25. manifolds

M a manifold

- induces a diffeological space $M_{\mathrm{Diff}}(-)$

$M_{\mathrm{Diff}}(U) := C^\infty(U, M)$

one can recover M from M_{Diff}

- underlying set $M_\infty(*)$

- then $M_{\mathrm{Diff}}(U) \subseteq \mathrm{Hom}_{\mathbf{Set}}(U, M_\infty(*))$ induced by

- $\phi \mapsto (u \mapsto u^*\phi)$ (here $u \in U$ is map $* \to U$)

- topology on $M(*)$: maximal such that all maps in $M_{\mathrm{Diff}}(U)$ are continuous

- smooth structure: characterize smooth functions: $f : M(*) \to \mathbb{R}$ is smooth if $\phi^* f : U \to \mathbb{R}$ is smooth for all ϕ in $M_{\mathrm{Diff}}(U)$

a map of manifolds $f : M \to N$ induces map $f_{\mathrm{Diff}} : M_{\mathrm{Diff}} \to N_{\mathrm{Diff}}$ of diffeological spaces

- one can recover f from f_{Diff}

Lemma 1.26. *We have a fully faithful unctor* $\mathbf{Mf} \to \mathbf{Mf}_{\mathrm{Diff}}$.

□

Example 1.27. many more examples of the following kind

- B Banach space

- $B_{\text{Diff}}(U) := C^\infty(U, B)$ makes sense

- get B_{Diff} - diffeological space □

Example 1.28. - X topological space

- $X_{\text{Diff}}(U) := \text{Hom}_{\mathbf{Top}}(U, X)$ makes sense

- get X_{Diff} - diffeological space

- in general can not recover X from X_{Diff}

– can recover underlying set as $X(*)$

– maximal topology such that all maps $\phi : U \to X(*)$ for $\phi \in X(U)$ are continuous is in general larger than original topology □

Example 1.29. Mapping spaces between manifolds

this example is the main reason to consider diffeological spaces

$\text{Hom}_{\mathbf{Mf}}(M, N)$ extends naturally to a diffeological space

- $\text{Hom}_{\mathbf{Mf}}(M, N)_{\text{Diff}}(U) := \text{Hom}_{\mathbf{Mf}}(U \times M, N)$

- apply to loop space $L(W) := \text{Hom}(S^1, W)$

- get $L(W)_{\text{Diff}}$

□

Example 1.30. can talk about smooth functions, or forms on diffeological spaces

$C^\infty(X) := \text{Hom}_{\mathbf{Mf}_{\text{Diff}}}(X, \mathbb{R}_{\text{Diff}})$

$\Omega^n(X) := \text{Hom}_{\mathbf{Sh}(\mathbf{Cart})}(X, \Omega^n)$

- de Rham complex $d : \Omega^n(X) \to \Omega^{n+1}(X)$ makes sense

□

use same idea to interpret metrics

- have sheaf S^2T in $\mathbf{Sh}(\mathbf{Cart})$

– $S^2T(U) = \Gamma(U, S^2TU)$

– has subsheaf $S^2_{\geq 0}T$ - non-negative symmetric tensors

– can not define sheaf of metrics $S^2_{>0}T$ since positivity is not preserved under pull-back

– can only define a notion of possibly degenerate metric

– this makes all construction problematic which use the inverse

Definition 1.31. *A possibly degenerate metric on a diffeological space M is a map $g : M \to S^2_{\geq 0}T$ in* $\mathbf{Sh}(\mathbf{Cart})$.

Example 1.32. If (M, g) is Riemannian

- get possibly degenerate metric on M_{Diff}

- can recover g from this □

Example 1.33. (W, g^W) - Riemannian

- $L(W)_{\mathrm{Diff}}$ has canonical possibly degenerate Riemannian structure

- the embedding $W_{\mathrm{Diff}} \to L(W)_{\mathrm{Diff}}$ (as constant loops) is isometric □

Example 1.34. $\gamma \mapsto E(\gamma)$ is a map $L(W)_{\mathrm{Diff}} \to \mathbb{R}_{\mathrm{Diff}}$

□

Remark 1.35. in order to model all aspects of tangent bundle diffeologically:

- must enlarge category **Cart** by adding fat points like $*^2 := \mathbb{R}[x]/(x^2)$

- $TM = \mathrm{Hom}(*^2, M)$ (in the sense of ringed spaces)

– element is a homomorphism $C^\infty(M) \to \mathbb{R}[x]/(x^2)$

– this is a point m and a derivation $X \in T_mM$:

– $f \mapsto f(m) + X(f)x$

- $* \to *^2$ corepresents projection $TM \to M$ □

1.5 Space of connections

$V \to M$ vector bundle

$\mathrm{Conn}(V)$ - set of connections

- can turn $\mathrm{Conn}(V)$ into diffeological space $\mathrm{Conn}(V)_{\mathrm{Diff}}$

consider vector bundle $W \to N \times M$

Definition 1.36. *A partical connection on W along M is a $\mathbb{R}$-linear map*

$$\nabla : \Gamma(N \times M, V) \to \Gamma(N \times M, \mathrm{pr}_M^* T^*M \otimes V)$$

satisfying the Leibnitz rule

$$\nabla_X(fv) = f\nabla_X v + X(f)$$

for all X in $\Gamma(N \times M, \mathrm{pr}_M^* TM)$, f in $C^\infty(N \times M)$, and v in $\Gamma(N \times M, V)$.

$\mathrm{Conn}_M(V)$ - set of partial connections

- is an affine space over $\Gamma(N \times M, \mathrm{pr}_1^* T^*M \otimes \mathrm{End}(V))$

Definition 1.37. *The diffeological space* $\mathrm{Conn}_{\mathrm{Diff}}(V)$ *is defined by*

$$\mathrm{Conn}_{\mathrm{Diff}}(V)(U) := \mathrm{Conn}_M(\mathrm{pr}_M^* V) \ .$$

assume: M is Riemannian and compact

- consider metric on V
- induces notion of adjoint
- get metric on $\mathrm{End}(V)$ by $\langle A, B\rangle := \mathrm{tr} A^* B$
- get metric on $\mathrm{pr}_1^* T^*M \otimes \mathrm{End}(V)$ by combining

get metric on $\mathrm{Conn}_{\mathrm{Diff}}(V)$:

- fix ∇ in $\mathrm{Conn}_{\mathrm{Diff}}(V)(U)$
- X, Y in $T_u U$
- $d_X \nabla(u) \in \Gamma(M, T^*M \otimes \mathrm{End}(V))$
- $g(X,Y) = \int_M \langle d_X \nabla(u)(m), d_Y \nabla(u)(m)\rangle dg^M$

in gauge theory

- consider functions like $\nabla \mapsto \int_M \|R^\nabla\|^2 dg^M$ (Yang-Mills functional)

– this is smooth function: $\mathrm{Conn}_{\mathrm{Diff}}(V) \to \mathbb{R}$

- metric allows to consider gradient and gradient flow

2 The group of Isometries

2.1 G-structures

recall:

M - manifold, $\dim(M) = n$

- have frame bundle $\mathrm{Fr}(M) \to M$

– a $GL(n,\mathbb{R})$-principal bundle

– m in M, e in $\mathrm{Fr}(M)_m$ is isomorphism $e : \mathbb{R}^n \to T_m M$

– $GL(n,\mathbb{R})$-action by $e \cdot g := e \circ g$

$f : M \to M'$ local diffeomorphism

- f induces $\mathrm{Fr}(f) : \mathrm{Fr}(M) \to \mathrm{Fr}(M')$

- $\mathrm{Fr}(f)(e) := Tf(\pi(e)) \circ e$

$\kappa : G \to GL(n,\mathbb{R})$ homomorphism of Lie groups

Definition 2.1. *A G-structure on M is a G-reduction (Q, r) of the frame bundle.*

recall notion of G-reduction :

- $Q \to M$ is G-principal bundle

- $r : Q \to \mathrm{Fr}(M)$ is G-equivariant bundle map:

$$\begin{array}{ccc} Q \times G & \longrightarrow & Q \\ \downarrow{\scriptstyle r\times\kappa} & & \downarrow{\scriptstyle r} \\ \mathrm{Fr}(M) \times GL(n,\mathbb{R}) & \longrightarrow & \mathrm{Fr}(M) \end{array}$$

notion of equivalence:

$$\begin{array}{ccc} Q & \xrightarrow{\cong} & Q' \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle r'} & \\ & \mathrm{Fr}(M) & \end{array}$$

consider special case: $\kappa : G \to GL(n, \mathbb{R})$ is inclusion of a sub Liegroup

- r identifies Q with a subbundle of $\mathrm{Fr}(M)$

Corollary 2.2. *If κ is an inclusion of a sub-Lie group, then a G-structure on M is a G-principal subbundle Q of $\mathrm{Fr}(M)$.*

(M, Q), (M', Q') manifolds with G-structures

$f : M \to M'$ local diffeomorphism

Definition 2.3. *f preserves the G-structures if $\mathrm{Fr}(f)(Q) = Q'$.*

Remark 2.4. If κ is not injective, then the notion of preservation of G-structure is additional structure

- a lift of $\mathrm{Fr}(f)$

$$\begin{CD} Q @>{\widetilde{\mathrm{Fr}(f)}}>> Q' \\ @V{r}VV @VV{r'}V \\ \mathrm{Fr}(M) @>{\mathrm{Fr}(f)}>> \mathrm{Fr}(M') \\ @VVV @VVV \\ M @>{f}>> M' \end{CD}$$

this applies e.g. to $Spin(n)$-structures

□

Example 2.5. Orientation is $GL(n, \mathbb{R})^+$-reduction

Example 2.6. choice of volume form is $SL(n, \mathbb{R})$ - reduction □

Example 2.7. choice of Riemannain metric is $O(n)$ - reduction □

Example 2.8. $U(n) \subseteq GL(n, \mathbb{C}) \subseteq GL(2m, \mathbb{R})$

reductions are called almost complex structures □

Example 2.9. $Sp(n) \subseteq GL(2n, \mathbb{R})$

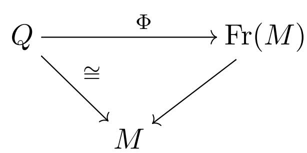
reductions are called symplectic structures □

Example 2.10. $Spin(n) \xrightarrow{2:1} SO(n) \to GL(n, \mathbb{R})$

a $Spin(n)$ - reduction is a spin structure □

Example 2.11. consider $G = 1$

- an 1-structure is a section Φ of $\mathrm{Fr}(M)$



- is a trivialization $\Phi : M \times \mathbb{R}^n \to TM$

□

general priciple:

V - real vector space

- $\mathcal{T}^k_l(V) := \underbrace{V \otimes \cdots \otimes V}_{k\times} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l\times}$
- $\mathrm{Aut}(V)$ acts on $\mathcal{T}^k_l(V)$ by functoriality

consider element $K \in \mathcal{T}^k_l(\mathbb{R}^n)$

- define $G \subseteq GL(n, \mathbb{R})$ as stabilizer of K

- given $Q \to M$ - a G-structure

– form $\mathcal{T}^k_l(TM) \cong Q \times_G \mathcal{T}^k_l(\mathbb{R}^n)$ - bundle of (k, l)-tensors

- K induces section $\mathcal{K}$ in $\Gamma(M, \mathcal{T}^k_l(TM))$:

– value at m in M: $\mathcal{K}(m) = [e, K]$ for any e in Q_m

– note $[eg, K] = [e, gK] = [e, K]$ for $g \in G$

– so $\mathcal{K}(m)$ well-defined independently of choice of e

given $\mathcal{K}$ can recover Q from section $\mathcal{K}$

- take subset of frames e in $\mathrm{Fr}(M)$ such that $[e, K] = \mathcal{K}(\pi(e))$

Example 2.12. Riemannian metrics

$K = \sum_{i=1}^{n} e_i^* \otimes e_i^*$ in $S^2(\mathbb{R}^{n,*}) \subseteq \mathcal{T}^2(\mathbb{R}^{n,*})$

– is positive definite

– all positive definite are isomorphic to this one

- stabilizer: $O(n)$

- a metric on M defines $O(n)$-structure $Q \subseteq \mathrm{Fr}(M)$

– $e \in Q_m$ if and only if $e : \mathbb{R}^n \to T_m M$ isometric

– Q is the subundle of orthogonal frames

□

Example 2.13. $K := e_1 \wedge \cdots \wedge e_n \in \Lambda^n \mathbb{R}^{n,*} \subseteq \mathcal{T}^n(\mathbb{R}^{n,*})$

- all volume forms are isomorphic to this one

- $SL(n, \mathbb{R})$ is stabilizer

- $SL(n, \mathbb{R})$ - structure on M is equivalent to datum of volume form $\mathcal{K} \in \Omega^n(M)$

□

Example 2.14. $\mathbb{R}^{2n} \cong \mathbb{C}$

- $I \in \mathrm{End}(\mathbb{R}^{2n})$ - multiplication by I

– $I^2 = -1$

– every endomorphism J of $\mathbb{R}^{2n}$ with $J^2 = -1$ is conjugated to I

- $\mathrm{End}(\mathbb{R}^{2n}) \cong \mathbb{R}^{2n,*} \otimes \mathbb{R}^{2n} = \mathcal{T}_1^1(\mathbb{R}^{2n})$

- stabilizer $GL(n,\mathbb{C})$

$GL(n,\mathbb{C})$-structure is the same as a section $\mathcal{I} \in \Gamma(M, \mathrm{End}(TM))$ with $\mathcal{I}^2 = -1$

- called an almost complex structure □

Example 2.15. almost symplectic structure

consider $\mathbb{R}^{2n}$

- $\omega = e_1^* \wedge e_{n+1}^* + \cdots + e_n^* \wedge e_{2n}^* \in \Lambda^2 \mathbb{R}^{n,*}$
- every non-degenerate alternating form is isomorphic to ω under $GL(2n,\mathbb{R})$
- stabilizer ist $Sp(n)$

- $Sp(n)$-structure is determined by form $\omega \in \Omega^2(M)$ everywhere non-degenerate □

fix tensor $K \in \mathcal{T}_l^k(\mathbb{R}^n)$, stabilizer G

G-structure on M determined by $\mathcal{K} \in \Gamma(M, \mathcal{T}_l^k(TM))$

- can one find coordinates locally such that $K = \mathcal{K}$
- in this case we call the G-structure flat

- always possible for $SL(n,\mathbb{Z})$-structure

- for almost symplectic:

– necessary and sufficient condition $d\omega = 0$ (Darboux theorem)

– in this case structure is called symplectic structure

- not always possible for Riemannian metric:

– necessary and sufficient condition: $R^{\nabla^{LC}} = 0$

– in this case (M,g) is called flat

- not always possible for almost complex structure

– $T^{0,1}M$ - consider -1-eigenspace of $\mathcal{I} \otimes 1$ on $TM \otimes_{\mathbb{R}} \mathbb{C}$

– this subbundle of $TM \otimes_{\mathbb{R}} \mathbb{C}$ must be involutive

— commutator of sections is again a section of the subbundle (Newlander-Nierenberg Theorem)

– in this case $(M, \mathcal{I})$ is called complex

– has charts with values in $\mathbb{C}^n$ and holomorphic transition maps

Example 2.16. T^*M has a symplectic structure

$\pi : T^*M \to M$

- $T\pi : T(T^*M) \to T^*M$

- define α in $\Omega^1(T^*M)$ - canonical 1-form

— $\xi \in T^*_m M$

— $X \in T_\xi(T^*M)$

– $\alpha(X) := \xi(T\pi(\xi)(X))$

— in fact: $T\pi(\xi)(X) \in T_m M$

— so can apply ξ

define: $\omega := d\alpha$

- clear $d\omega = 0$

- check: ω is non-degenerate

- local coordinates of M : $x_1, \ldots, x_n$

- local coordinates of T^*M : $x_1, \ldots, x_n, \xi^1, \ldots, \xi^n$

- $\pi(x, \xi) = x$

- $X = X^i \partial_{x^i} + Y_j \partial_{\xi_j}$

- $T\pi(\xi)(X) = X^i \partial_{x^i}$

- $\xi(T\pi(\xi)(X)) = \xi_i X^i$

- read off: $\alpha = \xi_i dx^i$

- $\omega = d\alpha = d\xi_i \wedge dx^i$ - this is obviously non-degenerated

– here flatness is clear: we have found suitable coordinates explicitly

□

2.2 Transformation groups

Definition 2.17. *A Lie transformation group is a triple (G, M, a) of a Lie group G, a manifold M and an effective action $a : G \times M \to M$.*

- effective means: $G \to \mathrm{Diff}(M)$ is injective

get map $\mathfrak{g} \to \mathcal{X}(M)$, $X \mapsto X^\sharp$

- $X^\sharp$ - fundamental vector field for X
- $X^\sharp(m) = d_1 a(e, m)(X)$

for X in $\mathcal{X}(M)$

- write $\exp(tX)m$ for the value of flow at time t with start in m
- recall: X is called complete if $\exp(tX)m$ exists for all t in $\mathbb{R}$ and m in M
- write $\mathcal{X}^c(M) := \{X \in \mathcal{X}(M) \mid X \text{ is complete}\}$ - set of complete vector fields

consider tranformation group (G, M, a),

- $\mathfrak{g} \subseteq \mathcal{X}(M)$

Lemma 2.18. *We have $\mathfrak{g} \subseteq \mathcal{X}^c(M)$.*

Proof. write e^{tX} for one-parameter group in G generated by X

- claim: $\exp(tX)m = e^{tX}m$ (exercise)
- claim shows assertion □

Lemma 2.19. *The map $\mathfrak{g} \to \mathcal{X}(M)$ is injective.*

Proof. assume: X in $\mathfrak{g}$ is in kernel

- then $e^X m = \exp(X)m = m$ for all m
- conclude: e^X acts trivially
- contradicts effectiveness

□

forming fundamental vector fields realizes $\mathfrak{g}$ as sub-Lie algebra of $\mathcal{X}(M)$

can reconstruct tranformation group (G, M, a) from $\mathfrak{g}$

Theorem 2.20 (Palais). *If $\mathfrak{g}$ is a finite-dimensional sub-Lie algebra of $\mathcal{X}^c(M)$, then there exists a unique Lie transfomation group (G, M, a) with G connected and Lie algebra $\mathfrak{g}$.*

Proof.

want to define G as group generated in $\mathrm{Aut}_{\mathbf{Mf}}(M)$ by $\exp(X)$ for X in $\mathfrak{g}$

- $\tilde{G}$ - simply connected Lie group with Lie algebra $\mathfrak{g}$

- want to see that $\tilde{G}$ acts on M such that $e^{tX}m = \exp(tX)m$

- obtain G as quotient $\tilde{G}/G_M$ where G_M - stabilizer of M

– But not clear that this action is well-defined!

in order to show this we use fibre bundle theory:

- consider $\tilde{G} \times M \to \tilde{G}$ as fibre bundle

– $\tilde{G}$ acts on $\tilde{G} \times M$ by $h(g, m) = (hg, m)$

define $\tilde{G}$-invariant connection on $\tilde{G} \times M$

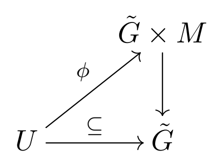
- give horizontal subbundle L of $T(\tilde{G} \times M)$

– generated at(g, m) by $(gX, X(m))$ for all X in $\mathfrak{g}$

– this subbundle is $\tilde{G}$-invariant

– check: this subbundle is involutive, i.e., defines a flat connection

in general for flat connection: for any (g, m) in M get unique local horizontal lift



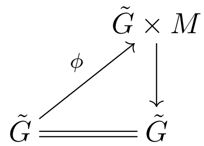
– U is open nbhd of g

– $\phi(g) = (g, m)$

now use: $\tilde{G}$ is simply connected

- for any (g, m) in M get unique global horizontal lift

-



– $\phi(g) = (g, m)$

write ψ_m for unique lift with $\psi_m(e) = (e, m)$

- identify set M with set these horizontal maps

- $\tilde{G}$ acts on this set

– so $\tilde{G}$ acts on M such that $g\psi_m = \psi_{g^{-1}m}$

show that this is the desired action

- X in $\mathfrak{g}$

– $(e^{tX}, \exp(tX)m)$ is horizontal curve which intersects (e, m)

- is in the image of $e^{tX}\phi_{\exp(tX)m}$

– conclude that $e^{tX}\phi_{\exp(tX)m} = \phi_m$

- replace m by $\exp(tX)m$

- conclude $e^{tX}\phi_m = \phi_{\exp(tX)m}$

– hence $e^{tX}m = \exp(tX)m$

$G_M \subseteq \tilde{G}$ stabilizer of M

- observe G_M is discrete

– $\exp(tX)m = m$ for all m implies $X = 0$

set $G := \tilde{G}/G_M$

- then G act effectively

- get desired transformation group

$\square$

consider the following situation

- M - manifold

- G - group

- G acts by diffeomorphisms on M

- $a : G \to \mathrm{Aut}_{\mathbf{Mf}}(M)$ - injective

What additional data makes (G, M, a) into a Lie transformation group?

$S := \{X \in \mathcal{X}^c(M) \mid (\forall t \in \mathbb{R} \mid \exp(tX) \in G)\}$

- at the moment this is just a subset

– In general we do not know that a linear combination of complete vector fields are a commutator of them is again complete!

– so not clear whether linear subspace or even sub-lieagebra

Theorem 2.21. *If S generates a finite-dimensional Lie algebra, then (G, M, a) has the structure of a Lie transformation group with Lie algebra S*

Proof. $\mathfrak{g}^*$ - Lie algebra generated by S (as subalgebra of $\mathcal{X}(M)$)

- is finite-dimensional by assumption

- want to show that $S = \mathfrak{g}$

have simply-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$

- the elements of $\mathfrak{g}$ have local flows

consider X, Y in $\mathfrak{g}^*$

- define $Z := \mathrm{Ad}(e^X)(Y)$ in $\mathfrak{g}^*$

Lemma 2.22. *If $X, Y \in S$, then $Z \in S$.*

Proof.

$$\exp(sX)\exp(tY)\exp(-sX)m = e^{sX}e^{tY}e^{-sX}m = e^{t\mathrm{Ad}(e^{sX})(Y)}m = \exp(t\mathrm{Ad}(e^{sX})(Y))m$$

for all small s, t (depending on m) and all m

- conclude: $\exp(sX)\exp(tY)\exp(-sX)m = \exp(t\mathrm{Ad}(e^{sX})(Y))m$ for all t, s, m

– conclude $\exp(X)\exp(tY)\exp(-X)m = \exp(tZ)m$ exists for all t and $\exp(tZ)$ belongs to G

— hence $Z \in S$ □

Lemma 2.23. *S spans $\mathfrak{g}$ as a vector space*

Proof. $V := \mathrm{span}_{\mathbb{R}}(S)$

- have seen above: $\mathrm{Ad}(e^S)(V) \subseteq V$

– differentiate in order to get $[S, V] \subseteq V$

– by linearity of bracket: $[V, V] \subseteq V$

— V is Lie algebra

– conclude from $S \subseteq V$ that $\mathfrak{g} \subseteq V$

– by construction $V \subseteq \mathfrak{g}$

— hence $g^* = V$

□

Lemma 2.24. $S = \mathfrak{g}$

Proof. consider $Y \in \mathfrak{g}$

- must show that $\exp(tY)m$ exists for all t and m and $\exp(tY)$ is in G

- suffices to show that there is δ in $(0, \infty)$ such that $\exp(tY)m$ exists for all t with $|t| \leq \delta$ and all m and $\exp(tY)$ is in G

$(X_i)_i$ - basis of $\mathfrak{g}$

- $\mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto e^{t_1 X_1} \ldots e^{a_n X_n} \in \tilde{G}$ local diffeo

– ex δ in $(0, \infty)$ such that for all t with $|t| \leq \delta$

— $e^{tY} = e^{a_1(t)X_1} \dots e^{a_n(t)X_n}$

— $t \mapsto (a_1(t), \dots, a_n(t))$ smooth

- $\exp(tY)m = \exp(a_1(t)X_1) \dots \exp(a_n(t)X_n)m$ for all t with $|t| \leq \delta$ and all m

- also clear: $\exp(tY)$ is in G

□

finish proof of Theorem

- use Theorem 2.20 for $\mathfrak{g}$

- get transformation group (G^*, M, a) with Lie algebra $\mathfrak{g}$

– $G^* = \tilde{G}/\tilde{G}_M$ - with $\tilde{G}_M$ stabilizer

consider $(V_\alpha)_\alpha$ system of open nbhds of 1 in G^*

- set $(hV_\alpha)_\alpha$ as system of open nbhds of h in G

- this defines topology on G

- $G^* \subseteq G$ is open, closed

- G becomes Lie group $1 \to G^* \to G \to \pi_0(G) \to 1$

- $G \times M \to M$ becomes smooth action

There is a gap here: $\pi_0(G)$ must be countable

□

Example 2.25. counterexample:

consider $M = \mathbb{R}$

- consider some uncountable subgroup G of $\mathbb{R}$ which is not equal to $\mathbb{R}$

– take any uncountable subset I

– let G be subgroup group generated by I

– then $S = 0$

– G is discrete □

2.3 Automorphism groups of structures

do not say G-structures since we use G to denote the automorphism group

M - manifold with 1-structure

- recall: this is a trivialization of TM

$G := \{f \in \mathrm{Aut}_{\mathbf{Mf}}(M) \mid f \text{ preserves 1-structure}\}$

need to consider non-connected manifolds M

- fix i in $\pi_0(M)$

- M_i component of M

– consider the subgroups $G_{(i)} \subseteq G_{\{i\}} \subseteq G$ of f which stabilize M_i point- and setwise

— define $G_i := G_{\{i\}}/G_{(i)}$

— G_i acts effectively on M_i

Example 2.26. Consider $M = \mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{Z}/2\mathbb{Z}$ and $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$

- components $0, 1, 2$

- write elements of M

– as (x, i), $x \in \mathbb{R}$, $i \in \mathbb{Z}/2Z$ for first two components

– and j in $\mathbb{Z}/2\mathbb{Z}$

- define action of G

– $(\sigma, 0, 0)(x, i) := (x, i + \sigma)$

– $(0, 0, \kappa)(x, i) := (x, i)$

– $(\sigma, 0, 0)j := j$

– $(0, 0, \kappa)j := j + \sigma$

– $(0, t, 0)(x, i) := (x + t, i)$

– $(0, t, 0)j := j$

- $G_{\{0\}} = \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$

- $G_{(0)} = \mathbb{Z}/2\mathbb{Z}$

- $G_0 = \mathbb{R}$

$\square$

Theorem 2.27. *Assume that M has finitely many components.*

1. *(G, M, a) refines to a Lie transformation group.*
2. $\dim(G) \leq |\pi_0(M)| \dim(M)$
3. *For every i in $\pi_0(M)$ we have an induced Lie transformation group (G_i, M_i, a).*
4. *For every i in $\pi_0(M)$ and m in M_i the map $G_i \to G_i m$ is an embedding onto a closed submanifold.*

Proof. (e_i) basis fields of 1-structure

$V := \mathrm{span}_{\mathbb{R}}((e_i)_i) \subseteq \mathcal{X}(M)$

- $V \ni v \mapsto \exp(v)m$ local diffeo near 0

- g in G preserves V

– $g_* v = v$ for every v in V

– conclude $g \exp(v) = \exp(v) g$

$\mathfrak{l} := \{X \in \mathcal{X}(M) \mid [V, X] = 0\}$

- $\mathfrak{l}$ is Lie subalgebra (by Jacobi identity)

- have decomposition $\mathfrak{l} = \bigoplus_{i \in \pi_0(M)} \mathfrak{l}_i$

fix i in $\pi_0(M)$ and m in M_i

Lemma 2.28. *The evaluation $\mathfrak{l}_i \to T_m M$ is injective.*

Proof. write $X = \sum_i a_i e_i$

- $0 = [e_j, X] = \sum_i e_j(a_i) e_i + \sum_i a_i [e_j, e_i]$

– system of homogeneous linear ode's for the a_j

consider m' in M_i

–solve ODE along a curve from m to m'

– $X(m) = 0$ - initial condition - implies $X(m') = 0$

- m' arbitrary in M_i

– conclude $X \equiv 0$ on M_i $\square$

conclude $\dim(\ell) \leq \dim(M)|\pi_0(M)|$

$S := \mathcal{X}^c(M) \cap \mathfrak{l}$ - set of complete elements in $\mathfrak{l}$

- S generates Lie algebra contained in $\mathfrak{l}$

- is also finite-dimensional

argue that $\exp(tX) \in G$ for all t:

- $\partial_t \exp(tX)_*(e_i) = \exp(tX)_*(e_i)[X, e_i] = 0$

- hence $\exp(tX)_* e_i = e_i$ for all t

– implies claim

conclude by Theorem 2.21 that G is part of Lie transformation group (G, M, a) with Lie algebra S

consider i in $\pi_0(M)$

- G_i acts on M_i and preserves (restriction of) 1-structure

apply to G_i and $S_i := S \cap \mathfrak{l}_i$

- conclude by Theorem 2.21 that G_i is part of Lie transformation group (G_i, M_i, a) with Lie algebra S_i

fix $m \in M_i$

Lemma 2.29. *$G_i m$ is closed in M_i*

Proof. $(g_k)_k$ sequence in G_i

- $g_k m \to m_0$

must find g in G_i with $gm = m_0$

want to define g by $m' \mapsto \lim_k g_k m'$

- consider set M_i' of m' in M_i such that $\lim_k g_k m'$ exists

- M_i' is open and closed:

– to see this: parametrize open neighbourhood of m' in M_i by $v \mapsto \exp(v)m'$

$$\lim_k g_k \exp(v)m' = \exp(v) \lim_k g_k m'$$

M_i is connected, hence $M_i' = M_i$

have by construction have $g \exp(v)m' = \exp(v) \lim_k g_k m'$ - this is smooth in v

- have g in G_i since it preserves V

□

fix m in M_i

Lemma 2.30. *$G_i \ni g \to gm \in M_i$ is injective*

Proof. M_i^g - fixed point set of g

- closed by continuity of action of g

- for m in M_i^g

– $g \exp(v)m = \exp(v)gm = \exp(v)g$

— M_i^g is also open

have two cases:

- $M_i^g = M_i$ and $g = 1$

- $M_i^g = \emptyset$

□

- $m \in M_i$
- by Lemma 2.28 $G_i \to Gm$ is immersion and hence embedding

□

Example 2.31. What happens if we drop the condition on finitely many components?

we consider the standard 1-structure on $\bigsqcup_{\mathbb{N}} \mathbb{R}$

$\prod_{n\in\mathbb{N}} \mathbb{R}$ acts

$(t_i)_{i\in\mathbb{N}}$ acts as $x \mapsto x + t_i$ on component with index i

is not a Lie transformation group □

2.4 The isometry group as a Lie transformation group

(M, g) - Riemannian manifold

- equivalently: $O(n)$ - structure $r : Q \to \mathrm{Fr}(M)$
- $I(M)$ - isometry group
- equivalently: group which preserves $O(n)$-structure

Theorem 2.32 (Myers-Steenrod 1939)**.** *We assume that M is connected.*

1. *$I(M)$ is part of a Lie transformation group $(I(M), M, a)$*
2. *For every m in M the stabiliser $I(M)_m$ is compact.*
3. *If M is compact, then $I(M)$ is compact.*

Proof. we use that Q has a canonical connection, the Levi-Civita connection

$\pi : \mathrm{Fr}(M) \to M$

- have tautological $\mathbb{R}^n$-valued 1-form θ in $\Omega^1(\mathrm{Fr}(M), \mathbb{R}^n)$

– $\theta(e)(X) := e^{-1}(T\pi(e)(X)) \in \mathbb{R}^n$ for all X in $T_e\mathrm{Fr}(M)$

f in $\mathrm{Aut}(M)$

- induces $\mathrm{Fr}(f) \in \mathrm{Aut}(\mathrm{Fr}(M))$

- $\mathrm{Fr}(f)^*\theta = \theta$

– indeed use $\pi \circ \mathrm{Fr}(f) = f \circ \pi$

$$\begin{aligned}
(\mathrm{Fr}(f)^*\theta)(e)(X) &= \theta(\mathrm{Fr}(f)(e))(T\mathrm{Fr}(f)(e)(X)) \\
&= \mathrm{Fr}(f)(e)^{-1}(T\pi(T\mathrm{Fr}(f)(e))(X)) \\
&= (Tf(\pi(e)) \circ e)^{-1}(Tf(\pi(e))(T\pi(e)(X))) \\
&= e^{-1}(T\pi(e)(X)) \\
&= \theta(e)(X)
\end{aligned}$$

$G \subseteq GL(n, \mathbb{R})$ sub Lie-group with finitely many components

consider G-reduction $Q \subseteq \mathrm{Fr}(M)$

- consider G-principal bundle automorphism

$$\begin{array}{ccc}
Q & \xrightarrow{\tilde{f}} & Q \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & M
\end{array}$$

Lemma 2.33. *If $\tilde{f}^*\theta_{|Q} = \theta_{|Q}$, then f preserves the G-structure and $\tilde{f} = \mathrm{Fr}(f)$.*

Proof. $J := \mathrm{Fr}(f)^{-1} \circ \tilde{f}$

- want to show: J is inclusion $Q \to \mathrm{Fr}(M)$

– know already

- $\pi \circ J = \pi$

- $J^*\theta = \theta$

- $\theta(J(e))(TJ(e)(X)) = J(e)^{-1}(T\pi(J(e))(TJ(e)(X))) = J(e)^{-1}(T\pi(e)(X))$

- $\theta(e)(X) = e^{-1}(T\pi(e)(X))$

– both together imply $J(e) = e$

– hence J is the canonical embedding

□

$\text{Aut}(M, Q)$ - group of G-structure preserving automorphisms of M

- consider principal bundle connection ω on Q

Definition 2.34. *Call f in $\text{Aut}(M, Q)$ affine if $\text{Fr}(f)^*\omega = \omega$.*

$\text{Aut}(M, Q, \omega)$ - subgroup of $\text{Aut}(M, Q)$ of affine transformations

Lemma 2.35. *$\text{Aut}(M, Q, \omega)$ is part of a Lie transformation group $(\text{Aut}(M, Q, \omega), M, a)$.*

Proof. $\theta \oplus \omega \in \Omega^1(Q, \mathbb{R}^n \oplus \mathfrak{g})$

- is a 1-structure on $\text{Fr}(M)$

- by Lemma 2.33 $\text{Aut}(M, Q, \omega)$ is 1-structure preserving automorphisms of $\text{Fr}(M)$

- Q has finitely many components

– by Theorem 2.27 get Lie transformation group $(\text{Aut}(M, Q, \omega), \text{Fr}(M), a')$

- action descends to action on M by Lemma 2.33

- get Lie transformation group $(\text{Aut}(M, Q, \omega), M, a)$

□

consider $G = O(n) \subseteq GL(n, \mathbb{R})$

- ω - Levi-Civita connection

- $I(M) = \text{Aut}(M, Q) = \text{Aut}(M, Q, \omega)$

- get Lie transformation group $(I(M), M, a)$

m in M

$I(M)_m$ stabilizer

- fix e in Q_m

- $I(M) \ni f \mapsto \text{Fr}(f)e$ is embedding onto closed submanifold

- $I(M)_m$ has image in fibre Q_m

– hence $I(M)_m$ is compact

if M is compact then Q is compact and hence $I(M)$ is compact

□

2.5 Manifolds with large isometry groups

M - manifold

- $n := \dim(M)$

- g^M - Riemannian metric

$I(M, g^M)$ - isometry group

Lemma 2.36. $\dim(I(M, g^M)) \leq \frac{n(n+1)}{2}$

Proof. $\dim O(TM) = \dim(O(n)) + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$

- have embedding $I(G, g^M)$ into $O(TM)$

– fix orthogonal frame e in $O(TM)$

— embedding is by $g \mapsto \mathrm{Fr}(g)e$

- hence estimate □

Lemma 2.37. *Let M be connected. If $\dim(I(M, g^M)) = \frac{n(n+1)}{2}$, then M is one of*

1. $\mathbb{R}^n$
2. S^n
3. $P^n(\mathbb{R})$
4. H^n

Proof. $I(M, g^M)e$ in $O(TM)$ closed

- and open by equal dimension

$O(TM)$ has one or two components

- if $O(TM)$ is connected: $I(M, g^M) = O(TM)$

- otherwise: $I(M, g^M)$ is component of $O(TM)$

- stabilizer of m: $I(M, g^M)_m$ is $O(T_m M)$

$I(M, g^M)_m$ acts transitively on 2-planes in $T_m M$

- sectional curvature is invariant

– hence have sectional curvature is constant in m

— can conclude: sectional curvature is constant (last semester)

- $I(M, g^M)$ acts transitively on points of M

– get uniform existence time of geodesic flow

– conclude: (M, g^M) is complete

- $\tilde{M} \to M$ universal covering

– $M = \tilde{M}/\Gamma$ - where Γ discrete subgroup of $I(\tilde{M}, g^{\tilde{M}})$

– has lifted metric $\tilde{g}$

– is also complete and has constant sectional curvature

consider Killing field X on M

- lifts to Killing field $\tilde{X}$ on $\tilde{M}$

- conclude: $\frac{n(n+1)}{2} = \dim(I(M, g^M)) \leq \dim(I(\tilde{M}, g^{\tilde{M}})) = \frac{n(n+1)}{2}$

– hence $X \mapsto \tilde{X}$ is isomorphism of Lie algebras

– $\tilde{X}$ is Γ-invariant

$I(\tilde{M}, g^{\tilde{M}})^0$ is generated by $\exp(\tilde{X})$ for all $\tilde{X}$

- these vector fields are Γ-invariant (no additional non-invariant ones by maximality of dimension of $I(M, g^M)$)

- all elements of $I(\tilde{M}, g^{\tilde{M}})^0$ commutes with Γ

now invoke classification of complete simply connected manifolds with constant sectional curvature

use

$K \geq 0$: S^n

- have group $\Gamma = C_2$ generated by antipodal involution

– the antipodal involution commutes with all isometries (is central in $I(S^n, g^{S^n}) \cong O(n{+}1)$)

– hence $\mathbb{RP}^n$ is non-simply connected example

– this is the only quotient of S^n by central isometries

$K = 0$: $\mathbb{R}^n$

- exclude quotients: $\mathbb{R}^n/\Gamma$:

– every isometry which commutes with all translations and rotations is trivial

$K < 0$: H^n

- exclude quotients H^n/Γ:

– every γ which commutes with all isometries is trivial

□

3 Construction of E examples from Lie groups

3.1 Symmetric spaces

(M, g^M) - Riemannian

Definition 3.1. (M, g^M) *is a symmetric space if every* m *in* M *is an isolated fixed point of an involutive isometry* θ_m.

note: $D\theta_m(m) = -1$

- otherwise $D\theta_m(m)$ would fix some nonzero X

– $\exp_m(tX)$ is then also fixed for all small t

– hence m not isolated

will provide the construction of Riemannian symmetric spaces using symmetric pairs

consider semisimple Lie group G

- $\mathfrak{g}$ - semisimple Lie algebra

- Killing form $B \in S^2(\mathfrak{g}^*)$

– $B(X,Y) := \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$

- semisimple is equivalent to: Killing form $B \in S^2(\mathfrak{g}^*)$ is non-degenerate

– recall further: G is compact if and only if B is negative definite

consider involution Θ on G

- set $K \subseteq G^\Theta$ open subgroup of fixed points

Definition 3.2. *A pair (G,K) of a Lie group and a closed subgroup is called a symmetric pair of there exists an involution Θ of G such that K is an open subgroup of G^Θ.*

– is a subgroup

- then $\mathfrak{k} \subseteq \mathfrak{g}$ - fixed points of induced involution $\theta := d\Theta$

– is sub Lie algebra of subgroup K

- $\mathfrak{p} := -1$-eigenspace of θ

– is not a Lie algebra in general:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{is called Cartan decomposition}$$

Lemma 3.3. *The Cartan decomposition is and* $\mathrm{Ad}(K)$*, θ-invariant, and B-orthogonal decomposition. We furthermore have*

$$[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k} \ .$$

Proof. θ-invariant by construction

$\mathrm{Ad}(K)$ commutes with θ

- implies $\mathrm{Ad}(K)$-invariance of decomposition

θ is automorphism of Lie algebras and preserves therefore B

- implies B-orthogonality of decomposition

commutator rules: apply automorphism θ

□

(G, K) - symmetric par

Definition 3.4. *We call* (G, K) *a Riemannian symmetric pair if* $\mathrm{Ad}(K) \subseteq \mathrm{Aut}(\mathfrak{p})$ *is compact.*

Corollary 3.5. *If* (G, K) *is a Riemannian symmetric pair, then* $\mathfrak{p}$ *admits* $\mathrm{Ad}(K)$*-invariant scalar product.*

Example 3.6. assume G semisimple, compact

- B is negative definite on $\mathfrak{g}$, $\mathrm{Ad}(G)$-invariant

– $-B_{|\mathfrak{p}}$ is positive definite, $\mathrm{Ad}(K)$-incvariant

say in this case: (G, K) is of compact type □

Example 3.7. assume G semisimple

- assume B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$

- then G is non-compact (necessarily)

– $B_{|\mathfrak{p}}$ is positive definite, $\mathrm{Ad}(G)$-invariant

say in this case: (G, K) is of non-compact type □

Example 3.8. Remaining case: $B = 0$

- $\mathfrak{g}$ is abelian

- G not semisimple

- say (G, K) is of Euclidean type □

Remark 3.9. (up to coverings) every Riemannian symmetric pair is a product of a non-compact, a compact, and an euclidean type □

consider Riemannian symmetric pair (G, K) (with involution Θ)

$M := G/K$ manifold

- G acts transitively on M from the left

- $G \to M$ is G-equivariant K-principal bundle

– $T_eG = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ decomposition

– $\mathfrak{k}$ - vertical

– $\mathfrak{p}$ - horizontal

– defines G-invariant principal bundle connection T^hG on $G \to M$ by equivariant extension

$$T^h_g G := L_{g,*}\mathfrak{p}$$

- check: this is right K-invariant:

– use identity: $R_{k,*}L_{g,*}X = L_{gk,*}\mathrm{Ad}(k^{-1}(X))$

– suggestive notation: $gXk = gkk^{-1}Xk = gk\mathrm{Ad}(k^{-1})(X)$

– define isomorphism of vector bundles over M: $G \times_K \mathfrak{p} \cong TM$

– $[g, X] \mapsto T(\pi)(L_{g,*}(X))$

— for well-definedness

— $[gk, X] \mapsto L_{gk,*}(X) = T(\pi)(L_{g,*}(\mathrm{Ad}(k)(X)))$

— $[g, \mathrm{Ad}(k)(X)] \mapsto T(\pi)(L_{g,*}(\mathrm{Ad}(k)(X)))$

any $\mathrm{Ad}(K)$-invariant metric $\langle -, - \rangle$ on $\mathfrak{p}$ defines G-invariant Riemannian metric g^M on M

- transitive G-action implies completeness of (M, g^M)

Lemma 3.10. *(M, g^M) is a Riemannian symmetric space.*

Proof. consider gK in M

- must find involutive isometry with isolated fixed point gK

- $\Theta_g := g\Theta g^{-1}$ is in $I(M, g^M)$

– fixes precisely point gK

– acts as -1 on T_{gM}

□

want to understand the Riemannian geometry of M in group-theoretic terms

the group G and Θ act by principal bundle automorphisms on $G \to M$

- nontrivially also on the base, i.e. not fibrewise

- for g in G: by left tranlation

- for $\Theta : g \mapsto \Theta(g)$

– note $\Theta(gk) = \Theta(g)\Theta(k) = \Theta(g)k$

- and $gK \mapsto \Theta(gK) = \Theta(g)K$

the connection T^hG on $G \to M$ is G- and Θ-invariant

- for G: by construction

- for Θ: $T\Theta(L_{g,*}X) = -L_{\Theta(g)}(X)$

Lemma 3.11. *1. The (principal bundle) curvature (at $e \in G$) of the connection is given by $\Omega(X, Y) = -[X, Y]$ for $X, Y \in \mathfrak{p}$.*

2. For X in $\mathfrak{p}$, k in K the curve $e^{tX}k$ is horizontal.

Proof. by definition $\Omega(X, Y)$ is the negative vertical part of $[X^h, Y^h](e)$

- here X^h, Y^h horizontal fields extending X, Y

- but for X in $\mathfrak{p}$ the corresponding left invariant field $g \mapsto L_{g,*}X$ is horizontal by definition

– commutator of left invariant fields is commutator in Lie algbera

– since $[X, Y] \in \mathfrak{k}$ this is already vertical

— conclude $\Omega(X,Y) = -[X,Y]$

$\partial_t \exp(tX)k = R_{k,*}L_{e^{tX},*}(X) = L_{e^{tX}k,*}(\mathrm{Ad}(k^{-1})(X)$ is horizontal

□

the principal bundle connection induces a vector bundle connection ∇ on $TM = G \times_K \mathfrak{p}$

- since $\mathrm{Ad}(K)$ acts isometrically on $\mathfrak{p}$ this connection is automatically metric

- this connection is G and Θ-invariant

Lemma 3.12. *1. We have $T^\nabla = 0$, i.e., ∇ is the Levi-Civita connection of (M, g^M).*

2. For X in $\mathfrak{p}$ the curve $\exp(tX)K$ is a geodesic.

3. Every G-invariant tensor on M is parallel.

Proof. show that torsion $T^\nabla = 0$ at $e \in G$

- then $T^\nabla = 0$ by G-invariance

T^∇ is Θ-invariant

- Θ acts by -1 on $\mathfrak{p} = T_{eK}$

- $T^\nabla(\Theta X, \Theta Y) = \Theta T(X,Y)$

implies $(1)^2 = -1$ or $T(X,Y) = 0$

curve $\partial_t e^{tX}K = L_{e^{tX},*}X$ in TM is parallel

- since it is image of horizontal curve $[e^{tX}, X]$ in G

Assertion 3: exercise

□

Corollary 3.13. *The Riemannian curvature at eK is given by $R^\nabla(X,Y) = -\mathrm{ad}([X,Y])$ in $\mathrm{End}(\mathfrak{p})$.*

for the next we assume that (G,K) is of compact or non-compact type

- $\langle -,- \rangle = cB_{|\mathfrak{p}}$

– $c >$ for non-compact type

– $c < 0$ for compact type

— we need that $\langle -,- \rangle$ is the restriction to $\mathfrak{p}$ of an $\mathrm{ad}(\mathfrak{g})$-invariant scalar product on $\mathfrak{g}$

Corollary 3.14. *The sectional curvature is given by* $K^{\nabla}(X,Y) = cB([X,Y],[X,Y])$.

Proof. R^{∇} by definition

- sectional curvature

- insert orthonormal X, Y:

– $K^{\nabla}(X,Y) = cB(-\mathrm{ad}([X,Y])Y, X)) = cB([Y,[X,Y]],X)) = -cB([X,Y],[Y,X]) = cB([X,Y],[X,Y])$

□

note that $[X,Y] \in \mathfrak{k}$ and $B_{|\mathfrak{k}}$ is negative definite

- hence $B([X,Y],[X,Y]) \leq 0$

consider maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$

Definition 3.15. $\dim(\mathfrak{a})$ *is called the rank of the symmetric space*

sectional curvature K^{∇} vanishes along $\mathfrak{a}$

- $\exp(\mathfrak{a})K$ is a flat submanifold in M

- the rank is the dimension of a maximal flat submanifold

Corollary 3.16. *If* (G,K) *is of compact (non-compact) type, then* (M,G) *has non-negative (non-positive) sectional curvature. If* $\mathrm{rk}(M) = 1$*, then it has positive (negative) sectional curvature.*

Proof. - if $\mathrm{rk}(M) = 1$, then $[X,Y] \neq 0$ for any two independent X, Y in $\mathfrak{p}$

- $B([X,Y],[X,Y]) \leq 0$

- $\pm cB([X,Y],[X,Y]) \leq 0$ depending on sign of c □

if (G,K) is a product of compact and non-compact factors, then the corresponding sectional curvature has no definite sign

3.2 Example S^n and $SO(n+1)$

we consider the group $G = SO(n+1)$

define Θ as conjugation by $\Theta := \text{diag}(1, -1, \dots, -1)$

- in blocks of size $(1, n)$
- $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$
- thus $K \subseteq \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$
- is compact

from orthogonality: $\det(D) = \pm 1$, i.e. $A = \det(D)$

have two choices for K: $SO(n)$, $O(n)$ (identified with D)

$SO(n+1)$ acts transitively on $S^n \subseteq \mathbb{R}^{n+1}$

- $SO(n)$ is precisely stabilizer of e_1 in $\mathbb{R}^{n+1}$
- $SO(n+1)/SO(n) = S^n$
- $SO(n) \cong SO(n+1)_N$ acts transitively on planes in north pole N
- sectional curvature of induced metric is constant

$(SO(n+1), \Theta)$ presents round sphere as symmetric space

- $\text{rk}(S^n) = 1$

Exercise: determine the value of the sectional curvature precisely

If we take $K = O(n)$, then get $\mathbb{RP}^n$

3.3 H^n and $SO(1, n)$

consider bilinear form on $\mathbb{R}^{n+1}$ represented by $B := \text{diag}(1, -1, \dots, -1)$

- $O(1, n)$ group of automorphisms
- $SO(1, n) \subseteq O(1, n)$ - singled out $\det(g) = 1$

$SO(1,n)$ has again two components

- $C := \{x \in \mathbb{R}^n \mid B(x,x) = 0\}$ light cone
- $C^* := C \setminus \{0\}$ has two componets
  – distinguished by sign of x_1
  – $SO(1,n)$ acts on C^*
- $SO(1,n)^+ \subseteq SO(1,n)$ subgroup which fixes the componets setwise

Exercise: Show that that $SO(1,n)$ contains elements which interchanges the components.

define Θ as conjugation by $\Theta := \mathrm{diag}(1,-1,\dots,-1)$

- in blocks of size $(1,n)$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

thus $K \subseteq \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

- conclude $D \in O(n)$
- from orthogonality: $\det(D) = \pm 1$, i.e. $A = \det(D)$

have again two choices for K: $SO(n)$, $O(n)$ (identified with D)

Exercise: Show that that $SO(n) = SO(1,n)^{+,\Theta}$.

consider $H^n := SO(1,n)^+ e_1$ (hyperboloid: $\{x \in \mathbb{R}^{n+1} \mid B(x,x) = 1 \,\&\, x_1 > 0\}$)

- stabilizer of e_1 is precisely $SO(n)$
- $H^n = SO(1,n)^+/SO(n)$

- projection $H^n \to \{0\} \times \mathbb{R}^n$ (last n coordinates) is a diffeomorphism
- $SO(n)$ acts transitively on planes at e_1
- sectional curvature of H^n is constant
- $(SO(1,n)^+, \Theta)$ defines presents hyperbolic space as symmetric space

Exercise: determine the value of the sectional curvature precisely

- have $\mathrm{rk}(H^n) = 1$

3.4 $\mathbb{CP}^n$ and $U(n+1)$

consider group $U(n+1)$

- define Θ as conjugation by $\Theta := \mathrm{diag}(1, -1, \ldots, -1)$
- in blocks of size $(1, n)$ (complex matrices)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

- thus $K \subseteq \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

- $A \in U(1)$, $D \in U(n)$
- $K = U(1) \times U(n)$ is compact

$U(n+1)$ acts transitively on $\mathbb{CP}^n$ (lines in $\mathbb{C}^{n+1}$)

- stabilizer of $\mathbb{C}e_1$ is precisely $U(1) \times U(n)$
- $U(n+1)/U(1) \times U(n) \cong \mathbb{CP}^n$

get Riemannian metric on $\mathbb{CP}^n$

- $(U(n+1), \Theta)$ presents $\mathbb{CP}^n$ as Riemannian symmetric space

in the following want to study this metric in detail

- K acts transitively on complex hyperplanes of $T_{\mathbb{C}e_1}\mathbb{CP}^n$, but not on all real ones

– can not conclude that sectional curvature is constant

- but know: sectional curvature is non-negative (since $U(n+1)$ is compact)

identify $\mathfrak{p}$ with $\mathbb{C}^n$

embed as $X \mapsto \begin{pmatrix} 0 & -\bar{X}^t \\ X & 0 \end{pmatrix}$ in $\mathfrak{u}(n+1)$

- consider family of hyperplanes H_s for s in $[0,1]$

– intersects all $U(1)\times U(n)$-orbits exercise

let $H(s)$ be generated by $E_{12}-E_{21}$ and $s^{1/2}i(E_{12}+E_{21})+(1-s)^{1/2}(E_{31}-E_{13})$

– H_1 is a complex plane

– H_0 is a real plane

$$\begin{aligned}
&[E_{12}-E_{21}, s^{1/2}i(E_{12}+E_{21})+(1-s)^{1/2}(E_{31}-E_{13})] \\
&\quad= s^{1/2}i(E_{11}-E_{22})+(1-s)^{1/2}E_{23}-s^{1/2}i(-E_{11}+E_{22})-(1-s)^{1/2}E_{32} \\
&\quad= 2s^{1/2}(E_{11}-E_{22})+(1-s)^{1/2}(E_{23}-E_{32})
\end{aligned}$$

- calculate with scalar product on $\mathfrak{u}(n+1)$ given by $\mathrm{tr}A^*A$

– this is $U(n+1)$-invariant

– proportional to Killing form, but easier to calculate

– A^*A:

$$\begin{aligned}
&(2is^{1/2}(E_{11}-E_{22})+(1-s)^{1/2}(E_{23}-E_{32}))^*(2is^{1/2}(E_{11}-E_{22})+(1-s)^{1/2}(E_{23}-E_{32})) \\
&\quad= (-2is^{1/2}(E_{11}-E_{22})-(1-s)^{1/2}(E_{23}-E_{32}))(2is^{1/2}(E_{11}-E_{22})+(1-s)^{1/2}(E_{23}-E_{32})) \\
&\quad= 4s(E_{11}+E_{22})+(1-s)(E_{22}+E_{33})+\text{off diagonal}
\end{aligned}$$

– $\mathrm{tr}A^*A$:

– the generators are orthogonal and have norm $\sqrt{2}$ (this is a similar calculation)

— $8s+2(1-s)=6s+2$

— $K(H(0))=1$

— $K(H(1))=4$

conclusion: minimal sectional curvature at real plane is $1/4$ of maximal sectional curvature of complex plane

- $\mathrm{rk}(\mathbb{CP}^n) = 1$

- this is a the scale invariant statement

3.5 G and $G \times G$

L - a compact Lie group

- $G := L \times L$

- Θ = flip: $\Theta(l, l') := (l', l)$

- $K := G^\Theta = L$ (diagonally embedded)

- $L = G/K$,

– projection $G \to L$: $(l, l') \mapsto ll'^{,-1}$

– metric on L is left-invariant metric determined by $\mathrm{Ad}(L)$-invariant scalar product on L

– every Lie group L with the left-invariant metric associated to the Killing form (or any other $\mathrm{Ad}(L)$-invariant metric) is Riemannian symmetric

– $\theta_e = (-)^{-1}$

note: every scalar product on $\mathfrak{l}$ induces left invariant metric

- get symmetric space property only for Ad-invariant metrics

4 Complex manifolds and the Kähler condition

4.1 Complex manifolds

recall from function theory:

- U open in $\mathbb{C}$

- $f : U \to \mathbb{C}$ smooth

Definition 4.1. *f is called holomorphic if $df(z)$ is complex linear.*

equivalently: df commutes with i

- check, that this is equivalent to Cauchy-Riemann equations

– $z = x + iy$

– $i\partial_x = \partial_y$, $i\partial_y = -\partial_x$

– $dxi = -dy\ dy = dxi$

– write $f = u + iv$

$df = \partial_x u dx + \partial_y u dy + i\partial_x v dx + i\partial_y v dy$

- $idf = -\partial_x v dx - \partial_y v dy + i\partial_x u dx + i\partial_y u dy$

- $dfi = \partial_x u dxi + \partial_y u dyi + i\partial_x v dxi + i\partial_y v dyi = -\partial_x u dy + \partial_y u dx - i\partial_x v dy + i\partial_y v dx$

– read off: $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$, $-\partial_x v = \partial_y u$, $\partial_y v = \partial_x u$

– these are the Cauchy-Riemann equations:

U open in $\mathbb{C}^n$

$f : U \to \mathbb{C}^m$ smooth

Definition 4.2. *f is holomorphic if df is complex linear.*

this is equivalent to: the components of f are holomorphic in each variable separately

globalize to manifolds:

M - manifold

- $n = 2m = \dim(M)$

- consider $GL(m, \mathbb{C})$-structure (represented by $I \in \Gamma(\mathrm{End}(TM))$, $I^2 = 1$)

– i.e. (M, I) is almost complex

Definition 4.3. *We say that M is a complex manifold if the almost complex structure is integrable.*

this means:

- we can find at every point m coordinates $z := (z_1, \ldots, z_m)$ in $\mathbb{C}^n$

– such that $Tz(m') \circ I_{m'} = iTz(m')$ in $\mathrm{Hom}(T_{m'}M, \mathbb{C}^n)$ for all m' near m

– this implies: the transition functions $z \mapsto z'(z)$ between two coordinate systems are holomorphic

Example 4.4. basic example: open subsets of $\mathbb{C}^n$ with standard coordinates □

(M, I), (M', I') almost complex manifolds

- $f : M \to M'$ smooth map

Definition 4.5. *We say that f is holomorphic if $Tf(m) \circ I(m) = I'(f(m)) \circ Tf(m)$ for all m in M.*

- can talk about holomorphic functions on complex manifold

- note: if (M, I) is only almost complex, then there might by only very few of them

Example 4.6. this is without proof:

recall $TM \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$ decomposition into ± 1-eigenspaces of $I \otimes \mathrm{id}_{\mathbb{C}}$

- $\mathcal{X}^{0,1}(M)$ and $\mathcal{X}^{1,0}(M)$ - sections of $T^{0,1}M$ and $T^{1,0}M$

Lemma 4.7. *$f : M \to \mathbb{C}$ is holomorphic if and only if $Xf = 0$ for all X in $\mathcal{X}^{0,1}(M)$.*

Theorem 4.8 (Newlander-Nierenberg)**.** *Integrability of I is equivalent to $[\mathcal{X}^{0,1}(M), \mathcal{X}^{0,1}(M)] \subseteq \mathcal{X}^{0,1}(M)$.*

Say that I is maximally non-integrable if for every m in M and every X in T_mM there are Y, Z in $\mathcal{X}^{0,1}(M)$ such that $[Y, Z](m) = X$.

- this is the extreme case

- exists locally

- if I is maximally non-integrable, then all holomorphic functions are constant

– in general: if I is not integrable, then there not enough holomorphic functions to build charts □

$f : (M, I) \to (M', I')$ - almost holomorphic

Proposition 4.9. *If m' is a regular value of f, then the restriction I'' of I to TN turns $N := f^{-1}(m')$ into an almost complex manifold. If (M, I) and (M', I') are complex, then (N, I'') is again complex.*

Proof. for m in N:

$Tf(m) \circ I(m) = I'(f(m)) \circ Tf(m)$ shows that I preserves $\ker(Tf(m)) = T_m N$

- can restrict I to I''

- for the second assertion we use that the implicit function theorem holds in the holomorphic context

Exercise: deduce the statement from the usual implicit function theorem

□

Example 4.10. . quadrics

- $f(z_1, \dots, z_n) = z_1^2 + \cdots + z_n^2 + 1$

– $(0, 0 \dots, 0)$ is only singular point of f

– 1 is only non-regular value

– 0 is regular value

– $f^{-1}(0)$ is a quadric

– make picture for $n = 2$ (real/imaginary part) □

Lemma 4.11. *(M, I) is a compact connected complex manifold, then every holomorphic function on M is constant.*

Proof. - by maximum principle

– $\phi : M \to \mathbb{C}$ holomorphic

– $|\phi|$ must have maximum at m

– ϕ is constant along every holomorphic map $\mathbb{C} \supseteq U \to M$ with $0 \to m$

– use holomorphic coordinates in order to produce many such linear (in coordinates) maps

– conclude that ϕ is constant near m

– use connectedness of M to conclude that ϕ is constant on M

□

Corollary 4.12. *If (M, I) is a compact connected complex manifold, then every holomorphic map $M \to \mathbb{C}^n$ is constant.*

in particular: there is no holomorphic embedding of M into $\mathbb{C}^n$ for any n

- this is in contrast to the real case

Example 4.13. complex torus

$A := \mathbb{C}^n/(\mathbb{Z}^n + i\mathbb{Z}^n)$

- is compact complex manifold (has even group structure)

- has no holomorphic embedding into $\mathbb{C}^n$ □

4.2 The complex projective space

$\mathbb{CP}^n$ - lines in $\mathbb{C}^{n+1}$

- $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ gives line $\mathbb{C}(z_0, \dots, z_n)$

- $(z_0', \dots, z_n')$ gives same line if an only if $(z_0, \dots, z_n) = \lambda(z_0', \dots, z_n')$ for $\lambda \in \mathbb{C}^*$

write $[z_0 : \dots : z_n]$ for equivalence class, i.e., the point in $\mathbb{CP}^n$

- $U_i := \{z_i \neq 0\}$ is open

- $\phi_i : U_i \to \mathbb{C}^n$ chart

– $\phi_i([z_0 : \dots : z_n]) := (\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots \frac{z_n}{z_i})$

– check coordinate transition

– say: $\phi_1 \circ \phi_0^{-1}$

– $(u_1, \dots, u_n) \mapsto [1, u_1, \dots, u_n] \mapsto (\frac{1}{u_1}, \frac{u_2}{u_1}, \dots, \frac{u_n}{u_1})$

– is holomorphic

the charts above determine a complex structure on $\mathbb{CP}^n$

Definition 4.14. *A complex manifold (M, I) is called projective if it admits a holomorphic embedding $(M, I) \to \mathbb{CP}^n$.*

- not every complex manifold is projective

- will see an obstruction later using Kähler class

4.3 The Fubini-Study metric

know $U(n+1)$ acts transitively on $\mathbb{CP}^n$

- $(u,[z]) \mapsto [uz]$ - this is matrix multiplication

– it acts by holomorphic transformations

– $U(1)\times U(n)$ stabilizes $[1,0,\dots,0]$

– want to determine Riemannian metric from symmetric space presentation at this point explicitly

– know: work with form $A \mapsto \mathrm{tr}(A^*A)$ on $\mathfrak{u}(n+1)$

$T_{[1,0,\dots,0]}\mathbb{CP}^n \cong \mathbb{C}^n$ using chart ϕ_0

- is identified with $\mathfrak{p}$ in $\mathfrak{u}(n+1)$ by

$(x_1,\dots,x_n) \mapsto A(x) := \sum_{i=1}^n x_i E_{i,0} - \bar{x}_i E_{0,i}$

$$\begin{aligned}
A(x)^*A(x) &= (\sum_{i=1}^n x_i E_{i,0} - \bar{x}_i E_{0,i})^*(\sum_{i=1}^n x_i E_{i,0} - \bar{x}_i E_{0,i}) \\
&= (\sum_{i=1}^n \bar{x}_i E_{0,i} - x_i E_{i,0})(\sum_{i=1}^n x_i E_{i,0} - \bar{x}_i E_{0,i}) \\
&= \sum_i |x_i|^2 E_{00} + \sum_i |x_i|^2 E_{ii} \\
\mathrm{tr}(A(x)^*A(x)) &= (n+1)\|x\|^2
\end{aligned}$$

- thus metric at $[1,0,\dots,0]$ in chart is (up to scale) standard metric on $\mathbb{C}^n$

- metric is completely determined by value at $T_{[1,0,\dots,0]}\mathbb{CP}^n$ and $U(n+1)$-invariance

Remark 4.15. for curiosity determine formula on all of $\mathbb{C}^n$ (image of the chart):

– $U(n+1)$ acts on S^{2n+1} in $\mathbb{C}^{n+1}$

– stabilizer of $(1,0,\dots,0)$ is $U(n)$

– $U(1)$ still acts from the right

– get $S^{2n+1} \to \mathbb{CP}^n$ - $U(1)$-principal bundle

– is necessarily Riemannian submersion if we equip S^{2n+1} with standard metric (by invariance)

on U_0 have split

– $s_0 : \mathbb{C}^n \to S^{2n+1}$

– $s_0(z_1, \dots, z_n) = \frac{(1,z_1,\dots,z_n)}{\sqrt{1+\|z\|^2}}$

– $ds_0 = \frac{(0,dz_1,\dots,dz_n)}{\sqrt{1+\|z\|^2}} - \frac{1}{(1+\|z\|^2)^{2/3}}(1, z_1, \dots, z_n) \otimes z \cdot dz$

– second component is vertical

- vertical part of first component is $\frac{(1,z_1,\dots,z_n)}{(1+\|z\|^2)^{3/2}}\bar{z} \cdot dz$

- horizontal component is

- $\frac{(0,dz_1,\dots,dz_n)}{\sqrt{1+\|z\|^2}} - \frac{(1,z_1,\dots,z_n)}{(1+\|z\|^2)^{3/2}}\bar{z} \cdot dz$

- metric is

$$\begin{aligned} &\frac{d\bar{z} \otimes dz}{1+\|z\|^2} + \frac{z \cdot d\bar{z} \otimes \bar{z} \cdot dz}{(1+\|z\|^2)^2} - 2\frac{z \cdot d\bar{z} \otimes \bar{z} \cdot dz}{(1+\|z\|^2)^2} \\ = \quad &\frac{d\bar{z} \otimes dz}{1+\|z\|^2} - \frac{z \cdot d\bar{z} \otimes \bar{z} \cdot dz}{(1+\|z\|^2)^2} \end{aligned}$$

□

4.4 Kähler geometry

(M, I) almost complex manifold

- g Riemannian metric

Definition 4.16. *We say that I and g are compatible if $I^* = -I$.*

Example 4.17. on $\mathbb{C}^n$ with standard metric $z \mapsto \Re(\bar{z} \cdot z)$

- multiplication by i satiesfies: $i^* = -i$

- hence the same on $\mathbb{CP}^n$ with Fubini-Study - $I^* = -I$ (the complex structure is antiselfadjoint)□

assume that g and I are compatible

Definition 4.18. *The form $\omega := g(I-,-)$ in $\Omega^2(M)$ is called the Kähler form.*

Definition 4.19. *(M, g, I) is called almost Kähler of $d\omega = 0$. It is Kähler if in addition I is integrable.*

assume (M, g, I) given

- g, I compatible

Lemma 4.20. *(M, g, I) is Kähler if and only if $\nabla I = 0$.*

Proof. only one conclusion feasable at this point:

$\nabla I = 0$ implies $d\omega = 0$:

- since $\nabla g = 0$ have
- $(\nabla_X \omega)(Y, Z) = g((\nabla_X I)Y, Z)$
- the following conditions are equivalent
  – $\nabla I = 0$
  – $\nabla \omega = 0$
  – $0 = (\nabla_X \omega)(Z, Y) = X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)$

$$\begin{aligned} d\omega(X, Y, Z) &= X(\omega(Y, Z)) - Y\omega(X, Z) + Z\omega(X, Y) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &= \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) - \omega(\nabla_Y X, Z) - \omega(X, \nabla_Y Z) + \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y) \\ &\quad -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &= 0 \end{aligned}$$

□

Lemma 4.21. *$\mathbb{CP}^n$ is Kähler.*

Proof. ω is $U(n+1)$ invariant (since I and g are) - ω is parallel

- $d\omega = 0$

- I is integrable

– have seen this independently (also $\nabla I = 0$ since I is invariant)

□

(M', g', I') - Kähler (e.g. $\mathbb{C}^n$ or $\mathbb{CP}^n$)

consider complex submanifold $i : M \subseteq M'$

- get an induced metric $g^M := i^* g^{M'}$

- complex structure integrable

- induced Kähler form $\omega^M = i^* \omega^{M'}$

– $d\omega^M = di^*\omega^{M'} = i^* d\omega^{M'} = 0$

Corollary 4.22. *A complex submanifold of a Kähler manifold is again Kähler (with the induced structure).*

affine or projective manifolds admit Kähler metrics

consider almost Kähler manifold (M, I, g)

- ω- Kähler form

- ω^n is a volume form

- i.e. ω is symplectic

- $GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})^+$ - i.e. complex manifolds are oriented

– M is closed, then $\int_M \omega^n > 0$

- $[\omega] \in H^2_{dR}(M)$

- $[\omega]^n \neq 0$

Corollary 4.23. *A closed almost almost Kähler manifold has a class c in $H^2_{dR}(M)$ such that $c^n \neq 0$.*

Example 4.24. S^{2n} for $n \geq 2$ does not have such a class

- has no almost Kähler metric □

5 De Rham cohomology

5.1 Basic theory

M- manifold

- consider chain complex

$$(\Omega^*(M), d) : \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \ .$$

- the de Rham complex, often denoted shortly by $\Omega^*(M)$

Definition 5.1. *The de Rham cohomology of M is the cohomology of the de Rham complex:*

$$H^k_{\mathrm{dR}}(M) := \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} \ .$$

by definition: $H^k_{\mathrm{dR}}(M)$ is a real vector space

Example 5.2.

$$H^k_{\mathrm{dR}}(*) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & else \end{cases}$$

Example 5.3. $H^0_{\mathrm{dR}}(M) = \mathbb{R}[\pi_0(M)]$

- $\ker(d : \Omega^0(M) \to \Omega^1(M))$ is vector space of locally constant functions □

consider smooth map $f : M \to M'$

- induces $f^* : (\Omega^*(M'), d) \to (\Omega^*(M), d)$

– morphism of chain complexes: $df^* = f^*d$

- get induced map: $f^* : H^k_{\mathrm{dR}}(M') \to H^k_{\mathrm{dR}}(M)$

– composition: $(f \circ g)^* = g^* \circ f^*$

Corollary 5.4. *We have a de Rahm cohomology functor $H^*_{\mathrm{dR}} : \mathbf{Mf}^{\mathrm{op}} \to \mathbf{Vect}^{\mathbb{Z}\mathbf{gr}}_{\mathbb{R}}$.*

consider smooth homotopy: $h : [0, 1] \times M \to M'$ between h_0 and h_1

- define $H : \Omega(M') \to \Omega(M)[-1]$

– degree -1-map

– $H(\omega) := \int_0^1 \iota_{\partial_t} h^*\omega dt$

– make clear that you understand the meaning of this formula

— here are the details

—- $h^*\omega(t, -) = \omega_0(t) + dt \wedge \omega_1(t)$

—- $\omega_i(t) \in \Omega^*(M)$

—- $\int_0^1 \iota_{\partial_t} h^*\omega dt = \int_0^1 \omega_1(t) dt$

- use Cartan formula: $\mathcal{L}_{\partial_t} = \iota_{\partial_t} d + d\iota_{\partial_t}$

$$\begin{aligned} dH\omega &= \int_0^1 d\iota_{\partial_t} h^*\omega dt \\ &= \int_0^1 (\mathcal{L}_{\partial_t} h^*\omega - \iota_{\partial_t} dh^*\omega) dt \\ &= h_1^*\omega - h_0^*\omega - Hd\omega \end{aligned}$$

$$dH + Hd = h_1^* - h_0^*$$

- H is chain homotopy between h_1^* and h_0^*

Corollary 5.5. *The functor H_{dR} is homotopy invariant: In the above situation $h_0^* = h_1^* : H_{\mathrm{dR}}(M') \to H_{\mathrm{dR}}(M)$.*

Example 5.6. $H^*_{\mathrm{dR}^n}(\mathbb{R}) \cong H^*_{\mathrm{dR}}(*)$

- the inclusion $i : * \to \mathbb{R}^n$ is a homotopy equivalence

- inverse $p : \mathbb{R}^n \to *$

– $p \circ i = \mathrm{id}_*$

– $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$

- $h(u, x) := ux$ is homotopy from $i \circ p$ to $\mathrm{id}_{\mathbb{R}^n}$ ☐

M manifold

- U, V open $u : U \to M$, $v : V \to M$ inclusions

- $U \cup V = M$

- $a : U \cap V \to U$, $b : V \cap U \to V$ inclusions

have exact sequence

$$0 \to \Omega(M) \xrightarrow{u^* \oplus v^*} \Omega(U) \oplus \Omega(V) \xrightarrow{a^* - b^*} \Omega(U \cap V) \to 0$$

Exercise: prove exactness

- exactness at $\Omega(M)$ and $\Omega(U) \oplus \Omega(V)$ is clear

– sheaf property of smooth sections of a vector bundle

— exactness as $\Omega(U \cap V)$:

— choose partition of unity (χ, κ) associated to (U, V)

—- assume $\alpha \in \Omega(U \cap V)$

—- consider $\kappa\alpha \oplus -\chi\alpha \in \Omega(U) \oplus \Omega(V)$

—- $a^*\kappa\alpha - b^*(-\chi\alpha) = (\kappa_{|U \cap V} + \chi_{|U \cap V})\alpha = \alpha$

Corollary 5.7 (Mayer-Vietoris sequence)**.** *We have a long exact sequence*

$$H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\partial} H^k_{\mathrm{dR}}(M) \xrightarrow{u^* \oplus v^*} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{a^* - b^*} H^k_{\mathrm{dR}}(U \cap V) \ .$$

Remark 5.8. here is an explicite description of the boundary operator using the partition of unity from above

- $[\omega] \in H^k_{\mathrm{dR}}(U \cap V)$

– $d\chi_{|U \cap V} - d\kappa_{|U \cap V}$ has compact support in $U \cap V$

– define $(d\chi_{|U \cap V} - d\kappa_{|U \cap V}) \wedge \omega$ in $\Omega^{k+1}(M)$ by extension by zero

get

$$\partial[\omega] = [(d\kappa_{|U \cap V} - d\chi_{|U \cap V}) \wedge \omega]$$

□

Example 5.9. decompose S^n into complements S^n_+ and S^n_- of south and north pole

- $S^n_\pm$ are homotopy equivalent to $*$

- $S^n_+ \cap S^n_-$ is homotopy equivalent to S^{n-1}

conclude inductively for $n \geq 1$

$$H^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & else \end{cases}$$

exercise: details

Example 5.10. assume M is oriented, closed

- $\dim(M) = n$

- $\int_M d\omega = 0$ by Stokes

- get $\int_M : H^n_{\mathrm{dR}}(M) \to \mathbb{R}$

- let ω be any volume form

– $\int_M \omega > 0$ shows: $H^n_{\mathrm{dR}}(M) \neq 0$ □

$\wedge : \Omega(M) \otimes \Omega(M) \to \Omega(M)$ is map of complexes

- $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^{|\alpha|}\alpha \wedge d\omega$

- get cup product $\cup : H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{dR}}(M)$

– is natural for maps

Example 5.11. $H^*_{\mathrm{dR}}(S^n) = \mathbb{R}[x]/(x^2)$

- $\deg(x) = n$

have map: $\Omega(M) \otimes \Omega(M') \to \Omega(M \times M')$

Proposition 5.12 (Küenneth formula)**.** *If one of the factors is compact, then induced map* $H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(M') \to H^*_{\mathrm{dR}}(M \times M')$ *is an isomorphism*

Proof. Note: $\Omega(M) \otimes \Omega(M') \to \Omega(M \times M')$ is not an isomorphism

cover M' by finitely open sets such that all multiple intersections are contractible

- choose Riemannian metric and take small convex geodesic balls

- argue by induction by the number of members of such a covering

- then argue by induction

– add one member of the covering in each step

- use Mayer-Vietoris and five Lemma

□

Example 5.13. $T^n = S^1 \times \cdots \times S^1$ - n factors

$H^*_{\mathrm{dR}}(S^1) \cong \mathbb{R}[x]$, x in degree 1 (therefore $x^2 = 0$)

$H_{\mathrm{dR}}(T^n) \cong \mathbb{R}[x_1] \otimes \cdots \otimes \mathbb{R}[x_n] \cong \mathbb{R}[x_1, \ldots, x_n]$ (this is $\Lambda^*\mathbb{R}^n$) □

5.2 Cohomology of quotients

Γ- finite group

- $\mathbb{R}[\Gamma]$ - group ring

- generated over $\mathbb{R}$ by elements of Γ subject to relation $\gamma \cdot \gamma' = \gamma\gamma'$

– here $\cdot$ - ring multiplication

Lemma 5.14. *We have an equivalence of categories:*

$$\mathbb{R}\text{-vector spaces with } \Gamma\text{-action} \simeq \mathbb{R}[\Gamma]\text{-modules}$$

Proof. - every action of Γ extends uniquely to an $\mathbb{R}[\Gamma]$-module structure

- as $\Gamma \subseteq \mathbb{R}[\Gamma]^\times$ - every $\mathbb{R}[\Gamma]$-module induces a Γ-action on the underlying $\mathbb{R}$-vector space □

Example 5.15. $\mathbb{R}$ has $\mathbb{R}[\Gamma]$-module structure corresponding to trivial Γ-action □

have functor $V \mapsto V^\Gamma$

- in the langue of $\mathbb{R}[\Gamma]$-modules: $V^\Gamma := \mathrm{Hom}_{\mathbb{R}[\Gamma]}(\mathbb{R}, V)$

Lemma 5.16. *The functor $V \mapsto V^\Gamma$ from real vector spaces with Γ-action to real vector spaces is exact.*

Proof. $P := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma$ in $\mathbb{R}[\Gamma]$

- is projection onto submodule V^Γ

- can decompose any exact sequence into a sum of images of P and $1 - P$

- these are exact too ◻

general: an exact functor like $(-)^\Gamma$ descends through cohomology

$\tilde{M}$ - with free action of Γ

- $M := \tilde{M}/\Gamma$

- $\pi : \tilde{M} \to M$

Lemma 5.17. $\pi^* : H_{\mathrm{dR}}(M) \to H_{\mathrm{dR}}(\tilde{M})^\Gamma$ *is an isomorphism.*

Proof. - $p^* : \Omega(M) \to \Omega(\tilde{M})^\Gamma$ is isomorphism

- hence $H^*_{\mathrm{dR}}(M) = H^*(\Omega(M)) \overset{p^*}{\cong} H^*(\Omega(\tilde{M})^\Gamma) \cong H^*(\Omega(\tilde{M}))^\Gamma = H^*_{\mathrm{dR}}(\tilde{M})^\Gamma$ ◻

Example 5.18. antipodal map acts on $H^n_{\mathrm{dR}}(S^n)$ by $(-1)^{n+1}$

- $H^*_{\mathrm{dR}}(\mathbb{RP}^{2n}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & else \end{cases}$

- $H^*_{\mathrm{dR}}(\mathbb{RP}^{2n+1}) \cong \begin{cases} \mathbb{R} & k = 0, 2n+1 \\ 0 & else \end{cases}$ ◻

G compact Lie group

- G_0 connected component of identity

$1 \to G_0 \to G \to \pi_0(G) \to 0$

assume G acts on M

- G_0 acts trivially on $H^*_{\mathrm{dR}}(M)$ by homotopy invariance

- $\pi_0(G)$ acts on $H^*_{\mathrm{dR}}(M)$

Lemma 5.19. *We have an isomorphism*

$$H^*(\Omega(M)^G) \cong H^*_{dR}(M)^{\pi_0(G)} \ .$$

Proof. 1.) show $H^*(\Omega(M)^{G_0}) \cong H^*_{dR}(M)$

2.) then apply $(-)^{\pi_0(G)}$ and conclude $H^*(\Omega(M)^G) \cong H^*_{dR}(M)^{\pi_0(G)}$

remains to show 1.)

define $P : \Omega(M) \to \Omega(M)$

$P\omega := \int_G g^*\omega dg$

-normalize dg such that $\int_G dg = 1$

- is chain map: $dP(\omega) := d\int_G g^*\omega dg = \int_G dg^*\omega dg = \int_G g^* d\omega dg = P(d\omega)$
- is projection into $\Omega(M)^G$

- cover G_0 by finitely many contractible sets $U_1, \ldots, U_r$

– can assume that all contain e

- choose partition of unity $\chi_1, \ldots, \chi_n$
- use homotopy formula applied to contraction of U_i to find
- $H(g)_i : \Omega(M) \to \Omega(M)[-1]$ for g in U_i (continuous in g)
- $dH(g)_i\omega - H(g)_i d\omega = g^*\omega - \omega$
- define $H := \sum_{i=1}^r \int_{G_0} \chi_i(g) H(g)_i dg$
- $dH\omega - Hd\omega = \sum_{i=1}^r \int_{G_0} \chi_i(g)(g^*\omega - \omega)dg = P(\omega) - \omega$

P is chain homotopic to identity

- inclusion $\Omega(M)^{G_0} \to \Omega(M)$ is chain homotopy equivalence

□

consider Riemann symmetric pair (G, K) of compact type

- assume that G is connected
- set $M := G/K$

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

Proposition 5.20. *We have an isomorphism of rings* $H^*_{\mathrm{dR}}(G/K) \cong (\Lambda^*\mathfrak{p}^*)^K$.

Proof. $H^*_{\mathrm{dR}}(G/K) \cong H^*(\Omega(G/K)^G)$ by Lemma 5.19

- every G-invariant form is determined by its value at e, this is an element in $\Lambda^*\mathfrak{p}^*$

- K still acts, hence G-invariance implies that value is in $(\Lambda^*\mathfrak{p}^*)^K$

- vice versa, every element in $(\Lambda^*\mathfrak{p}^*)^K$ extends uniquely to G-invariant form

– conclude $\Omega(G/K)^G \cong (\Lambda^*\mathfrak{p}^*)^K$ - is isomorphism of rings

– every G-invariant tensor is parallel

– every G-invariant form is parallel

— hence every G-invariant form is closed

— hence differential on $\Omega(G/K)^G$ is trivial

conclude

$$H^*(\Omega(G/K)^G) \cong (\Lambda^*\mathfrak{p}^*)^K$$

□

Example 5.21. want to calculate $H_{\mathrm{dR}}(S^n)$ using this method

$S^n \cong SO(n+1)/SO(n)$

- $\mathfrak{p} \cong \mathbb{R}^n$ with standard action of $SO(n)$

- $(\Lambda^*\mathbb{R}^{n,*})^{SO(n)} \cong \mathbb{R}[x]/(x^2)$

– $\deg(x) = n$

how to see this:

- $SO(n)$ acts degree-preserving

- can calculate invariants degree-wise

- $I^k_n := (\Lambda^k\mathbb{R}^{n,*})^{SO(n)}$

induction by n

$n = 0, 1$

- $I_0^* \cong \mathbb{R}[x]/(x^1)$, $\deg(x) = 0$

- $I_1^* \cong \Lambda^0\mathbb{R}^* \oplus \Lambda^1\mathbb{R}^* \cong \mathbb{R}[x]/(x^2)$, $\deg(x) = 1$

- for step $n-1 \to n$ (with $n \geq 2$) : have $SO(n-1)$-equivariant split exact sequence

-
$$0 \to \Lambda^{k-1}\mathbb{R}^{n-1,*} \xrightarrow{e^n \wedge} \Lambda^k\mathbb{R}^{n,*} \xrightarrow{res} \Lambda^k\mathbb{R}^{n-1,*} \to 0$$

- induces
$$0 \to I_{n-1}^{k-1} \to (\Lambda^k\mathbb{R}^{n,*})^{SO(n-1)} \to I_{n-1}^k \to 0$$

- have $I_n^k \subseteq (\Lambda^k\mathbb{R}^{n,*})^{SO(n-1)}$

have $I_n^0 = (\Lambda^0\mathbb{R}^{n,*})^{SO(n)} \cong \mathbb{R}$

- by induction I_{n-1}^{n-1} is generated by $e^1 \wedge \cdots \wedge e^{n-1}$

- the image of $e^n \wedge I_{n-1}^{n-1} \to (\Lambda^n\mathbb{R}^n)^{SO(n-1)}$ is generated by $e^1 \wedge \cdots \wedge e^n$ is $SO(n)$ invariant

– contributes to I_n^n

- $I_{n-1}^n = 0$

– together $I_n^n \cong \mathbb{R}$

- show $I_n^k = 0$ for $k = 1, \ldots, n-1$

– $k = 1$

— $(\Lambda^1\mathbb{R}^n)^{SO(n)} \cong 0$

- $k = 2, \ldots, n-2$:

– $I_{n-1}^{k-1} = 0$ and $I_{n-1}^k = 0$ by induction assumption

— conclude $I_n^k = 0$

remains $k = n-1$: $I_n^{n-1} = (\Lambda^{n-1}\mathbb{R}^{n,*})^{SO(n)} \cong (\Lambda^1\mathbb{R}^n)^{SO(n)} \cong 0$

□

Example 5.22. this example shows that compactness of G is relevant:

$H^n = SO(1,n)/SO(n)$

- $(\Lambda^*\mathfrak{p})^{SO(n)} = (\Lambda^*\mathbb{R}^n)^{SO(n)} \cong \mathbb{R}[x]/(x^2)$ -

- x in degree n

- but H^n is contractible

– $H^n_{\mathrm{dR}}(H^n) \cong 0$

– but $(\Lambda^n\mathbb{R}^n)^{SO(n)} \cong \mathbb{R}$

□

Example 5.23. want to calculate $H_{\mathrm{dR}}(\mathbb{CP}^n)$

claim: $H_{\mathrm{dR}}(\mathbb{CP}^n) \cong \mathbb{R}[x]/(x^{n+1})$ with $\deg(x) = 2$

- $\mathbb{CP}^n \cong U(n+1)/U(1) \times U(n)$

- $\mathfrak{p} \cong \mathbb{C}^n$ with standard action of $U(n)$ and $U(1)$

- $(\Lambda^*\mathbb{C}^n)^{U(n)\times U(1)}$ - note that we consider $\mathbb{C}^n$ as real vector space

- argue by induction

– $I_n^* := (\Lambda^*\mathbb{C}^n)^{U(n)\times U(1)}$

– use $\mathbb{C}^n \cong \mathbb{C}^{n-1} \oplus \mathbb{C}$

— this is $U(n-1) \times U(1)$-equivariant

- have inclusion $I_n^* \hookrightarrow I_{n-1}^* \otimes I_1^*$

- now $I_1^* \cong \mathbb{R} \oplus \mathbb{R}[2]$

– use $U(1) = SO(2)$, $\mathbb{C} \cong \mathbb{R}^2$

– show: $I_n^{2n} \cong I_{n-1}^{2n-2} \otimes I_1^2$ by showing that the elements of the r.h.s. are $U(n)$-invariant

– have restriction $I_n^* \to I_{n-1}^*$ whose kernel is $I_{n-1}^* \otimes I_1^{\geq 1}$

– show that this is surjective

– conclude above inclusion is surjective in all degrees (details exercise?)

□

(G, K) - symmetric pair

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

- recall: $\Omega(G/K)^G \cong (\Lambda^*\mathfrak{p}^*)^K$

How to construct elements in $(\Lambda^*\mathfrak{p})^K$?

- $R : \Lambda^2\mathfrak{p} \to \mathfrak{k}$, $R(X, Y) := [X, Y]$ is $\mathrm{Ad}(K)$-equivariant

- $R^* : S^*(\mathfrak{k}^*) \to S^*(\Lambda^2\mathfrak{p}^*) \to \Lambda^{\mathrm{ev}}\mathfrak{p}^*$

- restricts to $R^* : S^*(\mathfrak{k}^*)^K \to (\Lambda^{\mathrm{ev}}\mathfrak{p}^*)^K$

Example 5.24. Grassmannian $G(k, n, \mathbb{C})$:

manifold of k-dimensional subspaces of $\mathbb{C}^n$

- $G := U(n)$ acts transitively on $G(k, n, \mathbb{C})$

- stabilizer of $\mathbb{C}^k$: $K := U(k) \times U(n - k)$

- $G(k, n, \mathbb{C}) \cong U(n)/(U(k) \times U(n - k))$ as homogeneous space

is symmetric:

-use involution given by conjugation by $\mathrm{diag}(\underbrace{1, \dots, 1}_{k\times}, \underbrace{-1, \dots, -1}_{n-k\times})$

- block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\mathfrak{p} = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad B \in \mathrm{Mat}(k, n - k, \mathbb{C})$$

- adjoint action of $U(k) \times U(n - k)$ is $(u, v)B = uBv^{-1}$

construct elements of $S^*(\mathfrak{k})^K$

- consider $\mathfrak{u}(k)$

- $\mathfrak{u}(k) \ni X \mapsto \det(1 + tX) := 1 + te_1 + t^2e_2 + \cdots + t^ke_k$

– e_i are homogeneous polynomials on $\mathfrak{u}(k)$

– $\deg(e_i) = i$

– e_i is $\mathrm{Ad}(U(k))$ - invariant

— $e_i \in S^i(\mathfrak{u}(k)^*)^{U(k)}$

special cases:

– $e_1(X) = \mathrm{Tr}(X)$

– $e_k(X) = \det(X)$

– $S^*(\mathfrak{u}(k)^* \oplus \mathfrak{u}(n-k)^*) \cong S^*(\mathfrak{u}(k)^*) \otimes S(\mathfrak{u}(n-k)^*)$ contains

— $a_i := e_i \otimes 1$, $i = 1, \ldots, k$

— $b_j := 1 \otimes e_j$, $j = 1, \ldots, n-k$

— $c_i := R^* a_i$ in $i = 1, \ldots, k$,

— $\deg(c_i) = 2i$

— $d_j := R^* b_j$, $j = 1, \ldots, n-k$

— $\deg(d_j) = 2j$

get homomorphism of graded rings $\mathbb{R}[c_1, \ldots, c_k, d_1, \ldots, d_{n-k}] \to (\Lambda^* \mathfrak{p}^*)^K$

Proposition 5.25. *This map induces an isomorphism*

$$\frac{\mathbb{R}[c_1, \ldots, c_k, d_1, \ldots, d_{n-k}]}{(\sum_{i=0}^{l} c_i d_{i-l} = 0 \mid l = 1, \ldots, n)} \to (\Lambda^* \mathfrak{p}^*)^K \ .$$

- set $c_0 = 1$, $d_0 = 1$ and $c_i = 0$ for $i > k$ and $d_j = 0$ for $j > n-k$

proof and determination of relations goes beyond this course

- can calculate cohomology ring using algebraic topology (Serre spectral sequence)

- deduce proposition and relations from this

Corollary 5.26. *We have an isomorphism*

$$\frac{\mathbb{R}[c_1, \ldots, c_k, d_1, \ldots, d_{n-k}]}{(\sum_{i=0}^{l} c_i d_{i-l} = 0 \mid l = 1, \ldots, n)} \to H_{\mathrm{dR}}(G(k, n, \mathbb{C})) \ .$$

check case $k = 1$ (projective space)

generators: $c_1, d_1, \dots, d_{n-1}$

- relations: $c_1 + d_1 = 0$, $c_1 d_1 + d_2 = 0$, $\quad \dots \quad c_1 d_{n-2} + d_{n-1} = 0$

– can eliminate $c_1, d_2, \dots, d_{n-1}$

– $d_2 = d_1^2, \quad \dots \quad d_{n-1} = d_1^{n-1}$, $0 = d_1^n$

– $\mathbb{R}[d_1[/(d_1^n) \cong H^*_{\mathrm{dR}}(\mathbb{CP}^{n-1})$

□

in the next section we generalize this method

5.3 Chern-Weil theory - characteristic classes

G - Lie group

- $\pi : P \to M$ - G-principal bundle

Definition 5.27. *A form α in $\Omega(P)$ us called horizontal, if $\iota_X \alpha = 0$ for every vertical X in TP. It is called G-invariant, if $R_g^* \alpha = \alpha$ for all g in G*

$\Omega(P)^G_{\mathrm{hor}} \subseteq \Omega(P)$ - subspace of horizontal G-invariant forms

$\pi^* : \Omega^*(M) \to \Omega^*(P)$

Lemma 5.28. *π^* induces an isomorphism $\pi^* : \Omega^*(M) \to \Omega^*(P)^G_{\mathrm{hor}}$*

Proof. $\omega \in \Omega^*(M)$

- $\pi \circ R_g = \pi$ implies $R_g^* \pi^* \omega = \pi^* \omega$

– conclude $\pi^* \omega \in \Omega(P)^G$

X vertical

- $d\pi(X) = 0$

– $\iota_X \pi^* \omega = 0$

– conclude $\pi^* \omega \in \Omega(P)_{\mathrm{hor}}$

π is surjective submersion

- π^* is injective

- assume: $\alpha \in \Omega(P)^G_{\mathrm{hor}}$
- $s : U \to P$ local section
- $s^*\alpha$
- claim: $s^*\alpha$ is independent of the choice of section

– s' another section

– $s'(u) = s(u)g(u)$ for unique $g : U \to G$

—u in U, X in T_uM

$$
\begin{aligned}
s'^{,*}\alpha(u)(X) &= \alpha(s(u)g(u))(dR_{g(u)}(ds(u)(X))) + \alpha(s(u)g(u))(X^\sharp) \\
&= (R^*_{g(u)}\alpha)(s)(ds(u)(X)) \\
&= s^*\alpha(u)(X)
\end{aligned}
$$

where $X^\sharp = dg(u)(X)^\sharp(s(u))$ is vertical

- get globally defined ω in $\Omega(M)$ with $\omega_{|U} = s^*\alpha$
- $\pi^*\omega = \alpha$

□

choose connection ω in $\Omega^1(P, \mathfrak{g})^G$

- $\Omega := d\omega + [\omega, \omega]$ - curvature
- recall: $\Omega \in \Omega^2(P, \mathfrak{g})^G_{\mathrm{hor}}$

consider p in $S^*(\mathfrak{g}^*)^G$

- form $p(\Omega)$ in $\Omega^{\mathrm{ev}}(P)^G_{\mathrm{hor}}$
- interpret $\Omega : \Lambda^2 TP \to \mathfrak{g}$
- interpret $p : S^*(\mathfrak{g})^G \to \mathbb{R}$

- then $p(\Omega) := p \circ S^*(\Omega)) : S^*(\Lambda^2 TP) \to \mathbb{R}$

– or equivalently: $p(\Omega) \in S^*(\Omega^2(P)) \subseteq \Omega^{\mathrm{ev}}(P)$

– actually: $p(\Omega) \in \Omega^{\mathrm{ev}}(P)^G_{\mathrm{hor}}$

Lemma 5.29. *We have* $dp(\Omega) = 0$

Proof. note: $dp(X)([Y, X]) = 0$ by $\mathrm{Ad}(G)$-invariance

– differentiate identity $p(gXg^{-1}) = p(X)$ w.r.t g

- $\Omega = d\omega + [\omega, \omega]$

- $[\omega, [\omega, \omega]] = 0$ by Jacobi

- $d\Omega = 2[d\omega, \omega] = 2[\Omega, \omega]$

$$\begin{aligned} dp(\Omega) &= 2dp(\Omega)(d\Omega) \\ &= 2dp(\Omega)([\Omega, \omega]) \\ &= 0 \end{aligned}$$

□

let $c_p(\omega) \in \Omega(M)$ denote the closed form on M such that $\pi^* c_p(\omega) = p(\Omega)$

$f : M' \to M$

-

$$\begin{array}{ccc} P' & \xrightarrow{F} & P \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

- pull-back

- $F^*\omega := \omega'$ connection

- have $f^* c_p(\omega) = c_p(\omega')$

Lemma 5.30. *The class* $[c_p(\omega)]$ *in* $H_{\mathrm{dR}}(M)$ *does not depend on* ω*.*

Proof. ω'- second choice

$\tilde{P} := \mathrm{pr}_M^* P \to [0,1] \times M$

- $P_i = \tilde{P}_{|\{i\}\times M}$
- $P_i \cong P$ canonically

- arrange $\tilde{\omega}$ on $\tilde{P}$ such that $\tilde{\omega}_{|P_0} = \omega$ and $\tilde{\omega}_{|P_1} = \omega'$

– e.g $\tilde{\omega} = t\omega' + (1-t)\omega$

– $c_p(\tilde{\omega}) \in \Omega([0,1] \times M)$

– $c_p(\tilde{\omega})_{\{0\}\times M} = c_p(\omega)$ and $c_p(\tilde{\omega})_{\{1\}\times M} = \omega'$

– $d \int_{[0,1]\times M/M} c_p(\tilde{\omega}) = \omega' - \omega$

□

fix Lie group G

Definition 5.31. *A characteristic class* $\mathbf{c}$ *(of degree k) associates to every manifold M and G-principal bundle $P \to M$ a class $\mathbf{c}(P)$ in $H^k_{\mathrm{dR}}(M)$ such that for every pull-back*

$$\begin{array}{ccc} f^*P & \longrightarrow & P \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

we have $f^*\mathbf{c}(P) = \mathbf{c}(f^*P)$.

characteristic classes from a ring $\mathrm{ChW}(G)$

Remark 5.32. one can show that

$$\mathrm{ChW}(G) \cong H^*(BG; \mathbb{R}) \ .$$

□

let $\mathbf{c}$ be a characteristic class

Lemma 5.33. *If* $\deg(\mathbf{c}) > 0$ *and P is trivial, then* $\mathbf{c}(P) = 0$.

Proof. have pull-back

$$\begin{array}{ccc} P & \longrightarrow & G \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & * \end{array}$$

- $\mathbf{c}(G \to *) = 0$ (for degree-reasons)
- $\mathbf{c}(P) = f^*\mathbf{c}(G \to *) = 0$

□

consider p in $S^*(\mathfrak{g}^*)^G$

Definition 5.34. *We let* $\mathbf{c}_p(P) \in H_{\mathrm{dR}}(M)$ *denote the class of* $c_p(\omega)$*.*

this is the characteristic class $\mathbf{c}_p$ for G-principal bundles associated to p

- if p is homogeneous: $\mathbf{c}_P$ is of degree $2\deg(p)$

Corollary 5.35. *We have a homomorphism* $\mathbf{c} : S^*(\mathfrak{g}^*)^G \to \mathrm{ChW}(G)$ *(of degree 2)*

Example 5.36. $G = U(k)$

- $\det(1 + tX) = 1 + tc_1 + \cdots + t^k c_k$ defines $c_i \in S^i(\mathfrak{u}(k)^*)^{U(k)}$
- these are non-zero
- $\mathbf{c}_{c_i}$ has degree $2i$
- is called the ith Chern class for $U(k)$-bundles

- $U(n)/U(k) = V(k, n, \mathbb{C})$ is Stiefel manifold of k-dimensional subspaces with framed orthocomplement in $\mathbb{C}^n$
- get $U(k)$ principal bundle $U(n) \to V(k, n, \mathbb{C})$
- get classes $\mathbf{c}_{c_i} \in H^{2i}_{\mathrm{dR}}(V(k, n, \mathbb{C}))$

– one can show that they generate cohomology □

Example 5.37. $U(k) \times U(n-k)$

- $U(n)/U(k) \times U(n-k) = G(k, n, \mathbb{C})$ is Grassman manifold of k-dimensional subspaces in $\mathbb{C}^n$

- get $U(k) \times U(n-k)$ principal bundle $U(n) \to G(k,n,\mathbb{C})$

- we used the classes $\mathbf{c}_{a_i}$ and $\mathbf{c}_{b_j}$ in the calculation of $H_{\mathrm{dR}}(G(k,n,\mathbb{C}))$ □

5.4 Duality

M - manifold

$\Omega_c(M) \subseteq \Omega(M)$ subspace of compactly supported forms

- d preserves compact support

- get subcomplex $(\Omega_c(M), d)$ of $(\Omega(M), d)$

Definition 5.38. *The cohomology* $H^*_{c,\mathrm{dR}}(M) := H^*(\Omega_c(M), d)$ *is called the compactly supported de Rham cohomology*

- contravariant functorial for proper maps

– $f : M \to M'$ is proper if $f^{-1}(K)$ is compact for every compact K in M'

– $\mathrm{supp}(f^*\omega) = f^{-1}(\mathrm{supp}(\omega))$

— $\mathrm{supp}(\omega)$ compact implies $\mathrm{supp}(f^*\omega)$ is compact

- homotopy invariant for proper homotopies

– $h : [0,1] \times M \to M'$ is proper homotopy if f is proper

inclusion $\Omega_c(M) \to \Omega(M)$ induces

$$\iota : H^*_{c,\mathrm{dR}}(M) \to H^*_{\mathrm{dR}}(M)$$

– is ring homomorphism

– is an isomorphism if M is compact

wedge product : $\wedge : \Omega_c(M) \otimes \Omega_c(M) \to \Omega_c(M)$

- induces cup product

$$\cup : H^*_{c,\mathrm{dR}}(M) \otimes H^*_{c,\mathrm{dR}}(M) \to H^*_{c,\mathrm{dR}}(M)$$

(right) module structure $\Omega_c(M) \otimes \Omega(M) \to \Omega_c(M)$

- induces module structure

$$\cup : H^*_{c,\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(M) \to H^*_{c,\mathrm{dR}}(M)$$

new feature:

- $(\Omega_c(M), d)$ and therefore $H^*_{c,\mathrm{dR}}(-)$ are covariantly functorial for open embedding:

– extension by zero

– notation $i_!$

$M = U \cup V$ open decomposition

Lemma 5.39. *The complex*

$$0 \to \Omega_c(U \cap V) \xrightarrow{a_! \oplus b_!} \Omega_c(U) \oplus \Omega_c(V) \xrightarrow{u_! - v_!} \Omega_c(M) \to 0$$

is exact.

Proof. use partition of unity $\chi \in C_c(U)$, $\kappa = 1 - \chi \in C_c(V)$

check exactness:

$\Omega_c(U \cap V)$: is clear

$\Omega_c(U) \oplus \Omega_c(V)$:

- (α, β) in $\Omega_c(U) \oplus \Omega_c(V)$

– assume $u_!\alpha - v_!\beta = 0$

– implies $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta) \subseteq U \cap V$

– $(\alpha, \beta) = (a_!\alpha, b_!\alpha)$

$\Omega_c(M)$:

- consider γ in $\Omega_c(M)$

- $(u_! - v_!)(\chi\gamma, -\kappa\gamma) = \gamma$

□

Corollary 5.40. *We have a long exact Mayer-Vietoris sequence*

$$H^{k-1}_{c,\mathrm{dR}}(M) \stackrel{\partial}{\to} H^k_{c,\mathrm{dR}}(U\cap V) \to H^k_{c,\mathrm{dR}}(U)\oplus H^k_{c,\mathrm{dR}}(V) \to H^k_{c,\mathrm{dR}}(M)\ .$$

formula for ∂:

- $[\gamma]$ in $H^{k-1}_{c,\mathrm{dR}}(M)$

- claim: $\partial[\gamma] = [d\chi\wedge\gamma]$

– $d(\chi\gamma, -\kappa\gamma) = (d\chi\wedge\gamma, -d\kappa\gamma)$

– $\mathrm{supp}(d\chi\wedge\gamma)\subseteq \mathrm{supp}(d\chi)\cap\mathrm{supp}(\gamma)$

— is closed subset of $\mathrm{supp}(\gamma)$ and hence compact in M

— is contained $U\cap V$

—- hence $\mathrm{supp}(d\chi\wedge\gamma)$ is compact in $U\cap V$

consider $M\times\mathbb{R}$

- integration map

$$P:\int_{M\times\mathbb{R}/M}\Omega_c(\mathbb{R}\times M)\to\Omega_c(M)[-1]$$

- note that differential in $\Omega(M)[n]$ is $(-1)^n d$

Stokes: P is chain map

- must check $dP = Pd$

– decompose $\omega = \omega_0 + dt\wedge\omega_1$

— $-dP(\omega) = -d\int_{\mathbb{R}\times M/M}\omega = -\int_{\mathbb{R}} d\omega_1(t)dt$

— $Pd(\omega) = \int_{\mathbb{R}}\partial_t\omega_0(t) - \int_{\mathbb{R}} d\omega_1(t)dt = -\int_{\mathbb{R}} d\omega_1(t)dt$

Lemma 5.41. *P induces an isomorphism*

$$H_{c,\mathrm{dR}}(\mathbb{R}\times M)\to H_{c,\mathrm{dR}}(M)[-1]\ .$$

Proof. let $\chi \in C^\infty(\mathbb{R})$

- $\chi \equiv 1$ for $t \geq 1$

- $\chi \equiv 0$ for $t < 1$

- $d\chi \in \Omega^1_c(\mathbb{R})$

define $E : \Omega_c(M)[-1] \to \Omega_c(\mathbb{R} \times M)$, $\omega \mapsto d\chi \wedge \mathrm{pr}_M^* \omega$

claim: E is a homotopy inverse

$P(E(\omega)) = \omega$ is clear

- construct chain homotopy $\mathrm{id}_{\Omega_c(\mathbb{R}\times M)} \sim E \circ P$

- $h : \mathbb{R} \times \mathbb{R} \times M \to \mathbb{R} \times M$

- $h(u, t, m) = (u + t, m)$

- define $H : \Omega_c(\mathbb{R} \times M) \to \Omega_c(\mathbb{R} \times M)$

- $H(\omega)(t, m) := \int_{-\infty}^0 \iota_{\partial_u} h^*(\omega)(u, t, m)du - \chi(t)E(\omega)$

– first term term is also $\int_{-\infty}^t \omega_1(u)du$

– second term term is also $\chi(t) \int_{-\infty}^\infty \omega_1(u)du$

– get $H(\omega)(m, t) = 0$ for $|t| >> 0$

- $(dH + Hd)(\omega) = \omega - d\chi \wedge E(P(\omega))$

– H is desired chain homotopy

– alltogether this shows: E is chain homotopy inverse to P

□

Corollary 5.42. $H^k_{c,\mathrm{dR}}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = n \\ 0 & else \end{cases}$

Proof. induction starting with $k = 0$ □

from now on assume: M is oriented

define duality map $D : \Omega_c(M) \to \Omega(M)^*[-n]$

- $\omega \mapsto (\alpha \mapsto \int_M \omega \wedge \alpha)$

check: this is chain map

$$\begin{aligned} D(d\omega)(\alpha) &= \int_M d\omega \wedge \alpha \\ &= \int_M d(\omega \wedge \alpha - (-1)^{\deg(\omega)}\omega \wedge d\alpha) \\ &= -(-1)^{\deg(\omega)} D(\omega)(d\alpha) \\ &= (-1)^n D(\omega)((-1)^{\deg(\alpha)} d\alpha) \\ &= dD(\omega)(\alpha) \end{aligned}$$

- dualization $V \mapsto V^*\mathrm{Hom}(V, \mathbb{R})$ is exact functor on $\mathbb{R}$-vector spaces
- descends to cohomology
- for chain complex $C$of real vector spaces: $H^k(C^*) \cong H^{-k}(C)^*$

– apply to de Rham complex: $H^k(\Omega(M)^*[-n]) \cong H^{n-k}_{\mathrm{dR}}(M)^*$

get induced duality map

- $D : H^k_{c,\mathrm{dR}}(M) \to H^k(\Omega(M)^*[-n]) \cong H^{n-k}_{\mathrm{dR}}(M)^*$

Example 5.43. $D : H^k_{c,\mathrm{dR}}(\mathbb{R}^n) \to H^{n-k}_{\mathrm{dR}}(\mathbb{R}^n)^*$ is an isomorphism □

$i : M' \to M$ open embedding

Lemma 5.44.

$$\begin{array}{ccc} H^*_{c,\mathrm{dR}}(M') & \xrightarrow{i_!} & H^*_{c,\mathrm{dR}}(M) \\ \downarrow D & & \downarrow D \\ H^{n-k}_{\mathrm{dR}}(M') & \xrightarrow{(i^*)^*} & H^{n-k}_{\mathrm{dR}}(M) \end{array}$$

commutes

Proof. $\int_{M'} \alpha \wedge i^*\omega = \int_M i_!\alpha \wedge \omega$ □

M manifold

- $\mathcal{U} = (U_\alpha)_\alpha$ a covering

- $\mathcal{U}$ is called a good covering if all intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}$ are diffeo to $\mathbb{R}^n$

Lemma 5.45. *If M admits a finite good covering, then $D : H^*_{c,\mathrm{dR}}(M) \to H^*_{\mathrm{dR}}(M)^*$ is an isomorphism.*

Proof. induction by the size of covering

start: one set

- this is Example 5.43

induction:

May-Vietoris

- add one set

- $M' \cup U = M$

– induction hypothesis applies to M' and $M' \cap U$

$$\begin{array}{ccccccc} H^{k-1}_{c,\mathrm{dR}}(M) & \xrightarrow{\partial} & H^k_{c,\mathrm{dR}}(U \cap M') & \longrightarrow & H^k_{c,\mathrm{dR}}(U) \oplus H^k_{c,\mathrm{dR}}(M') & \longrightarrow & H^k_{c,\mathrm{dR}}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-k+1}_{\mathrm{dR}}(M)^* & \xrightarrow{\partial^*} & H^{n-k}_{\mathrm{dR}}(U \cap M')^* & \longrightarrow & H^{n-k}_{\mathrm{dR}}(U)^* \oplus H^{n-k}_{\mathrm{dR}}(M')^* & \longrightarrow & H^{n-k}_{\mathrm{dR}}(M)^* \end{array}$$

must check that square involving boundary maps commutes

$a : U \to M$, $b : M' \to M$, $j : U \cap M' \to M$ inclusions

$[\gamma] \in H^{k-1}_{c,\mathrm{dR}}(M)$

- $\omega \in H^{n-k}_{\mathrm{dR}}(U \cap M)$

– choose $\chi \in C_c(U)$ such that $1 - \chi \in C_c(M')$

– then $\partial[\gamma] = [d\chi \wedge \alpha]$

$$D(\partial[\gamma])([\omega]) = \int_{U \cap M'} d\chi \wedge \alpha \wedge \omega$$

– then $\partial[\omega] = [d\chi \wedge \omega]$

$$\begin{aligned} \partial^* D[\gamma](\omega) &= (-1)^{n+k-1} D[\alpha](\partial\omega) \\ &= (-1)^{n+k-1} \int_M \alpha \wedge d\chi \wedge \omega \\ &= \int_M d\chi \wedge \alpha \wedge \phi \\ &= D(\partial[\gamma])([\omega]) \end{aligned}$$

finish argument by Five Lemma

□

Corollary 5.46 (Poincar'e duality)**.** *If M is n-dimensional, compact and oriented, then $D : H_{\mathrm{dR}}(M) \to H_{\mathrm{dR}}(M)^*[-n]$ is an isomorphism.*

Corollary 5.47. *If M is n-dimensional, compact, oriented and connected, then $H^n_{\mathrm{dR}}(M) \cong \mathbb{R}$.*

Example 5.48. $H_{\mathrm{dR}}(S^n) = \mathbb{R}[x]/(x^2)$

- duality: $(p, q) \mapsto \partial_x pq_{|x=0}$

$H_{\mathrm{dR}}(\mathbb{CP}^n) = \mathbb{R}[x]/(x^{n+1})$, $\deg(x) = 2$

- duality: $(p, q) \mapsto (\frac{1}{n!}\partial_x^n pq)_{|x=0}$

$H_{\mathrm{dR}}(T^n) = \mathbb{R}[x_1, \ldots, x_n]$, $\deg(x_i) = 1$

- duality: $(p, q) \mapsto \int_B pq$

– Berezin integral: takes coefficient at $x_1 \ldots x_n$

□

Example 5.49. signature

M compact, connected, oriented, $n = 4m$-dimensional

- $D : H^{2n}_{\mathrm{dR}}(M) \cong H^{2n}_{\mathrm{dR}}(M)^*$ - duality

- $(x, y)_M := D(x)(y) = \int_M x \cup y$

– $(-,-)_M$ is symmetric bilinear form on $H^{2n}_{\mathrm{dR}}(M)$

– this is called the intersection form of M

– it is non-degenerated by Poincaré duality

classification of bilinear forms over $\mathbb{R}$:

$(-,-)_M$ is determined by $b^{\pm}_{2m}$:

- $b^+_{2m} + b^-_{2m} = b_{2m}$ - Betti number

Definition 5.50. $\mathrm{sign}(M) := b^+_{2m} - b^-_{2m}$ *is called the signature of* M

- $\mathrm{sign}(M)$ is oriented homotopy invariant of M

- $\mathrm{sign}(M^{\mathrm{op}}) = -\mathrm{sign}(M)$ (orientation change)

- $\mathrm{sign}(S^{4m}) = 0$

- $\mathrm{sign}(S^{2m} \times S^{2m}) = 1$

- $\mathrm{sign}(T^{4m}) = 0$

- $\mathrm{sign}(\mathbb{CP}^{2n}) = 1$

□

6 Riemannian geometry and de Rham cohomology

6.1 Hodge $*$

M manifold

- $n := \dim(M)$

- g Riemannian metric

- induces metrics $(-,-)$ on $\Lambda^k T^*M$

at a point:

- V - euclidean vector space

– $e_1, \ldots, e_n$ - ONB of V

– $e^1, \dots, e^n$ - dual basis of V^*

– $e^{i_1} \wedge \cdots \wedge e^{i_k}$ for $i_1 < \cdots < i_k$ forms ONB of $\Lambda^k V^*$

assume M is oriented

- metric induces volume form vol in $\Omega^n(M)$

at a point:

$e^1, \dots, e^n$ oriented ONB

- $\mathrm{vol} = e^1 \wedge \cdots \wedge e^n$

have non-degenerate pairing

- $\langle -, - \rangle : \Lambda^k T^*M \otimes \Lambda^{n-k} T^*M \xrightarrow{\wedge} \Lambda^n T^*M \xrightarrow{\mathrm{vol}^{-1}} M \times \mathbb{R}$

at a point:

- $i = i_1 < \dots i_k$

- $j = j_1 \cdots < j_k$

- i' complementary sequence to i

– $(2,4)' = (1,3)$ (if $n = 4$)

- $\sigma(i)$ - sign of permutation which orders concatenation $i \sharp i'$

– $\sigma((1,3)) = -1$

– $\sigma((3,4)) = 1$

$\langle e^i, e^j \rangle = \sigma(i) \delta_{i,j}$

- this shows non-degeneracy

there exists a uniquely determined $* : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ such that

$(\alpha, *\omega) = \langle \alpha, \beta \rangle$

Definition 6.1. $*$ *is called the Hodge* $*$*-operator* at a point:

$*e^i = \sigma(i)e^{i'}$

- check:

– $(e^i, *e^{j'}) = (e^i, e^j) = \delta_{i,j}$

– $\langle e^i, e^{j'} \rangle = \sigma(i, j')\delta_{i,j}$

$** e^i = \sigma(i)\sigma(i')e^i$

- $\sigma(i)\sigma(i') = (-1)^{k(n-k)}$

$n = 4$

- $*e^1 = e^2 \wedge e^2 \wedge e^3$

- $*e^2 = -e^1 \wedge e^3 \wedge e^4$

- $*e^1 \wedge e^2 = e^3 \wedge e^4$

6.2 The Hodge decomposition

M manifold, vol - volume measure

- $E \to M$ vector bundle

- h metric on E

- get pairing on sections

– $\phi \in \Gamma(M, E)$, $\psi \in \Gamma_c(M, E)$

$(\phi, \psi) := \int_M h(m)(\phi(m), \psi(m))\text{vol}(m)$

F second vector bundle with metric

$D : \Gamma(M, E) \to \Gamma(M, F)$ - differential operator

- preserves supports

– restricts to

- $D : \Gamma_c(M, E) \to \Gamma_c(M, F)$

Definition 6.2. *A formal adjoint of D is a differential operator $D^* : \Gamma(M, F) \to \Gamma(M, E)$ such that $(D\phi, \psi) = (\phi, D^*\psi)$ for all $\phi \in \Gamma(M, E)$ and $\psi \in \Gamma_c(M, F)$.*

a formal adjoint exists and is unique

locally

- in chart of M

- trivialization of E, F,

– $e := \dim(E)$, $f := \dim(F)$

– $D = \sum_{k=0}^{d} \sum_{i \in I_k} a_i \partial^i$

– where

— $I_k = \{i_1 \leq \cdots \leq i_k\}$ - set of multi-indices

— $a_i \in C^\infty(M, \mathrm{Mat}(f, e))$

— $\mathrm{vol} = v dx$

$$\begin{aligned}
(D\phi, \psi) &= \int_M (\sum_{k=0}^{d} \sum_{i \in I_k} a_i \partial^i \phi)^* \psi v dx \\
&= \sum_{k=0}^{d} \sum_{i \in I_k} \int_M \partial^i \phi^* \cdot a_i^* \cdot \psi v dx \\
&= \sum_{k=0}^{d} \sum_{i \in I_k} (-1)^k \int_M \phi^* \cdot v^{-1} \partial^i (v a_i^* \cdot \psi) v dx \\
&= \int_M \phi^* \cdot (\sum_{k=0}^{d} \sum_{i \in I_k} (-1)^k v^{-1} \partial^i (v a_i^* \cdot \psi)) v dx \\
&= (\phi, D^* \psi)
\end{aligned}$$

read off:

$D^* \psi = \sum_{k=0}^{d} \sum_{i \in I_k} (-1)^k v^{-1} \partial^i (v a_i^* \cdot \psi) = \sum_{k=0}^{d} \sum_{i \in I_k} a_i' \partial^i \psi$

- use Leibnitz rule for second equality

consider $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$

Definition 6.3. *The formal adjoint of d_k is $\delta_k : \Omega^{k+1}(M) \to \Omega^k(M)$.*

note $d_{k+1} \circ d_k = 0$ implies

$\delta_k \circ \delta_{k+1} = d_k^* \circ d_{k+1}^* = (d_{k+1} \circ d_k)^* = 0$

Lemma 6.4. $\delta_k = (-1)^{k+1} * d_{n-k-1} *$.

Proof. $\deg(\alpha) = k$, $\deg(\omega) = k+1$

$$\begin{aligned}
(d_k\alpha, \omega) &= (-1)^{(k+1)(n-k-1)}(d_k\alpha, **\omega) \\
&= (-1)^{(k+1)(n-k-1)} \int_M d_k\alpha \wedge *\omega \\
&= (-1)^{(k+1)(n-k-1)+k+1} \int_M \alpha \wedge d_{n-k-1} * \omega \\
&= (-1)^{(k+1)(n-k)+(n-k+1)(k+1)} \int_M \alpha \wedge ** d_{n-k-1} * \omega \\
&= (\alpha, (-1)^{k+1} * d_{n-k-1} * \omega)
\end{aligned}$$

□

general fact:

D differential operator between vector bundles E and F

- E, F with metrics

- M with volume

Lemma 6.5. *We have* $\ker(D) = \operatorname{im}(D^*)^\perp$.

Proof. $\phi \in \ker(D)$ implies $(\phi, D^*\psi) = (D\phi, \psi) = 0$ for all ψ, hence $\phi \in \operatorname{im}(D^*)^\perp$

$\phi \in \operatorname{im}(D^*)^\perp$ implies $(\phi, D^*\psi) = (D\phi, \psi) = 0$ for all ψ, hence $\phi \in \ker(D)$ □

- d in $\mathbb{N}$

Definition 6.6. *We say that* $\operatorname{ord}(D) \leq d$ *if for any* $f_0, \ldots, f_d$ *in* $C^\infty(M)$ *we have*

$$[f_d, [f_{d-1}, \ldots [f_0, D] \ldots]] = 0 \ .$$

Lemma 6.7. *We have* $\mathrm{ord}(D) \leq d$ *if and only if* M *can be covered by charts and trivializations of the bundles such that locally*

$$D = \sum_{k=0}^{d} \sum_{i \in I_k} a_i \partial^i \ .$$

Proof. exercise $\square$

if $\mathrm{ord}(d) \leq d$, then

$$e^{tf} D e^{-tf} = \sigma_d(D)(f) t^d + \cdots + \sigma_0(D)(f)$$

Lemma 6.8. *Assume that* $\mathrm{ord}(D) \leq d$. *Then* $\sigma_d(D)(f)(x)$ *only depends on* $df(x)$. *We have* $\sigma_d(D) \in \Gamma(M, S^d(T^*M) \otimes \mathrm{Hom}(E, F))$.

Proof. estimate of order of $e^{tf} D e^{-tf}$ in t

- $(\partial_t^k)_{|t=0} e^{tf} D e^{-tf} = [f, [f, \dots [f, D] \dots]] = 0$ (k commutators) for $k \geq d+1$

locally

- $\sigma_d(D)(f)(x) = \sum_{(i_1 \leq \cdots \leq i_d) \in I_d} \partial_{i_1} f(x) \dots \partial_{i_d} f(x) a_{i_1 \leq \cdots \leq i_d}$

$\square$

Definition 6.9. $\sigma_d(D)$ *is called the principal symbol of order* d *of* D.

Definition 6.10. *A differential operator* D *of order* $\leq d$ *is called elliptic if* $\sigma_d(D)(\xi) : E_x \to F_x$ *is invertible for all* ξ *in* $T_x^*M \setminus 0$ *and* x *in* M.

Theorem 6.11 (from analysis, without proof!)**.** *If* D *is an elliptic differential operator and* M *is compact, then*

$$\Gamma(M, E) \cong \ker(D) \oplus \mathrm{im}(D^*) \ .$$

Moreover $\dim \ker(D) < \infty$ *and* $\ker(D) = \mathrm{im}(D^*)^\perp$.

note that D^* is also elliptic and hence $\Gamma(M, F) \cong \ker(D^*) \oplus \mathrm{im}(D)$.

M Riemannian manifold

Definition 6.12.

$$\Delta_k := \delta_{k-1} d_{k-1} + \delta_k d_k : \Omega^k(M) \to \Omega^k(M)$$

is called the Hodge Laplacian.

have $\mathrm{ord}(\Delta_k) = 2$

Lemma 6.13. *We have* $\sigma_2(\Delta_k)(\xi) = 2\|\xi\|^2$.

Proof. $[f, d] = -\epsilon_{df}$

- $\epsilon_\xi := \xi\wedge$

$[f, \delta] = [f, d^*] = [d, f]^* = \epsilon^*_{df} = i_{df}$

here is the argument for last equality

- $i \in \{1, \ldots, n\}$, $j \in I_{k-1}$, $h \in I_k$
- $(\epsilon_{e^1}(e^j), e^h) = \delta_{\{1\}\sharp j,h} = (e^j, i_{e^1} e^h)$

$$\begin{aligned} [f,[f,\delta d]] &= [f,[f,\delta]d] + [f, \delta[f,d] \\ &= [f, i_{df} d] + [f, \delta\epsilon_d] \\ &= i_{df}\epsilon_{df} + i_{df}\epsilon_{df} \\ &= 2 i_{df}\epsilon_{df} \end{aligned}$$

analoguously

$$[f,[f,d\delta]] = 2\epsilon_{df} i_{df}$$

have

$$2\epsilon_{df} i_{df} + 2\epsilon_{df} i_{df} = 2\|df\|^2$$

here is the argument

- $(\epsilon_{e^1} i_{e^1} + \epsilon_{e^1} i_{e^1})e^j = e^j$

– if j contains 1, then first term contributes, other wise second term contributes hence

$\sigma_2(\Delta_k) = \|df\|^2$ □

Corollary 6.14. *The Hodge Laplacian Δ_k is elliptic.*

Theorem 6.15. *Let M be a compact Riemannian manifold. Then we have decompositions $\Omega^k(M) \cong \mathrm{im}(d_{k-1}) \oplus \mathrm{im}(d_{k-1})^\perp$ and $\Omega^k(M) \cong \mathrm{im}(\delta_k) \oplus \mathrm{im}(\delta_k)^\perp$.*

Proof. show first assertion, the second is similar

consider Laplace operator $\Delta_k := \delta_{k-1}d_{k-1} + \delta_k d_k : \Omega^k(M) \to \Omega^k(M)$

- elliptic and formally selfadjoint op

- Theorem 6.11 gives $\Omega^k(M) \cong \ker(\Delta^k) \oplus \mathrm{im}(\Delta^k)$

- $\omega \in \Omega^k$

- $\omega = \omega_0 + \Delta_k \omega'$ with

$\Delta\omega_0 = 0$

the following is the desired decomposition: $\omega = d_{k-1}\delta_{k-1}\omega' + (\omega_0 + \delta_k d_k \omega')$

- $d_{k-1}\delta_{k-1}\omega' \in \mathrm{im}(d_{k-1})$

- show: $(\omega_0 + \delta_k d_k \omega') \in \mathrm{im}(d_{k-1})^\perp$

– $(d_{k-1}\beta, \omega_0 + \delta_k d_k \omega') = (\beta, \delta_{k-1}\omega_0 + \delta_{k-1}\delta_k d_k \omega')$

– $\delta_{k-1}\delta_k d_k \omega' = 0$ is clear

$$\begin{aligned}
(\delta_{k-1}\omega_0, \delta_{k-1}\omega_0) &\leq (\delta_{k-1}\omega_0, \delta_{k-1}\omega_0) + (d_k\omega_0, d_k\omega_0) \\
&= (\omega_0, (d_{k-1}\delta_{k-1} + \delta_k d_k)\omega_0) \\
&= (\omega_0, \Delta_k \omega_0) \\
&= 0
\end{aligned}$$

□

the following is for compact M:

Definition 6.16. *We define the space of harmonic forms by* $\mathcal{H}^k(M) := \ker(\delta_{k-1}) \cap \ker(d_k)$*.*

Theorem 6.17. *Assume that* M *is compact. We have a decomposition*

$$\Omega^k(M) \cong \operatorname{im}(d_{k-1}) \oplus \mathcal{H}^k(M) \oplus \operatorname{im}(\delta_k) \ .$$

Furthermore, $\operatorname{im}(d_{k-1}) \oplus \mathcal{H}^k(M) = \ker(d_k)$ *and* $\mathcal{H}^k(M) = \ker(\Delta_k)$*.*

Proof. $\Omega^k(M) \cong \operatorname{im}(d_{k-1}) \oplus \operatorname{im}(d_{k-1})^{\perp}$ and $\Omega^k(M) \cong \operatorname{im}(\delta_k) \oplus \operatorname{im}(\delta_k)^{\perp}$.

show orthogonality $\alpha = d_{k-1}\alpha' \in \operatorname{im}(d_{k-1})$

$\omega = \delta_k \omega' \in \operatorname{im}(\delta_k)$

$\gamma \in \ker(\delta_{k-1}) \cap \ker(d_k)$

- $(\alpha, \gamma) = (d_{k-1}\alpha', \gamma) = (\alpha', \delta_k \gamma) = 0$
- $(\omega, \gamma) = (\delta_k \omega', \gamma) = (\omega', d_k \gamma) = 0$
- $(\alpha, \omega) = (d_{k-1}\alpha', \delta_k \omega') = (\alpha', \delta_{k-1}\delta_k \omega') = 0$

completeness:

$\theta \in \Omega^k(M)$

- $\theta = \alpha + \sigma$ with $\alpha = d_{k-1}\alpha' \in \operatorname{im}(d_{k-1})$, $\sigma \in \operatorname{im}(d_{k-1})^{\perp}$
- $\sigma = \gamma + \delta_k \omega'$ with $\gamma \in \operatorname{im}(\delta_k)^{\perp}$

$\theta = d_{k-1}\alpha' + \gamma + \delta_k \omega'$ is desired decomposition

must show that γ is harmonic

- claim: $\delta_{k-1}\gamma = 0$

— $\ker(\delta_{k-1}) = \operatorname{im}(d_{k-1})^{\perp}$

— implies $\delta_{k-1}\sigma = 0$.

— since also $\delta_{k-1}\delta_k \omega' = 0$ conclude:

— $\delta_{k-1}\gamma = 0$

must show: $d_k\gamma = 0$

- $\ker(d_k) = \mathrm{im}(\delta_k)^{\perp} \ni \gamma$

$\mathrm{im}(d_{k-1}) \oplus \mathcal{H}^k(M) \subseteq \ker(d_k)$

$\mathrm{im}(\delta_k) \perp \ker(d_k)$ implies

$\mathrm{im}(d_{k-1}) \oplus \mathcal{H}^k(M) = \ker(d_k)$

- $\Delta_k(\mathcal{H}^k) = 0$ is clear
- vice versa: assume $\Delta_k\omega = 0$
- then $0 = (\Delta_k\omega, \omega) = \|d_k\omega\|^2 + \|\delta_{k-1}\omega\|^2$
- hence $\omega \in \ker(d_k \cap \ker(\delta_k) = \mathcal{H}^k(M)$

□

Corollary 6.18. *If M is compact, then $\mathcal{H}^k(M) \to \ker(d_k) \to H^k_{\mathrm{dR}}(M)$ is an isomorphism. In particular, every class $[\omega]$ in $H^k_{\mathrm{dR}}(M)$ has a unique representative ω in $\mathcal{H}^k(M)$ characterized by the additional equation $\delta_{k-1}\omega = 0$.*

Example 6.19. $M = G/K$ compact symmetric

- $\mathcal{H}^k(M) = \Omega^k(M)^G$

speciality: $\mathcal{H}^*(M)$ is an algebra

in general: the wedge product of harmonic forms is not necessarily harmonic

□

Definition 6.20. *M is called formal if there exists a zig-zag of quasi-isomorphisms of differential-graded algebras*

$$H^*_{\mathrm{dR}}(M) \to A_1 \leftarrow A_2 \to \cdots \to \Omega^*(M) \ .$$

Corollary 6.21. *If M is closed and admits a Riemannian metric such that $\mathcal{H}^*(M)$ is an algebra under $\wedge$, then M is formal.*

Corollary 6.22. *Compact symmetric spaces are formal.*

M compact, oriented Rimannian

Proposition 6.23. *The Hodge $*$-operator preserves harmonic forms and $* : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}$ is the Poincaré duality isomorphism: $* = (-1)^{k(n-k)}D$*

Proof. $\omega \in \mathcal{H}(M)$

$d * \omega = \pm * * d * \omega = \pm * \delta\omega = 0$

$\delta * \omega = \pm * * d * \omega = * d\omega = 0$

- hence $*\omega \in \mathcal{H}(M)$

$\omega \in \mathcal{H}^k(M)$, $\alpha \in \mathcal{H}^{n-k}(M)$

$$\begin{aligned} (*\omega, \alpha) &= (\alpha, *\omega) \\ &= \int_M \alpha \wedge \omega \\ &= (-1)^{k(n-k)} D(\omega)(\alpha) \end{aligned}$$

□

Example 6.24. $\dim(M) = 4m$

- $* : \mathcal{H}^{2n}(M) \to \mathcal{H}^{2n}(M)$

- $\mathrm{sign}(M) = \mathrm{sign}(*)$ *on* $\mathcal{H}^{2n}(M)$

6.3 De Rham cohomology of complex manifold

M - manifold

$T_{\mathbb{C}}M := TM \otimes \mathbb{C}$

- $\otimes := \otimes_{\mathbb{R}}$

use $\Lambda^k_{\mathbb{C}}(\mathbb{R}^n \otimes \mathbb{C}) \cong \Lambda^k_{\mathbb{R}}\mathbb{R}^n \otimes \mathbb{C}$

set: $A^k(M) := \Gamma(M, \Lambda^k_{\mathbb{C}} T_{\mathbb{C}} M) \cong \Gamma(M, \Lambda^k_{\mathbb{R}} T^*M \otimes \mathbb{R}) \cong \Omega^k(M) \otimes \mathbb{C}$

- complex differential forms

- $d : A^k(M) \to A^{k+1}(M)$ - complex linear extension of de Rham differential

- $-\otimes \mathbb{C}$ is exact functor

$H^k_{\mathrm{dR},\mathbb{C}}(M) := H^k(A^*(M), d) \cong H^k(\Omega^*(M) \otimes \mathbb{C}, d) \cong H^k(\Omega(M), d) \otimes \mathbb{C} \cong H^k_{\mathrm{dR}}(M) \otimes \mathbb{C}$

assume now that (M, I) is almost complex manifold

- write also I for induced complex structure on $\mathrm{End}(T^*M)$

- $T^*_{\mathbb{C}}M \cong T^{*.1,0}M \oplus T^{*.0,1}M$

– $T^{*,1,0}M$ - i-eigenspace of $I \otimes \mathrm{id}_{\mathbb{C}}$

– $T^{*,0,1}M$ - $-i$-eigenspace of $I \otimes \mathrm{id}_{\mathbb{C}}$

complex conjugation: $\overline{(-)} : T^*_{\mathbb{C}}M \to T^*_{\mathbb{C}}M$

- $\overline{(-)} : T^{*,1,0}_{\mathbb{C}}M \xrightarrow{\cong} T^{*,0,1}_{\mathbb{C}}M$

- $\overline{(-)} : T^{*,0,1}_{\mathbb{C}}M \xrightarrow{\cong} T^{*,1,0}_{\mathbb{C}}M$

$T^*_{\mathbb{C}}M \cong T^{*,1,0}M \oplus T^{*,0,1}M$ induces

$$\Lambda^k T^*_{\mathbb{C}}M \cong \bigoplus_{p+q=k} \Lambda^p T^{*,1,0}M \otimes \Lambda^q T^{*,0,1}M$$

– define $\Lambda^{p,q}T^*_{\mathbb{C}}M := \Lambda^p T^{*,1,0}M \otimes \Lambda^q T^{*,0,1}M$

– set $A^{p,q}(M) := \Gamma(M, \Lambda^{p,q}T^*_{\mathbb{C}}M)$

— then $A^k(M) \cong \bigoplus_{p+q=k} A^{p,q}(M)$

how does d interact with this decomposition

Lemma 6.25. $d : A^{p,q}(M) \subseteq A^{p-1,q+2}(M) + A^{p,q+1}(M) + A^{p+1,q}(M) + A^{p+2,q-1}(M)$

Proof. local argument

- choose basis $e^1, \ldots, e^n$ of $T^{*,1,0}M$

- apply $\overline{(-)}$ - get basis $\bar{e}^1, \ldots, \bar{e}^n$ of f$T^{*,1,0}$

- $fe^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_q}$ - in $A^{p,q}(M)$

$$\begin{aligned} df &= \sum_k \partial_k f dx^k \wedge e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_q} \\ &\quad + f \sum_{l=1}^{p} (-1)^l e^{i_1} \wedge \cdots \wedge de^{i_k} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_q} \\ &\quad + f \sum_{h=1}^{q} (-1)^{h+p} e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \wedge d\bar{e}^{j_h} \wedge \cdots \wedge \bar{e}^{j_q} \end{aligned}$$

$dx^k \in A^{0,1}(M) + A^{1,0}(M)$

- first term in $A^{p+1,q}(M) + A^{p,q+1}(M)$

$de^k, d\bar{e}^k$ is just 2-form, any bidegree

- second term in $A^{p-1,q+2}(M) + A^{p,q+1}(M) + A^{p+1,q}(M)$

- third term in $A^{p,q+2}(M) + A^{p+1,q}(M) + A^{p+2,q-1}(M)$ □

assume now that I is integrable

- study consequences for de Rham complex

- here is one

Lemma 6.26. *If I is integrable, then $d : A^{p,q}(M) \subseteq A^{p+1,q}(M) \oplus A^{p,q+1}(M)$.*

local structure

- by assumption on I have complex coordinates $z_k = x_k + iy_k$

- $dz^k := dx^k + idy^k$

- $d\bar{z}^k := dx^k - idy^k$

- $\partial_i := \partial_{z^i} := \frac{1}{2}(\partial_{x^i} - i\partial_{y^i})$

- $\bar{\partial}_i := \partial_{\bar{z}^i} := \frac{1}{2}(\partial_{x^i} + i\partial_{y^i})$

- basis of $\Lambda^{p,q} T_{\mathbb{C}} M$ is $dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$

- but now $ddz^i = 0$ and $dd\bar{z}^i = 0$

– this shows Lemma 6.26

$d = \sum_{i=1}^n (\epsilon_{dx^i}\partial_{x_i} + \epsilon_{dy^i}\partial_{y_i}) = \sum_{i=1}^n (\epsilon_{dz^i}\partial_i + \epsilon_{d\bar{z}^i}\bar{\partial}_i)$

- set $\partial := \sum_{i=1}^n \epsilon_{dz^i}\partial_i$ and $\bar{\partial} := \sum_{i=1}^n \epsilon_{d\bar{z}^i}\bar{\partial}_i$

$\partial : A^{p,q}(M) \to A^{p+1,q}(M)$

$\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$

have $[\partial_i, \partial_j] = 0$ and $[\bar{\partial}_i, \bar{\partial}_j] = 0$

- hence $\partial^2 = 0$ and $\bar{\partial}^2 = 0$

- hence $0 = d^2 = (\partial + \bar{\partial})^2 = \bar{\partial}\partial + \partial\bar{\partial}$

get double complex $(A^{*,*}(M), \partial, \bar{\partial})$

- interesting homological algebra, spectral sequences

Definition 6.27. *The pth Dolbeault-complex of M is the complex $(\mathcal{A}^{p,*}(M), \bar{\partial})$.*

Definition 6.28. *For $p, q \in \mathbb{N}^2$ we define the Dolbeault cohomology $H^{p,q}(M) := H^q((\mathcal{A}^{p,*}(M), \bar{\partial})$ and the Hodge numbers $h^{p,q}(M) := \dim H^{p,q}(M)$.*

Remark 6.29.

$$\Omega^p_{\text{hol}}(M) := \ker(\bar{\partial} : A^{p,0}(M) \to A^{p,1}(M))$$

is the space of holomorphic p-forms

- complex of sheaves $(A^{p,*}, \bar{\partial})$ is a soft resolution of sheaf $\Omega^p_{\text{hol}}(M)$

– $H^{p,q}(M) \cong H^q_{\text{sheaf}}(M, \Omega^p_{\text{hol}})$

– Dolbeault cohomology calculates sheaf cohomology of the sheaf of holomorphic p-forms

□

Example 6.30. M compact complex surface $\dim_{\mathbb{R}}(M) = 2$

- also called curve in algebraic geometry since $\dim_{\mathbb{C}} M = 1$

- g - genus

Riemann Roch Theorem:

$h^{0,0}(M) - h^{0,1}(M) = 1 - g$

- $h^{0,0} = 1$ (holomorphic functions are constant), $h^{0,1}(M) = g$

$h^{1,0}(M) - h^{1,1}(M) = 2g - 2 + 1 - g = g - 1$

- What can one say about $h^{1,0}(M)$ and $h^{1,1}(M)$ separately?

Serre duality - see later

$\square$

$f : M \to M'$ holomorphic.

Proposition 6.31. *f induces map of double complexes $f^* : (A^{*,*}(M'), \partial, \bar{\partial}) \to (A^{*,*}(M), \partial, \bar{\partial})$ and $f^* : H^{*,*}(M') \to H^{*,*}(M)$.*

Proof. df commutes with I

- it restricts to

- $df \otimes \mathrm{id}_{|T^{1,0}M} : T^{1,0}M \to f^*T^{1,0}M'$ and same for $(0,1)$

- hence f^* restricts to $f^* : A^{p,q}(M') \to A^{p,q}(M)$

- f^* preserves d and hence ∂, ∂' $\square$

Remark 6.32. $\Omega^k(M) \otimes \mathbb{C}$ has decreasing filtration

- $F^l\Omega^k(M) \otimes \mathbb{C} := \bigoplus_{p+q=k, p \geq l} A^{p,q}(M)$

- compatible with d

- $d : F^l\Omega^k(M) \subseteq F^l\Omega^{k+1}(M)$

- get filtration of $H_{\mathrm{dR}}(M) \otimes \mathbb{C}$ by images of $\bigoplus_{p+q=k, p \geq l} H^{p,q}(M)$

the spectral sequence associated to this filtration is called the Hodge-de Rham spectral sequence.

zero page

- $A^{p,q}(M)$,

- $d_0 = \bar{\partial}$

first page:

- $E_1^{p,q} := H^{p,q}(M)$

- $d_1 := \partial : H^{p,q}(M) \to H^{p+1,q}(M)$

conclude: estimate of Betti numbers

$$b^k(M) \leq \sum_p h^{p,k-p}(M)$$

check for surfaces:

- $1 = b^0(M) \leq h^{0,0}(M) = 1$

- $b^1(M) \leq 2g \leq h^{1,0} + h^{0,1}(M) = g + h^{1,0}(M)$

– hence $h^{1,0}(M) \geq g$

– will see that we have equality here later

- $1 = b^2(M) \leq h^{1,1}(M)$

- $h^{1,1}(M) = h^{1,0}(M) + 1 - g \geq 1$ is compatible with □

Riemannian metric on M

M compact

- induces hermitean metric on $\Lambda^{p,q} T_{\mathbb{C}} M$

- can define $\bar{\partial}^*$ - formal adjoint of $\bar{\partial}$

- $\bar{\Delta} := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$

- $\bar{\Delta}$ is elliptic

Theorem 6.33. *If M is compact, then we have a decomposition $A^{p,q}(M) \cong \operatorname{im}(\bar{\partial}_{q-1}) \oplus \mathcal{H}^{p,q}(M) + \operatorname{im}(\bar{\partial}_q^*)$. Furthermore we have an isomorphism $H^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$ and $h^{p,q} = \dim H^{p,q}(M) < \infty$.*

- in general: $h^{p,q}$ is sensitive to complex structure, difficult to calculate

6.4 The Kähler package

M manifold

- $2n := \dim_{\mathbb{R}}(M)$

- I almost complex structure

- g metric such that $I = -I^*$

- $\omega = g(I, -, -)$ - Kähler form in $\Omega^2(M)$

define $L : \Lambda^k T^*M \to \Lambda^{k+2} T^*M$

- $L(\alpha) = \omega \wedge \alpha$

Lemma 6.34. $L(\Lambda^{p,q} T^*_{\mathbb{C}} M) \subseteq \Lambda^{p+1,q+1} T^*_{\mathbb{C}} M)$

Proof. must show: $\omega \in A^{1,1}(M)$

choose local ONB of the form $(e^j, Ie^j)_{j=1,\dots,n}$ of TM

$g = \sum_{j=1}^n (e^j \otimes e^j + Ie^j \otimes Ie^j)$

$$\begin{aligned}
\omega &= \sum_{j=1}^n (Ie^j \otimes e^j - e^j \otimes Ie^j) \\
&= \sum_{j=1}^n Ie^j \wedge e^j \\
&= \sum_{j=1}^n (ie^j + Ie^j) \wedge e^j \\
&= i \sum_{i=1}^n (ie^j + Ie^j) \wedge (-ie^j + \frac{1}{2}(ie^j + Ie^j)) \\
&= \frac{i}{2} \sum_{j=1}^n (ie^j + Ie^j) \wedge (-ie^j + Ie^j)
\end{aligned}$$

- $I(ie^j + Ie^j) = Iie^j - e^j = i(ie^j + Ie^j)$ hence $(ie^j + Ie^j) \in T^{*,1,0}M$

- $I(-ie^j + Ie^j) = -Iie^j - e^j = -i(-ie^j + Ie^j)$ hence $(-ie^j + Ie^j) \in T^{*,0,1}M$

– conclude: $\omega \in A^{1,1}(M)$

□

define $\Lambda := L^* : \Lambda^{k+2}T^*M \to \Lambda^k T^*M$

X in TM

- notation: $\widehat{X} = g(X, -)$ - the dual 1-form to X

Lemma 6.35. *We have* $[\iota_X, \Lambda] = 0$ *and* $[\iota_X, L] = \epsilon_{\widehat{IX}}$.

Proof. $[\iota_X, \Lambda]$ since Λ is even

$[\iota_X, L] = \epsilon_{\iota_X \omega} = \epsilon_{g(IX,-)} = \epsilon_{\widehat{IX}}$

□

recall: (M, g, I) Kähler if $d\omega = 0$ (then also I is integrable)

Lemma 6.36. *If* (M, I, g) *is Kähler, then* $[L, d] = 0$ *and* $[\Lambda, \delta] = 0$.

Proof. $[L, d] = -d\omega \wedge - = 0$

take adjoints to get $[\Lambda, \delta] = 0$ □

$*$ - Hodge $*$

- consider $\mathbb{C}$-linear extension to $\Lambda^* T^*_{\mathbb{C}} M$

Lemma 6.37. $*$ *restricts to maps* $* : \Lambda^{p,q} T^*_{\mathbb{C}} M \to \Lambda^{n-q,n-p} T^*_{\mathbb{C}} M$.

Proof. use basis $dz^1, \dots, dz^n, d\bar{z}^1, \dots, d\bar{z}^n$ □

recall: $d = \partial + \bar{\partial}$

- $\partial : A^{p,q}(M) \to A^{p+1,q}(M)$

- $\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$

– consider formal adjoints: ∂^* and $\bar{\partial}^*$

– define $\Delta^{\partial} := \partial\partial^* + \partial^*\partial, \qquad \Delta^{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$

- both preserve summands $A^{p,q}(M)$ of $A^{p+q}(M)$

Theorem 6.38. *If* (M, I, g) *is Kähler, then* $\Delta = 2\Delta^{\partial} = 2\Delta^{\bar{\partial}}$. *In particular* Δ *preserves* $A^{p,q}(M)$, define

$$d^c := [L, \delta] : \Omega(M) \to \Omega(M)[1]$$

$$\delta^c := (d^c)^* = [d, \Lambda] : \Omega(M)[1] \to \Omega(M) \ .$$

calculate local formula at some point p

$(e^i)_{i=1,\ldots,2n}$ - local basis dual to $(e_i)_{i=1,\ldots,2n}$

recall: $d = \sum_i \epsilon_{e^i} \nabla_{e_i}$

- calculate formal adjoint of ∇_{e_i}

– use definition diverence div : $\mathcal{X}(M) \to C^\infty(M)$

– div is formal adjoint of grad : $C^\infty(M) \to \mathcal{X}(M)$

$(\mathrm{div}(X), f) = (X, \mathrm{grad}(f))$ forall f in $C_c^\infty(M)$

$$\begin{aligned}\int_M (\nabla_{e_i}\alpha, \beta)\mathrm{vol} &= \int_M e_i(\alpha,\beta)\mathrm{vol} - \int_M (\alpha, \nabla_{e_i}\beta)\mathrm{vol} \\ &= \int_M (\alpha,\beta)\mathrm{div}(e_i)\mathrm{vol} - \int_M (\alpha, \nabla_{e_i}\beta)\mathrm{vol}\end{aligned}$$

- hence $\nabla_{e_i}^* = -\nabla_{e_i} + \mathrm{div}(e_i)$

- get $\delta = \sum_i(-\nabla_{e_i} + \mathrm{div}(e_i))\iota_{e^i}$

– want to switch ι_{e^i} to the left

- claim: $\delta = -\sum_i \iota_{e^i}\nabla_{e_i}$

– consider $u := \sum_i(-\nabla_{e_i} + \mathrm{div}(e_i))\iota_{e^i} - (-\sum_i \iota_{e^i}\nabla_{e_i})$

– u is bundle endomorphism

– is independent of choice of basis (e_i)

— fix point p in M

— at p can assume that e_i (and hence e^i) are parallel

— at this point $\mathrm{div}(e_i) = 0$ and $[\iota_{e^i}, \nabla_{e^i}] = 0$

— hence $u(p) = 0$

— since p is arbitrary conclude $u = 0$

M Kähler implies $\nabla\omega = 0$ and hence $[L, \nabla] = 0$

$$d^c = [L, \delta] = -\sum_i [L, \iota_{e_i} \nabla_{e_i}] = -\sum_{i=1}^{2n} [L, \iota_{e_i}] \nabla_{e_i} = \sum_{i=1}^{2n} \epsilon_{Ie_i} \nabla_{e_i}$$

now use complex coordinates

- write $z^k = x^k + iy^k$

- $e^i := dx^i$, $Ie^i := dy^i$

use $-i\epsilon_{e^i+iIe^i} = \epsilon_{Ie^i+iIIe^i}$

$\partial = \sum_{i=1}^n \epsilon_{dz^i}\partial_i = \sum_i \epsilon_{(e^i+iIe^i)}\frac{1}{2}(\partial_{e_i} - i\partial_{Ie_i}) = \frac{1}{2}(\sum_{i=1}^n \epsilon_{(e^i+iIe^i)}\partial_{e_i} + \sum_{i=1}^n \epsilon_{(Ie^i+iIIe^i)}\partial_{Ie_i})$

use $i\epsilon_{e^i-iIe^i} = \epsilon_{Ie^i-iIIe^i}$

$\bar\partial = \sum_{i=1}^n \epsilon_{d\bar z^i}\bar\partial_i = \sum_i \epsilon_{(e^i-iIe^i)}\frac{1}{2}(\partial_{e_i} + i\partial_{Ie_i}) = \frac{1}{2}(\sum_{i=1}^n \epsilon_{(e^i-iIe^i)}\partial_{e_i} + \sum_{i=1}^n \epsilon_{(Ie^i-iIIe^i)}\partial_{Ie_i})$

get

$i(\bar\partial - \partial) = \sum_{i=1}^n \epsilon_{Ie^i}\partial_{e_i} + \epsilon_{IIe^i}\partial_{Ie_i} = d^c$

$i(\partial^* - \bar\partial^*) = \delta^c$

$[L, \partial^* + \bar\partial^*] = [L, \delta] = d^c = i(\bar\partial - \partial)$

part $A^{p,q}(M) \to A^{p+1,q}$: $[L, \bar\partial^*] = -i\partial$,

part $A^{p,q}(M) \to A^{p,q+1}$: $[L, \partial^*] = i\bar\partial$

$[L, \partial + \bar\partial] = [L, d] = 0$

part $A^{p,q}(M) \to A^{p+2,1}$: $[L, \partial] = 0$

part $A^{p,q}(M) \to A^{p+1,q+2}$: $[L, \bar\partial] = 0$

take adjoints and get identities

$$[\Lambda, \bar\partial] = -i\partial^*, \quad [\Lambda, \partial] = i\bar\partial^*, \quad [\Lambda, \partial^*] = 0\ , \quad [\Lambda, \bar\partial^*] = 0$$

use $\bar\partial^2 = 0$

$-i(\bar\partial\partial^* + \partial^*\bar\partial) = \bar\partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\bar\partial = 0$

analogously $-i(\partial\bar\partial^* + \bar\partial^*\partial) = 0$

$$\begin{aligned}\Delta &= d\delta + \delta d \\ &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= \Delta^{\partial} + \Delta^{\bar\partial}\end{aligned}$$

remains to show: $\Delta^{\partial} = \Delta^{\bar\partial}$

$$\begin{aligned}-i\Delta^{\partial} &= -i(\partial\partial^* + \partial^*\partial) \\ &= \partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\partial \\ &= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial \\ &= \partial\Lambda\bar\partial + \bar\partial\partial\Lambda - \Lambda\partial\bar\partial - \bar\partial\Lambda\partial \\ &= [\partial, \Lambda]\bar\partial + \bar\partial[\partial, \Lambda] \\ &= -i\bar\partial^*\bar\partial - i\bar\partial\bar\partial^* \\ &= -i\Delta^{\bar\partial}\end{aligned}$$

Lemma 6.39. *If (M, I, g) is Kähler, then $[\Delta, L] = 0$.*

Proof. $d\omega = 0$

- part in $A^{2,1}(M)$ is $\partial\omega = 0$

use $\Delta = 2\Delta^{\partial}$

$[\Delta, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2\partial[\partial^*, L] + [\partial^*, L]\partial$

- already shown: $[\partial^*, L] = -i\bar\partial$
- $\bar\partial\partial + \partial\bar\partial = 0$

get $[\Delta, L] = 0$

$\square$

Corollary 6.40. *1. If (M, I, g) is a compact Kähler manifold of complex dimension n, then we have an orthogonal decomposition $\mathcal{H}^k(M) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$*

2. *$*$ induces an isomorphism $\mathcal{H}^{p,q}(M) \cong \mathcal{H}^{n-q,n-p}(M)$ (Serre duality). In particular, $h^{p,q}(M) = h^{n-q,n-p}(M)$.*

3. *We have $b^k(M) = \sum_{p+q=k} h^{p,q}(M)$ for every $k \in \mathbb{N}$.*

4. *We have $0 \neq [\omega^l] \in \mathcal{H}^{l,l}(M)$ for $l = 0, \ldots, n$. In particular, $h^{l,l}(M) \geq 1$*

5. *Complex conjugation induces an isomorphism $\mathcal{H}^{p,q}(M) \to \mathcal{H}^{q,p}(M)$ and $h^{p,q}(M) = h^{q,p}(M)$. In particular, $b^{2k+1}(M) \in 2\mathbb{Z}$.*

6. *The Hodge de-Rham spectral sequence degenerates at the E_1-term.*

Example 6.41. Hodge numbers for connected complex curve, genus g

- $h^{0,1} = h^{1,0} = g$

- $h^{0,0} = h^{1,1} = 1$ $\square$

6.5 Lefschetz theory

start with some linear algebra with hermitean vector spaces

$\mathbb{C}^n$ - with standard $\mathbb{C}$-basis $(e_j)_{j=1,\ldots,n}$

- $\mathbb{R}$-basis is $(e_j, ie_j)_{j=1,\ldots,n}$

- $(e^j)_{j=1,\ldots,n}$ - dual $\mathbb{C}$-basis

- $(e^j, ie^j)_{j=1,\ldots,n}$ - dual $\mathbb{R}$-basis

euclidean metric: $g = \sum_{j=1}^n e^j \otimes e^j + ie^j \otimes ie^j$

- Kähler form $\omega(-,-) = g(i-,-) = \sum_{j=1}^n \left(ie^j \otimes e^j - e^j \otimes ie^j\right) = \sum_{i=1}^n ie^j \wedge e^j$

consider operators on $\Lambda^* \mathbb{R}^{2n,*} \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda^*_{\mathbb{R}} \mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$

- $L := \omega \wedge - = \sum_{j=1}^{n} \epsilon_{ie^j}\epsilon_{e^j}$

- $\Lambda := L^* = \sum_{j=1}^{n} \iota_{e_j}\iota_{ie_j}$

- $\deg : \Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$ - degree operator

– for $\alpha \in \Lambda^k_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$: $\deg(\alpha) = k$

- $N := \deg - n$

Lemma 6.42. *We have* $[L, \Lambda] = N$, $[N, L] = 2L$, $[N, \Lambda] = -2\Lambda$.

Proof. L increases degree by 2

- $[\deg, L] = 2L$

- take adjoint: $[\Lambda, \deg] = 2\Lambda$

- implies: $[N, L] = 2L$, $[N, \Lambda] = -2\Lambda$

let $n_{e^h} = \epsilon_{e^h}\iota_{e_h} \in \mathrm{End}(\Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})$

- $n_{e^h}(e^{i_1} \wedge \cdots \wedge e^{i_r} \wedge ie^{j_1} \wedge \cdots \wedge ie^{j_s}) = \begin{cases} e^{i_1} \wedge \cdots \wedge e^{i_r} \wedge ie^{j_1} \wedge \cdots \wedge ie^{j_s} & h = i_l \text{ for some } l \\ 0 & else \end{cases}$

- $n_{ie^h}(e^{i_1} \wedge \cdots \wedge e^{i_r} \wedge ie^{j_1} \wedge \cdots \wedge ie^{j_s}) = \begin{cases} e^{i_1} \wedge \cdots \wedge e^{i_r} \wedge ie^{j_1} \wedge \cdots \wedge ie^{j_s} & h = j_l \text{ for some } l \\ 0 & else \end{cases}$

- $\iota_{e_h}\epsilon_{e^h} = 1 - n_{e^h}$, $\iota_{ie_h}\epsilon_{ie^h} = 1 - n_{i^h}$

$$
\begin{aligned}
[L,\Lambda] &= \sum_{j,k=1}^{n} (\epsilon_{ie^j}\epsilon_{e^j}\iota_{e_k}\iota_{ie_k} - \iota_{e_k}\iota_{ie_k}\epsilon_{ie^j}\epsilon_{e^j}) \\
&= \sum_{j=1}^{n} (\epsilon_{ie^j}\epsilon_{e^j}\iota_{e_j}\iota_{ie_j} - \iota_{e_j}\iota_{ie_j}\epsilon_{ie^j}\epsilon_{e^j}) \\
&= \sum_{j=1}^{n} \epsilon_{ie^j}\iota_{ie_j} n_{e^j} - \sum_{j=1}^{n} \iota_{e_j}\epsilon_{e^j}(1-n_{ie^j}) \\
&= \sum_{j=1}^{n} n_{ie^j} n_{e^j} - \sum_{j=1}^{n} (1-n_{e^j})(1-n_{ie^j}) \\
&= \sum_{j=1}^{n} (n_{e^j} + n_{ie^j}) - n \\
&= \deg - n \\
&= N
\end{aligned}
$$

□

recall Lie algebra $sl(2,\mathbb{R})$

- linear generators: L, Λ, N

- relations: $[N, L = 2L]$, $[N,\Lambda] = -2\Lambda$, $[L,\Lambda] = N$

Corollary 6.43. $\Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$ *carries a representation of* $sl(2,\mathbb{R})$.

$sl(2,\mathbb{R})$ is semisimple

- every finite-dimensional complex representation completely decomposes into irreducible representations

- the list of irreducible representations up to isomorphism is $(V_k)_{k\in\mathbb{N}}$

– $\dim(V_k) = k+1$

– weights (eigenvalues of N) in V_k are $-k, -k+2, \ldots, k-2, k$

– lowest weight vector v_{-k}

– $(L^r v_{-k})_{r=0,\ldots,k}$ is $\mathbb{C}$-basis of V_k

— weight of $L^r v_{-k}$ is $2r - k$

– $L^{k+1} v_{-k} = 0$

provide explicit description

$SL(2,\mathbb{R})$ acts on $\mathbb{C}^2$ - usual matrix multiplication

- acts on $S^k(\mathbb{C}^{2,*})$ - homogenous polynomials on x, y of degree k

- basis $x^k, x^{k-1}y, \ldots, xy^{k-1}, x^k$

- get action by $sl(2,\mathbb{R})$ by differentiation

$N := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $L := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\Lambda := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$e^{tN} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

- $e^{tN}x = e^t x$, $e^{tN}y = e^{-t}y$

- $e^{tN}x^{k-l}y^l = e^{t(k-2l)}x^{k-l}y^l$

- $Nx^{k-l}y^l = (k-2l)$

- $e^{tL} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

- $e^{tL}x = x$, $Lx = 0$, $Ly = x$

- $Lx^{k-l}y^l = lx^{k-l+2}y^{l-2}$

y^k - generates $S^k(\mathbb{C}^{2,*})$ under powers of L

- is lowest weight vector of weight $-k$

- conclude irreducibility

- $V_k \cong S^k(\mathbb{C}^{2,*})$ as $sl(2,\mathbb{R})$- representations

W any finite-dimensional representation of $sl(2,\mathbb{R})$

- have canonical $sl(2,\mathbb{R})$ -equivariant decompostion $W \cong \bigoplus_{k\in\mathbb{Z}} W_k$

– W_k is isomorphic to a finite sum of copies of V_k

– want to describe this explicitly

$W \cong \bigoplus_{k\in\mathbb{Z}} W(k)$ - weight decomposition (eigenvalues of N)

for $k \in \mathbb{N}$

- $W(k)_- := \ker(L^{k+1} : W(-k) \to W(k+2))$ is the subspace of lowest weight vectors of W_k

- get canonical $sl(2,\mathbb{R})$-equivariant isomorphism $W(k)_- \otimes V_k \xrightarrow{\cong} W_k$

– uniquely determined by $w \otimes v_{-k} \to w$

– explicitly:

$$\oplus_{r=0}^{k} L^r : \bigoplus_{r=0}^{k} W(k)_- \xrightarrow{\cong} W_k$$

– get $sl(2,\mathbb{R})$-equivariant isomorphism

$$\bigoplus_{k=0}^{\infty} \bigoplus_{r=0}^{k} W(k)_- \otimes V_k \xrightarrow{\cong} W$$

– explicitly

$$\oplus_{k=0}^{\infty} \oplus_{r=0}^{k} L^r : \bigoplus_{k=0}^{\infty} \bigoplus_{r=0}^{k} W(k)_- \xrightarrow{\cong} W$$

apply this to $\Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$

recall: $N = \deg - n$

- $(\Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})(k) = \Lambda^{n+k}_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$

for $k \le n$ set

- set $(\Lambda^{n-k}_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})_{\text{prim}} := (\Lambda^*_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})(-k)_-$

Corollary 6.44. *1. For $k \le n$ we have an isomorphism*

$$L^r : (\Lambda^{n-k}_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}) \to (\Lambda^{n+k}_{\mathbb{R}}\mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

2. *We have a decomposition*

$$\oplus_{k=0}^{\infty} \oplus_{r=0}^{k} L^r : \bigoplus_{k=0}^{\infty} \bigoplus_{r=0}^{k} (\Lambda_{\mathbb{R}}^{n-k} \mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})_{\mathrm{prim}} \xrightarrow{\cong} \Lambda_{\mathbb{R}}^{*} \mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$$

note:

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2, l/2)} (\Lambda_{\mathbb{R}}^{l-2r} \mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C})_{\mathrm{prim}} \xrightarrow{\cong} \Lambda_{\mathbb{R}}^{l} \mathbb{C}^{n,*} \otimes_{\mathbb{R}} \mathbb{C}$$

applies to $\Lambda^* T^*_{\mathbb{C}} M$ fibrewise

$L, \Lambda, N := \deg - n$ define an action of $sl(2, \mathbb{R})$ by bundle endomorphisms

$[L, d] = 0$ from Kähler condition

Corollary 6.45. 1. *For $k \leq n$ the operator $L^k : A^{n-k}(M) \to A^{n+k}(M)$ is an isomorphism.*

2. *It induces an isomorphism (Hard Lefschetz)*

$$H^{2n-k}_{\mathrm{dR},\mathbb{C}}(M) \to H^{2n+k}_{\mathrm{dR},\mathbb{C}}(M)$$

3. *It restricts to isomorphisms*

$$H^{n-p,n-q}_{\mathrm{dR},\mathbb{C}}(M) \to H^{n+q,n+p}_{\mathrm{dR},\mathbb{C}}(M) \ .$$

Definition 6.46. *For $k \leq n$ Define*

$$A^{n-k}(M)_{\mathrm{prim}} := \ker(L^{k+1} : A^{n-k}(M) \to A^{n+k+1}(M) \ ,$$

$$H^{n-k}(M)_{\mathrm{dR},\mathbb{C},\mathrm{prim}} := \ker(L^{k+1} : H^{n-k}_{\mathrm{dR},\mathbb{C}}(M) \to H^{n+k+2}_{\mathrm{dR},\mathbb{C}}(M)$$

and for $p + q = n - k$

$$H^{p,q}(M)_{\mathrm{dR},\mathbb{C},\mathrm{prim}} := \ker(L^{k+1} : H^{p,q}_{\mathrm{dR},\mathbb{C}}(M) \to H^{p+k+1,q+k+1}_{\mathrm{dR},\mathbb{C}}(M) \ .$$

L commutes with Δ

Corollary 6.47. *We have an isomorphisms*

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2,l/2)} A^{l-2r}_{\mathrm{prim}}(M) \xrightarrow{\cong} A^l(M) \ ,$$

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2,l/2)} H^{l-2r}_{\mathrm{dR},\mathbb{C},\mathrm{prim}}(M) \xrightarrow{\cong} H^l_{\mathrm{dR},\mathbb{C}}(M)$$

and for and for $p+q=l$

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2,l/2)} H^{p-r,q-r}_{\mathrm{prim}}(M) \xrightarrow{\cong} H^{p,q}(M) \ .$$

Corollary 6.48.

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2,l/2)} \mathcal{H}^{l-2r}_{\mathrm{prim}}(M) \xrightarrow{\cong} \mathcal{H}^l(M)$$

and for and for $p+q=l$

$$\oplus_r L^r : \bigoplus_{r=0}^{\min(n-l/2,l/2)} \mathcal{H}^{p-r,q-r}_{\mathrm{prim}}(M) \xrightarrow{\cong} \mathcal{H}^{p,q}(M) \ .$$

[Hel01][KN96a][KN96b][Voi07][Kob95]

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