Differential Geometry

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1 Prerequisites - what do participants know?

topological spaces

- Hausdorff
- second countable
- basis of topology
- compact subset

diffential calculus in many variables

- differentiability, partial derivatives
- Schwarz Lemma
- implicit function theorem
- submanifolds

DGL

- vector fields on $\mathbb{R}^n$

- existence, uniqueness

- dependence of parameters and initial conditions

- flows

tensor algebra for vector spaces

- $V \otimes W$

- $S^2(V)$

- $\Lambda^3 V^*$

- $SO(n)$,

differential forms

- de Rahm

- integration of Stokes?

mathematical language

- category

- functor

- cartesian product

physics:

- lagrange and Hamilton formalism for classical mechanics

- electro-magnetism, Maxwell

2 Smooth manifolds

2.1 Topological and smooth manifolds manifolds

2.1.1 Topological notions

M - topological space:

consider following conditions:

- Hausdorff

– unicity of limits

Example 2.1. A non-Hausdorff space

form push-out

$$\begin{array}{ccc} (-\infty, 0) & \xrightarrow{\text{incl}} & \mathbb{R} \\ \downarrow{\scriptstyle \text{incl}} & & \downarrow \\ \mathbb{R} & \longrightarrow & M \end{array}$$

every $x \geq 0$ gives rise to x_+ and x_- in M

- $(-\frac{1}{n})_n$ has two limits 0_+ and 0_-

- 0_+ and 0_- can not be separated by opens

-M is not Hausdorff

- but locally homeomorphic to $\mathbb{R}$

□

- regular

– can separate points from closed subsets

- paracompact: Every covering $\mathcal{U} = (U_i)_{i \in I}$ of M has locally finite subcovering

– locally finite: Every m in M admits open nbhd $m \in U \subseteq M$ such that $\{i \in I \mid U \cap U_i \neq \emptyset\}$ is finite.

— this is stronger then to require: $\{i \in I \mid x \in U_i\}$ is finite for every x

– paracompact implies existence of continuous partitions of unity

- second countable: M has a countable base of topology.

– can work with sequences instead of nets in order to define closures or check continuity of functions

– if M is locally compact and second countable, then it admits an exhaustion by compact subsets

Example 2.2. a (non)second countable space

$\bigsqcup_{i \in I} \mathbb{R}$ is second countable if and only if I is countable. □

Proposition 2.3 (Urysohn's metrization theorem)**.** *The following conditions on M are equivalent:*

1. *M is paracompact, second-countable regular space.*
2. *M is metrizable.*

will combine paracompact, second-countable regular by saying metrizable

2.1.2 Locally euclidean spaces and topological manifolds

general principle: some conditions holds locally, if every point admits a nbhd on which this condition holds

call the spaces $\mathbb{R}^n$ for $n \geq 0$ euclidean spaces

M - a topological space

Definition 2.4. *M is locally euclidean if every m in M admits an open nbhd $m \in U \subseteq M$ such that U is homeomorphic to an euclidean space.*

Example 2.5. $\mathbb{R}^n$ is locally euclidean: take $\mathbb{R}^n$ as neigbourhood. □

Lemma 2.6. *An open subset of $\mathbb{R}^n$ is is locally euclidean.*

Proof. $V \subseteq \mathbb{R}^n$ open

- can not take $\mathbb{R}^n$

$x \in V \subseteq \mathbb{R}^n$

- choose $\epsilon > 0$ such that $U := B(x, \epsilon) \subseteq V$ (open ball)

- there exists homeomorphism $B(x, \epsilon) \to \mathbb{R}^n$

– $y \mapsto \phi(\|y - x\|)(y - x)$

– $\phi : [0, \epsilon) \to [0, \infty)$ continuous, monotoneous surjective, e.g. $t \mapsto \frac{t}{\epsilon - t}$

□

M - locally euclidean, $m \in M$,

- $\phi : U \to \mathbb{R}^n$ homeomorphism for neighbourhood U of m

- define the dimension of M at m by $\dim_m(M) := n$

Proposition 2.7. *For every point m in M the number $\dim_m(M)$ is well-defined.*

Proof. must show that it does not depend on choice of homeomorphism

- $\phi' : U' \to \mathbb{R}^{n'}$ a second choice

- get homeomorphism $\phi'\phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ between opens of euclidean spaces

- apply

Theorem 2.8 (invariance of the dimension)**.** *If an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^{n'}$, then $n = n'$*

- this is usually shown in an algebraic topology course using homology

□

Corollary 2.9. *The function $m \mapsto \dim_m(M)$ is locally constant.*

if it is constant, then its value is called the dimension of M

Definition 2.10. *M is a topological manifold if if is metrizable and locally euclidean.*

Definition 2.11. *A morphism between topological manifolds is just a continuous map.*

get category $\mathbf{Mf}^{\mathrm{top}}$ of topological manifolds and continuous maps

- it is not easy to provide examples of topological manifolds which do not come from smooth ones

- therefore no specific examples here

2.1.3 Smooth manifolds

M - topological manifold

- a smooth structure on M is an additional datum

- a topological chart is pair (U, ϕ) of

– $U \subseteq M$ open

– $\phi : U \to \mathbb{R}^n$ (for some n) homeomorphism on image

- $\mathcal{A}^{\mathrm{top}} := \{(U, \phi)\}$ - set of topopogical charts

- since M is topological manifold: $\bigcup_{(U,\phi)\in\mathcal{A}^{\mathrm{top}}} U = M$

Definition 2.12. *A subset $\mathcal{A}$ of $\mathcal{A}^{\mathrm{top}}$ is an atlas if $\bigcup_{(U,\phi)\in\mathcal{A}} U = M$.*

- $(U, \phi), (U', \phi') \in \mathcal{A}^{\mathrm{top}}$

- define transition function: $\phi'\phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$

- is homeomorphism between open subsets of euclidean spaces by construction

Definition 2.13. *A subset $\mathcal{A}$ of $\mathcal{A}^{\mathrm{top}}$ is called smooth if all transition functions between charts in $\mathcal{A}$ are smooth.*

Note that atlasses on M from a poset w.r.t. inclusion

Definition 2.14. *A smooth structure on M is a maximal smooth atlas.*

Lemma 2.15. *Every smooth atlas is contained in a uniquely determined maximal one.*

Proof. $\mathcal{A}$ - smooth atlas

Existence:

- call (U, ϕ) in $\mathcal{A}^{\text{top}}$ compatible with $\mathcal{A}$ if $\mathcal{A} \cup \{(U, \phi)\}$ is compatible

- show: if $\mathcal{A}'$ is smooth, $\mathcal{A} \subseteq \mathcal{A}'$ and (U, ϕ) compatible with $\mathcal{A}$, then also with $\mathcal{A}'$

– must check that $\phi' \phi^{-1}$ is smooth for all $(U', \phi') \in \mathcal{A}'$

— consider $m \in U \cap U'$

— consider chart (V, ψ) in $\mathcal{A}$ at m

— factorize as $(\phi' \psi^{-1})(\psi \phi^{-1})$ - is defined near $\phi(m)$

— get smoothness of $\phi' \phi^{-1}$ near m

- let $\bar{\mathcal{A}}$ consist of all (U, ϕ) which are compatible with $\mathcal{A}$

- conclude: $\bar{\mathcal{A}}$ is smooth atlas

– $\bar{\mathcal{A}}$ is maximal, since it already contains all charts which could possibly added

unicity:

- let $\bar{\mathcal{A}}'$ is any maximal smooth atlas containing $\mathcal{A}$

- then $\bar{\mathcal{A}}' \cup \bar{\mathcal{A}}$ is smooth

– by maximality conclude $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$ □

we say that $\mathcal{A}$ generates the smooth structure $\bar{\mathcal{A}}$

Definition 2.16. *A smooth manifold is a pair $(M, \mathcal{A})$ of a topological manifold with a smooth structure.*

- we use maximal atlas in order to have a good notion of equality of manifolds

- in order to describe a manifold it suffices to provide any generating smooth atlas

Definition 2.17. *A smooth map between smooth manifolds $(M, \mathcal{A}) \to (M', \mathcal{A}')$ is a continuous map such that composition $\phi' f \phi^{-1} : \phi(f^{-1}(U') \cap U) \to \phi'(U')$ is smooth for every pair of charts $(U, \phi) \in \mathcal{A}$ and $(U', \phi') \in \mathcal{A}'$.*

Remark 2.18. It suffices to check the condition on f for charts in generating atlasses.

Exercise! □

get category **Mf** of smooth manifolds and smooth maps

have forgetful functor $\mathbf{Mf} \to \mathbf{Mf}^{\mathrm{top}}$

Example 2.19.

$\mathbb{R}^n$

- generating atlas $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$

any open subset $U \subseteq \mathbb{R}^n$

- generating atlas $(U, U \to \mathbb{R}^n)$

morphisms between these examples are smooth maps in the usual sense

□

Example 2.20. open subsets of smooth manifolds are smooth manifolds □

M - smooth manifold

Definition 2.21. *A smooth function on M is a morphism $M \to \mathbb{R}$.*

- the smooth functions on M form the $\mathbb{R}$-algebra $C^\infty(M)$

Definition 2.22. *A curve in M is a morphism $\gamma : I \to M$ with I an open interval in $\mathbb{R}$.*

2.2 Examples and constructions of smooth manifolds

2.2.1 Regular submanifolds

$U \subseteq \mathbb{R}^n$ open

$g : U \to \mathbb{R}^k$ smooth

u in U

- have differential $dg(u) : \mathbb{R}^n \to \mathbb{R}^k$, linear map

Definition 2.23. *g is regular in u if $dg(u)$ is surjective.*

consider subspace $M \subseteq \mathbb{R}^n$

- is a metrizable topological space

Definition 2.24. *M is a regular if for every m in M there exists a neighbourhood U of m and a smooth function $g : U \to \mathbb{R}^k$ such that $M \cap U = g^{-1}(0)$ and g is regular at M.*

call g a defining function of M at m

- set $T_m M := \ker(dg(m))$ - linear subspace fo $\mathbb{R}^n$

Remark 2.25. $T_m M$ does note depend on choice of defining function g of M at m

Exercise! □

Theorem 2.26 (Implizit function theorem)**.** *There exist open neigbourhoods $0 \in V \subseteq T_m M$ and $m \in U' \subseteq U$ such that:*

1. *For every v in V there exists a unique point $\psi(v)$ in $T_m M^\perp$ such that $v + \psi(v) + m \in M \cap U'$.*
2. *$\psi : V \to T_m M^\perp$ is smooth.*

the map $V \ni v \mapsto v + \psi(v) + m \in W := U' \cap M$ homeomorphism.

- inverse: $W \ni \phi(x) := x \mapsto \mathrm{pr}_{T_m M^\perp}(x - m)$

take $\mathcal{A} := \{(W, \phi)\}$ - set of all charts defined in this way

- domains cover M

Corollary 2.27. *M is topological manifold.*

Proposition 2.28. *$\mathcal{A}$ is a smooth atlas.*

Proof. is an atlas by construction

- $\mathcal{A}$ is a smooth:

- consider transition function

$v \mapsto \phi' \phi^{-1}(v) = \mathrm{pr}'_{T_{m'} M^\perp}(v + \psi(v) + m - m')$ - this map is obviously smooth □

Definition 2.29. *Call M with the smooth manifold structure constructed above a regular submanifold*

note that $\dim_m(M) = n - k$ (when $g : U \to \mathbb{R}^k$ is defining at m)

Example 2.30. detection of smooth maps into and from a regular submanifold

$f : N \to M$ is smooth iff $f : N \to M \to \mathbb{R}^n$ is smooth

$f : M \to N$ is smooth if it extends to a smooth function $\tilde{f} : \mathbb{R}^n \to N$

Exercise!

2.2.2 Explicit examples of regular submanifolds

$S^n \subset \mathbb{R}^{n+1}$ defined by $f(x) = \|x\|^2 - r$

the following examples have group structures

$GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ - open subset

$SL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ - defined by $A \mapsto \det(A) - 1$

$O(n) \subseteq \mathbb{R}^{n^2}$ - defined by $A \mapsto A^t A \in S^2(\mathbb{R}^n) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$, $\dim(O(n)) = \frac{n(n-1)}{2}$

$SO(n) \subseteq O(n)$ open

$U(n) \subseteq \mathbb{R}^{2n^2}$ - defined by $A \mapsto A^* A \in \{\textit{hermitean matrices}\} \cong \mathbb{R}^{n(n-1)+n}$, $\dim(U(n)) = n^2$

2.2.3 Cartesian products

Proposition 2.31. *The category* **Mf** *admits cartesian products.*

Proof. $M, M' \in$ **Mf**

- consider topological space $M \times M'$

- is topological manifold

– a product of metrizable spaces is metrizable (take product metric)

- $M \times M'$ is locally euclidean

– $(m, m') \in M \times M'$

– (U, ϕ) chart at m, (U', ϕ') chart at m'

– $(U \times U', \phi \times \phi')$ is a chart of $M \times M'$ at (m, m')

— call this chart product chart

define smooth structure on $M \times M'$ as generated by product charts of charts of the smooth structures

- check: this is compatible atlas

check

$p : M \times M' \to M$ and $p' : M \times M' \to M'$ are smooth

- check smoothness using product charts in domain

- use $\phi_1 p(\phi_0 \times \phi')^{-1} = \phi_1 \phi_0^{-1}$

check that $(M \times M', p, p')$ satisfies the universal property

$$\mathrm{Hom}_{\mathbf{Mf}}(N, M \times M') \overset{(p,p')}{\to} \mathrm{Hom}_{\mathbf{Mf}}(N, M) \times \mathrm{Hom}_{\mathbf{Mf}}(N, M')$$

is bijection

- injective:

– is clear since we have cartesian products of underlying sets

- surjective:

– $f : N \to M$, $f' : N \to M'$ given

- $f \times f' : N \to M \times M'$ is continuous (since work with cartesian product in topological spaces)

- check smoothness using product charts:

– $(\phi_1 \times \phi_1')(f \times f')(\phi_0 \times \phi_0')^{-1} = (\phi_1 f \phi_0^{-1}, \phi_1' f' \phi_0'^{,-1})$ is smooth

□

Example 2.32. $\mathbb{R}^n \times \mathbb{R}^{n'} \cong \mathbb{R}^{n+n'}$ (as manifolds)

$S^1 \times \cdots \times S^1 =: T^n$ (n factors) is called the n-torus

$M \subseteq \mathbb{R}^n$ regular, $M' \subseteq \mathbb{R}^{n'}$ regular, then $M \times M' \subseteq \mathbb{R}^{n+n'}$ is regular $\square$

2.2.4 Lie groups

existence of cartesian products in a category $\Rightarrow$ can talk about groups in this category:

general:

- $\mathcal{C}$ category with cartesian products

- $*$ - empty cartesian product

– $\mathrm{pr}_C : * \times C \overset{\cong}{\to} C$ - will often be used implicitly

idea: write group axioms in terms of diagrams of maps

Definition 2.33. *A group in $\mathcal{C}$ is a triple $(C, \mu : C \times C \to C, e : * \to C)$ such that*

$$\begin{array}{ccc} C \times C \times C & \xrightarrow{\mu(\mu \times \mathrm{id}_C)} & C \times C \\ \downarrow{\scriptstyle \mu(\mathrm{id}_C \times \mu)} & & \downarrow{\scriptstyle \mu} \\ C \times C & \xrightarrow{\mu} & C \end{array} \qquad \textit{(associativity)}$$

$$\begin{array}{ccccc} C & \xrightarrow{e \times \mathrm{id}_C} & C \times C & \xleftarrow{\mathrm{id}_C \times e} & C \\ & {\scriptstyle \mathrm{id}_C} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \mathrm{id}_C} & \\ & & C & & \end{array} \qquad \textit{unit}$$

commute and the shear map $s : C \times C \overset{(\mathrm{id}_C, \mu)}{\to} C \times C$ is an isomorphism.

- shear maps s encodes inverses $I : C \overset{\mathrm{id}_C \times e}{\to} C \times C \overset{s^{-1}}{\to} C \times C \overset{\mathrm{pr}_2}{\to} C$

– advantage of using shear map: being a group is a property of (C, μ, e) - no additional datum required

groups in **Set** are usual groups

groups in **Top** are topological groups

specialize to **Mf**

in **Mf**: $* \cong \mathbb{R}^0$

- $\mathrm{Hom}(*, M) \cong$ underlying set of M

Definition 2.34. *A group in* **Mf** *is called a Lie group.*

Example 2.35. $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$, $U(n)$, all with matrix multiplication, are Lie groups and unit given by identity matrix (interpreted as map $* \to M$)

- matrix multiplication $\mathrm{End}(\mathbb{R}^n) \times \mathrm{End}(\mathbb{R}^n) \to \mathrm{End}(\mathbb{R}^n)$ is smooth and associative, compatible with identity relation

- restricts to the structures on the submanifolds

- shear map is an isomorphism:

– use that $A \mapsto A^{-1}$ is smooth on $GL(n,\mathbb{R})$

— either by formula involving determinants of adjuncts

— or by inverse function theorem

– inverse of shear map $(A, B) \mapsto (A, AB)$ is $(A, B) \mapsto (A, A^{-1}B)$

□

Example 2.36. $\mathbb{R}^n$ with $+$ is a Lie group □

if G is Lie group, then $I : G \to G$, $g \mapsto g^{-1}$ is smooth

actions:

general: $\mathcal{C}$ - category with cartesian products

- (G, μ, e) a group in $\mathcal{C}$

- C an object

Definition 2.37. *An action of G on C is a map $a : G \times C \to C$ such that*

$$\begin{array}{ccc} G \times G \times C & \xrightarrow{\mathrm{id}\times a} & G \times C \\ \downarrow{\scriptstyle \mu\times \mathrm{id}_C} & & \downarrow{\scriptstyle a} \\ G \times C & \xrightarrow{a} & C \end{array} \qquad \textit{associativity}$$

and

$$\begin{array}{ccc} C & \xrightarrow{e\times \mathrm{id}_C} & G\times C \\ & {\scriptstyle \mathrm{id}_C}\searrow & \downarrow{\scriptstyle a} \\ & & C \end{array} \qquad \textit{unit}$$

commute.

Example 2.38. G acts on itself with $a = \mu$ □

Example 2.39. in **Mf**:

$GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication

$O(n)$ acts on S^{n-1} □

2.3 Tangent vectors

idea:

- a tangent vector on a manifold M at m is a direction of an infinitesimal curve starting at m

- can consider the derivative of functions in this direction

- axiomatization of the properties of this derivative $\Rightarrow$ notion of a derivation

- will turn this idea up-side-down and use derivations in order to to define tangent vectors

2.3.1 Derivations

- k - a field

- consider commutative unital k-Algebras (e.g. k)

Definition 2.40. *An augmented k-algebra is a pair (A, e) of a k-algebra A with a homomorphism $e : A \to k$.*

A homomorphism of augmented k-algebras $\phi : (A, e) \to (A', e')$ is a homomorphism of k-algebras $\phi : A \to A'$ such that $e'\phi = e$.

Example 2.41. M a manifold

m in M

- $C^\infty(M)$ - is a $\mathbb{R}$-algebra

- $\mathrm{ev}_m : C^\infty(M) \to \mathbb{R}$ given by $\mathrm{ev}_m(f) := f(m)$ is an augmentation

$F : M \to M'$ smooth map of manifolds,

- $m' := F(m)$

- get homomorphism $F^* : (C^\infty(M'), \mathrm{ev}_{m'}) \to (C^\infty(M), \mathrm{ev}_m)$ of augmented $\mathbb{R}$-algebras

□

(A, e) - augmented k-algebra

Definition 2.42. *A derivation of (A, e) is a k-linear map $X : A \to k$ such that for all a, b in A we have $X(ab) = X(a)e(b) + e(a)X(b)$.*

write $\mathrm{Der}(A, e)$ for k-vector space of derivations of (A, e)

Example 2.43. partial derivatives are derivations

consider $C^\infty(\mathbb{R}^n)$ with augmentation ev_0

$i \in \mathbb{N}$

- $\partial_i(0) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ given by $f \mapsto (\partial_i f)(0)$ is a derivation □

Example 2.44. derivations annihilate constants

(A, e) - augmented k-algebra

for X in $\mathrm{Der}(A, e)$

- we have $X(1_A) = 0$:

– $X(1_A) = X(1_A^2) = 2X(1_A)e(1_A) = 2X(1_A)$

unit: $k \to A$, $\lambda \mapsto \lambda 1_A$

- these elements are called the constants

- $e(\lambda 1_A) = \lambda$

- by linearity: $X(\lambda 1_A) = 0$ □

consider homomorphism $\phi : (A, e) \to (A', e')$ of augmented k-algebras it induces a homomorphism

$\mathrm{Der}(\phi) : \mathrm{Der}(A', e') \to \mathrm{Der}(A, e)$ given by $\mathrm{Der}(\phi)(X)(a) := X(\phi(a))$

- check:

$$\begin{aligned} \mathrm{Der}(\phi)(X)(ab) &= X(\phi(ab)) = X(\phi(a))e'(\phi(b)) + e'(\phi(a))X(\phi(b)) \\ &= \mathrm{Der}(\phi)(X)(a)e(b) + e(a)\mathrm{Der}(\phi)(X)(b) \end{aligned}$$

- Der is contravariant functor from augemented k-algebras to k-vector spaces

M - a manifold

- m in M

- consider poset $\mathcal{U}_m$ of open neighbourhoods of M

- for $U \subseteq V$ in $\mathcal{U}_m$ get restriction map $(C^\infty(V), \mathrm{ev}_m) \to (C^\infty(U), \mathrm{ev}_m)$

Definition 2.45. *The augmented $\mathbb{R}$-algebra of germs at m of smooth functions on M is defined by $(C^\infty_m(M), \mathrm{ev}_m) := \mathrm{colim}_{U \in \mathcal{U}_m^{\mathrm{op}}}(C^\infty(U), \mathrm{ev}_m)$ in augmented $\mathbb{R}$-algebras.*

we will work with the following explicit description:

- an element of $C^\infty_m(M)$ is represented by a pair (V, f) of $V \in \mathcal{U}_m$ and $f \in C^\infty(M)$

- if $U \subseteq V$ in $\mathcal{U}_m$, then $(U, f_{|U})$ represents the same element

for the moment we write $[V, f]$ for the element represented by (V, f)

- the algebra structure is defined as follows:

- $[V, f] + \lambda[V', f'] = [V \cap V', f_{|V\cap V'} + \lambda f'_{|V\cap V'}]$

- $[V, f] \cdot [V', f'] = [V \cap V', f_{|V\cap V'} f'_{|V\cap V'}]$

Check: well-definedess

augmentation $\mathrm{ev}_m : C^\infty_m(M) \to \mathbb{R}$: $\mathrm{ev}_m([V, f]) = f(m)$

Check: well-definedess

properties

1. $C^\infty(M) \to C^\infty_m(M), \qquad f \mapsto [M, f]$ is surjective

   Exercise!

2. $m \in U \subseteq M$ open:

   - restriction $C^\infty_m(M) \to C^\infty_m(U)$ is isomorphism preserving augmentation

   Exercise!

3. $U \subseteq M$ open, $m \in U$,

   $U' \subseteq M'$ open, $\phi : U \to U'$ isomorphism

   - $\phi^* : (C^\infty_{\phi(m)}(U'), \mathrm{ev}_{\phi(m)}) \to (C^\infty_m(U), \mathrm{ev}_m)$ is isomorphism

   Exercise!

from now on instead of $[U, f]$ write f (the precise domain of f is irrelevant)

$n := \dim(M)$

- conclude using a chart with $\phi(m) = 0$: $(C^\infty_m(M), \mathrm{ev}_m) \cong (C^\infty_0(\mathbb{R}^n), \mathrm{ev}_0)$

Example 2.46. have derivation $\partial_i(0) : C^\infty_0(\mathbb{R}^n)$ is defined by $\partial_i(0)(f) := (\partial_i f)(0)$

Check: is well-defined

Proposition 2.47. *The derivations* $(\partial_i(0))_{i=1,\ldots,n}$ *form a basis of* $\mathrm{Der}(C^\infty_0(\mathbb{R}^n), \mathrm{ev}_0)$.

Proof.

$(\partial_i(0))_{i=1,\ldots,n}$ is linearly independent:

- assume that $\sum_{i=1}^n \lambda_i \partial_i(0) = 0$

– for every j:

— $0 = (\sum_{i=1}^n \lambda_i \partial_i(0))(x^j) = \sum_{i=1}^n \lambda_i (\partial_i x^j)_{|x=0} = \lambda_j$

$(\partial_i(0))_{i=1,\ldots,n}$ spans:

- X in $\mathrm{Der}(C^\infty_0(\mathbb{R}^n))$ given

– set $\mu_i := X(x^i)$

– set $Y := \sum_{i=1}^n \mu_i \partial_i(0)$

- we will show that $X = Y$

– consider $f \in C_0(\mathbb{R}^n)$

– Taylor: there exists $g_i \in C_0^\infty(\mathbb{R}^n)$ with $g_i(0) = 0$ such that

$$f = f(0) + \sum_{i=1}^{n} (\partial_i f)(0) x^i + \sum_{i=1}^{n} x^i g_i$$

calculate:

$$\begin{aligned} X(f) &= X(f(0)) + X(\sum_{i=1}^{n} (\partial_i f)(0) x^i) + X(\sum_{i=1}^{n} x^i g_i) \\ &= \sum_{i=1}^{n} (\partial_i f)(0) X(x^i) + \sum_{i=1}^{n} (X(x^i) g_i(0) + x^i(0) X(g^i)) \\ &= \sum_{i=1}^{n} (\partial_i f)(0) \mu_i \\ &= Y(f) \end{aligned}$$

□

M smooth, $m \in M$

Corollary 2.48. $\dim_m(M) = \dim \operatorname{Der}(C_m^\infty(M), \mathrm{ev}_m)$.

Example 2.49. consider germs of continuous functions $C_0(\mathbb{R}^n)$

- then $\operatorname{Der}(C_0(\mathbb{R}^n), \mathrm{ev}_0) \cong 0$

– consider X in $\operatorname{Der}(C_0(\mathbb{R}^n), \mathrm{ev}_0)$

– $f \in C_0(\mathbb{R}^n)$

– $g := \sqrt[3]{f - f(0)} \in C_0(\mathbb{R}^n)$

– $f = f(0) + g^3$

– $X(f) = X(f(0)) + X(g^3) = 0 + 3g(0)^2 X(g) = 0$

this shows: the concept of tangent space using derivations does not extend to topological manifolds

□

2.3.2 Tangent vectors

Definition 2.50. *The vector space* $T_mM := \mathrm{Der}(C^\infty_m(M), \mathrm{ev}_m)$ *is called the tangent space of* M *at* m*. Its dual* T^*_mM *is called the cotangent space of* M *at* m*.*

m in M

- $\dim T_mM = \dim_m(M) = \dim T^*_mM$

$f \in C^\infty_m(M)$

- defines element $df(m) \in T^*_mM$ by $df(m)(X) := X(f)$ for all X in T_mM

Definition 2.51. $df(m) \in T^*_mM$ *is called the derivative of* f *at* m*.*

note Leibnitz rule:

$$d(ff')(m) = df(m)f'(m) + f(m)df'(m)$$

- verification:

$$d(ff')(m)(X) = X(ff') = X(f)f'(m) + f(m)X(f') = df(m)(X)f'(m) + f(m)df'(m)(X)$$

(U, ϕ) - a chart

Definition 2.52. *The components* $x^i : U \to \mathbb{R}$ *of* ϕ *(i.e.,* $\phi = (x^1, \ldots, x^n)$*) are called the coordinate functions on* U *associated to* ϕ*.*

Corollary 2.53. $(dx^i(m))_{i=1,\ldots,n}$ *is a basis of* T^*_mM

we let $(\partial_i(m))_{i=1,\ldots,n}$ be the dual basis of T_mM

- i.e.: $\partial_i(m)(x^j) = \delta_i{}^j$

- every tangent vector X in T_mM can uniquely be written as $X = \sum_{i=1}^n \mu_i \partial_i(m)$

– must set $\mu_i := X(x^i)$

- note: these bases of T_mM and T_m^*M depend on the choice of the chart (U, ϕ)

$F : M \to M'$ morphism of manifolds

set $m' := F(m)$

- get $F_m^* : (C_{m'}^\infty(M), \mathrm{ev}_{m'}) \to (C_m^\infty(M), \mathrm{ev}_m)$ - pull-back

- homomorphism of augmented $\mathbb{R}$-algebras

Definition 2.54. *The differential of F at m is the linear map $TF(m) := \mathrm{Der}(F_m^*) : T_mM \to T_{m'}M'$.*

- often also denoted by $dF(m)$ or $DF(m)$

- explicitly: for $X \in T_mM$ the derivation $TF(m)(X)(f) := X(F_m^* f)$

- note: F must only be defined near m in order to get $TF(m)$

- observe chain rule: for $F' : M' \to M''$:

$$T(F'F)(m) = TF'(F(m))TF(m) : T_mM \to T_{m''}M''$$

Exercise!

$f \in C^\infty(M)$

$df(m) = \mathrm{can} \circ df(m)$

$F : M' \to M,\ F(m') = m$

chain rule implies:

Lemma 2.55. *We have $d(F^* f)(m') = df(m)TF(m')$*

Proof. for X' in $T_{m'}M'$

$$
\begin{aligned}
d(F^*f)(m')(X') &= X'(F^*f) \\
&= TF(m')(X')(f) \\
&= df(m)TF(m')(X')
\end{aligned}
$$

□

V - f.d. vector space

- v in V

- as a consequence of Proposition 2.47:

Corollary 2.56. *We have a canonical identification* $\mathrm{can} : V \xrightarrow{\cong} T_vV$ *which sends* X *in* V *to the derivation* $f \mapsto \frac{d}{dt}_{|t=0} f(v + tX)$.

we often do not write can in formulas, be careful

consider map $L_w : V \to V$, $L_w(v) := v + w$ - translation by w

- this commutes:

$$
\begin{array}{ccc}
V & = & V \\
\cong \downarrow \mathrm{can} & & \cong \downarrow \mathrm{can} \\
T_vV & \xrightarrow{dL_w(v)} & T_{v+w}V
\end{array}
$$

2.3.3 Change of coordinates

(U, ϕ) - a chart of M at m

can consider ϕ as isomorphism $\phi : U \to \phi(U)$

- get isomorphism $T\phi(m) : T_mM \to T_{\phi(m)}\mathbb{R}^n \cong \mathbb{R}^n$ (canonical iso implicitly used)

– characterized by $T\phi(m)(\partial_i(m)) = e_i$ (standard basis vector) for all i

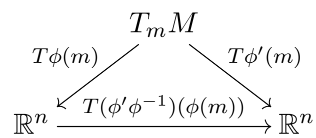
- (U', ϕ') second chart

- have $T(\phi'\phi^{-1})(\phi(m)) \in GL(n, \mathbb{R})$

– Jacobi matrix of $\phi'\phi^{-1}$ at $\phi(m)$

– chain rule for $\phi' = (\phi'\phi^{-1}) \circ \phi$ says:

Corollary 2.57.



denote charts by ϕ instead of (U, ϕ)

set $\rho_{\phi',\phi}(m) := T(\phi'\phi^{-1})(\phi(m))$

- is smooth function $U \cap U' \to GL(n, \mathbb{R}^n)$

- satisfy the cocyle relations:

– $\rho_{\phi,\phi} = 1$

– $\rho_{\phi'',\phi'}\rho_{\phi',\phi} = \rho_{\phi'',\phi}$ (product in $GL(n, \mathbb{R})$, on $U \cap U' \cap U''$))

— a consequence: $\rho^{-1}_{\phi',\phi} = \rho_{\phi,\phi'}$ (inverse in $GL(n, \mathbb{R})$)

2.3.4 geometric tangent vectors at regular submanifolds

$M \subseteq \mathbb{R}^n$ - regular submanifold

- define $T^{\mathrm{geom}}_m M := \ker(dg(m))$ for defining function g of M at m

- call this geometric tangent space

a curve in M at m is a curve $\gamma : I \to M$ with $0 \in I$ and $\gamma(0) = m$

- interpret $(\partial_t)_{|t=0}\gamma$ as vector in $\mathbb{R}^n$

Lemma 2.58. *For every X in $T^{\mathrm{geom}}_m M$ there exists a curve γ in M at m such that $(\partial_t)_{|t=0}\gamma = X$.*

Proof. apply Implicit Function Theorem 2.26

get

- suitable neighbourhood of $0 \in V \subseteq T_m^{\text{geom}} M$

- map $\psi : V \to T_m M^\perp$ such that $v + \psi(v) + m$ is parametrization of M near m

claim: $d\psi(0) = 0$

- $g(v + \psi(v) + m) \equiv 0$ implies

– $d_{T_m M} g(m) + d_{T^m M^\perp} g(m) d\psi(0) = 0$

– $d_{T^m M^\perp} g(m) d\psi(0) = 0$ since $d_{T_m M} g(m) = 0$ by definition of $T_m M$

– $d_{T^m M^\perp} g(m)$ is isomorphism by regularity of g at m

— conclude $d\psi(0) = 0$

- define $\gamma(t) := tX + \psi(tX) + m$

- then

$$(\partial_t)_{|t=0}\gamma = X + d\psi(0)(X) = X$$

□

M manifold, m in M (not necessarily submanifold)

- a curve γ in M at m induces a tangent vector $\gamma'(0) := T\gamma(\partial_1(0)) \in T_m M$

Proposition 2.59. *There is an isomorphism $T_m^{\text{geom}} M \cong T_m M$ uniquely determined by the condition that $(\partial_t)_{|t=0}\gamma$ is sent to $\gamma'(0)$ for any curve in M at m.*

Proof. observe:

- if γ_0, γ_1 are two curves in M at m and $(\partial_t)_{|t=0}\gamma_0 = (\partial_t)_{|t=0}\gamma_1$, then also $\gamma_0'(0) = \gamma_1'(0)$.

– $f \in C^\infty(M)$

– has smooth extension $\tilde{f}$ to nbhd

– chain rule

– $\gamma = \gamma_0, \gamma_1$

– $df(m)(\gamma'(0)) = \partial_1(0)(f\gamma) = \frac{d}{dt}_{|t=0} f(\gamma(t)) = \frac{d}{dt}_{|t=0} \tilde{f}(\gamma(t)) = d\tilde{f}(m)((\partial_t)_{|t=0}\gamma_i)$

– use: definition of derivative $df(m)$, definition of partial derivative $\partial_1(0)$, that $\tilde{f}$ extends f, and classical chain rule for functions between euclidean spaces

– implies $df(\gamma_0'(0)) = df(\gamma_1'(0))$

– f arbitrary (note that $C^\infty(M) \to C^\infty_m(M)$ is surjective): $\gamma_0'(0) = \gamma_1'(0)$

define map $\kappa : T_m^{\mathrm{geom}}M \to T_m M$ such that it sends X in $T_m^{\mathrm{geom}}M$ to $\gamma'(0)$ for any curve γ in M at m with $(\partial_t)_{|t=0}\gamma = X$

- formula: $\kappa(X)(f) = d\tilde{f}(m)(X)$

- is linear in X, hence κ is linear

κ is isomorphism:

- $\mathrm{pr}_{T_m^{\mathrm{geom}}M} : M \to T_m^{\mathrm{geom}}M$ - orthogonal projection

- calculate: $T\mathrm{pr}_{T_m^{\mathrm{geom}}M}(m)(\kappa(X)) = (\partial_t)_{|t=0}\mathrm{pr}_{T_m^{\mathrm{geom}}M}(tX + \psi(tX) + m) = X$

for dimension reasons κ and $T\mathrm{pr}_{T_m^{\mathrm{geom}}M}(m)$ are inverse to each other

□

2.3.5 Discussion

$f \in C^\infty(M)$

- get $m \mapsto df(m) \in T_m^*M$

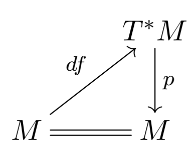
- want to say that this depends smoothly on m

– how?

form set $T^*M := \bigsqcup_{m \in M} T_m^*M$

- have canonical map $p : T^*M \to M$

- want to interpret df as a map $df : M \to T^*M$, $m \mapsto df(m)$ such that $p\,df = \mathrm{id}_M$



must equip T^*M with a suitable manifold structure

consider family of derivations $X = (X(m))_{m \in M}$, $X(m) \in T_m M$

- say: X is a smooth vector field if $m \mapsto X(m)(f)$ is smooth for every f in $C^\infty(M)$

– how can one formulate this in terms of the family X alone?

form set $TM := \bigsqcup_{m \in M} T_m M$

- have map $p : TM \to M$

- interpret X as map

$$\begin{array}{ccc} & & TM \\ & \overset{X}{\nearrow} & \downarrow p \\ M & = & M \end{array}$$

- must equip TM with manifold structure

Example 2.60. $T^{\mathrm{geom}}M$ as regular submanifold

$M \subseteq \mathbb{R}^n$ - regular submanifold

- define $T^{\mathrm{geom}}M := \bigcup_{m \in M} \{m\} \times T^{\mathrm{geom}}_m M \subseteq \mathbb{R}^{2n}$ - just a subset

Lemma 2.61. *$T^{\mathrm{geom}}M$ is a regular submanifold.*

Proof. construct local defining functions

$(m, X) \in T^{\mathrm{geom}}M$

- g on U defining function of M near m

– $(g, dg) : (x, \xi) \mapsto (g(m), dg(m)(\xi))$ defines $T^{\mathrm{geom}}M$ on $U \times \mathbb{R}^n$

– check regularity:

– $d(g, dg)(m, X) = \begin{pmatrix} dg(m) & 0 \\ d^2 g(m)(X, -) & dg(m) \end{pmatrix}$

- is surjective since $dg(m)$ is so □

□

2.4 Fibre bundles

2.4.1 Bundles and bundle morphisms

B a manifold (the base)

F - a manifold (typical fibre)

Definition 2.62. *A fibre bundle over B with typical fibre F is a smooth map $\pi : M \to B$ such that there exists:*

1. *$(U_\alpha)_\alpha$ - an open covering of B*
2. *a collection of diffeomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ (called local trivializations) such that*

$$\begin{array}{ccccc} U_\alpha \times F & \xleftarrow{\psi_\alpha} & \pi^{-1}(U_\alpha) & \xrightarrow{\text{incl}} & M \\ \downarrow{\scriptstyle \text{pr}} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ U_\alpha & = & U_\alpha & \xrightarrow{\text{incl}} & B \end{array}$$

commutes.

Example 2.63. the trivial bundle $\text{pr} : B \times F \to B$

- local trivialization is $\psi = \text{id}_{B\times F}$ defined on all of B □

later: $TM \to M$ and $T^*M \to M$ will be fibre bundles with typical fibre $\mathbb{R}^n$

Definition 2.64. *A morphism of fibre bundles is a commutative square*

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \quad .$$

If the lower map is id_B, then we call this a morphism of fibre bundles over B.

2.4.2 Fibre bundles and cocycles

write $U_{\alpha,\beta} := U_\alpha \cap U_\beta$

the local trivializations determine maps (of sets) $\rho_{\alpha,\beta} : U_{\alpha,\beta} \to \text{Aut}_{\mathbf{Mf}}(F)$ such that the following map is smooth

$$U_{\alpha,\beta} \times F \to U_{\alpha,\beta} \times F \ , \quad \psi_\alpha \psi_\beta^{-1}(u, f) = (u, \rho_{\alpha,\beta}(u)(f))$$

- we have cocycle condition

– $\rho_{\alpha,\beta}\rho_{\beta,\gamma} = \rho_{\alpha,\gamma}$ on $U_{\alpha,\beta,\gamma}$ for all α, β, γ

– $\rho_{\alpha,\alpha} \equiv \mathrm{id}_F$

vice versa: a smooth cocycle is a family $\rho = (\rho_{\alpha,\beta})$ of maps $\rho_{\alpha,\beta} : U_{\alpha,\beta} \to \mathrm{Aut}_{\mathbf{Mf}}(F)$ such that

- $(u, f) \mapsto (u, \rho_{\alpha,\beta}(u)(f))$ is smooth

- cocyle conditions are satified

want to construct fibre bundles from cocycles

Example 2.65. B - a manifold of dimension n

$F := \mathbb{R}^n$

$\mathcal{A}$ - the smooth structure of B

- gives covering by domains of smooth charts (U, ϕ)

- get cocyle with values in $GL(n, \mathbb{R}) \subseteq \mathrm{Aut}_{\mathbf{Mf}}(\mathbb{R}^n)$: $\quad \rho_{\phi',\phi} := T(\phi'\phi^{-1})\phi$

the fibre bundle constructed from this data is the tangent bundle TB of B

could consider new cocycle $(\Lambda^3(\rho^{*,-1}_{\alpha,\beta}))_{\alpha,\beta}$ with values in $\mathrm{Aut}(\Lambda^3\mathbb{R}^{n,*})$

- associated fibre bundle is bundle of 3-forms $\Lambda^3 T^*B \to B$

□

Construction 2.66. start with the construction of $\pi : M \to B$ from the following data:

- $(U_\alpha)_\alpha$ an open covering of B

- a smooth cocycle $\rho = (\rho_{\alpha,\beta})$ with values in $\mathrm{Aut}_{\mathbf{Mf}}(F)$

underlying set of M:

$$M := \bigsqcup_{\alpha \in A} U_\alpha \times F/\sim$$

- thereby $(u, f) \in U_\alpha \times F$ and $(u', f') \in U_{\alpha'} \times F$ are equivalent if $u = u'$ and $f' = \rho_{\alpha',\alpha}(u)f$

- is equivalence relation by cocycle condition (check)

- write points in M as $[u,f]_\alpha$

$\pi : M \to B$ sends $[u,f]_\alpha$ to u

- check: is well-defined

local trivializations:

$$\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$$

- $[u,f]_\alpha \mapsto (u,f)$
- check well-defineness:

– for every α: the map $U_\alpha \times F \ni (u,f) \mapsto [u,f]_\alpha \in M$ is injective

– this follows since $\rho_{\alpha,\beta}$ has values in automorphisms

check:

$$\begin{array}{ccccc} U_\alpha \times F & \xleftarrow{\psi_\alpha} & \pi^{-1}(U_\alpha) & \xrightarrow{\text{incl}} & M \\ \downarrow{\scriptstyle \text{pr}} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ U_\alpha & = & U_\alpha & \xrightarrow{\text{incl}} & B \end{array}$$

commutes

check:

$$\psi_\alpha \psi_\beta^{-1}(u,f) = (u, \rho_{\alpha,\beta}(u)(f))$$

define topology on M: minimal such that all ψ_α are continuous

- by definition: $h : X \to M$ continuous if $\psi_\alpha h$ is continuous for all α

claim: ψ_α is a homeomorphism

- ψ_α is bijective amd continuous
- remains to show that ψ_α^{-1} is continuous

– this follows from: $\psi_\beta \psi_\alpha^{-1}$ is continuous for all β

Lemma 2.67. *$f : M \to X$ continuous if $f\psi_\alpha^{-1}$ is continuous for all α*

Proof. $\Rightarrow$: clear

$\Leftarrow$:

U open in X

- must check that $f^{-1}(U)$ is open in M

- consider $m \in f^{-1}(U)$

– chose α s.t. $m \in \pi^{-1}(U_\alpha)$

- since $f\psi_\alpha^{-1}$ is continuous there is open nbhd V of $\psi_\alpha(m)$ such that $f(\psi_\alpha^{-1}(V)) \subseteq U$

- then $\psi_\alpha^{-1}(V)$ is open nbhd of m in $f^{-1}(U)$

conclude: $f^{-1}(U)$ is open

□

π is continuous:

- use $\pi\psi_\alpha^{-1} = \mathrm{pr} : U_\alpha \times F \to U_\alpha$ is continuous for all α

M is Hausdorff

- $m \neq m'$

– if $\pi(m) \neq \pi(m')$

— use B is Hausdorff: find open V, V' in B with: $\pi(m) \in V$, $\pi(m') \in V'$, $V \cap V' = \emptyset$

— then $\pi^{-1}(V)$ and $\pi^{-1}(V')$ separate m and m'

– if $\pi(m) = \pi(m') \in U_\alpha$, $\psi_\alpha(m) = (u, f)$, $\psi_\alpha(m') = (u, f')$, $f \neq f'$

— use that F is Hausdorff: find opens W, W' in F with $f \in W$, $f' \in W'$ and $W \cap W' = \emptyset$

— then $\psi_\alpha^{-1}(U_\alpha \times W)$ and $\psi_\alpha^{-1}(U_\alpha \times W')$ separate m and m'

M is locally euclidean: M is locally a product of topological manifolds

M is second countable:

- can cover B by a countable subcover of the given cover

- F is second countable

Proposition 2.68. *A second countable locally euclidean Hausdorff space is regular and paracompact, hence a topological manifold.*

Exercise: find proof by google

smooth structure:

for every chart (U, ϕ) of B and chart (W, κ) of F define chart $(\phi, \kappa)\psi_\alpha : \psi_\alpha^{-1}((U \cap U_\alpha) \times W) \to \phi(U \cap U_\alpha) \times \kappa(W)$

- these from an atlas

- transition functions are smooth

– given by $(x, v) \mapsto (\phi' \phi^{-1}(x), \kappa'(\rho(\phi^{-1}(x))(\kappa^{-1}(v))))$

equip M with smooth structure generated by this atlas

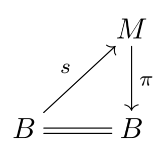
ψ_α is smooth by construction

- check: π is smooth □

2.4.3 Sections

Definition 2.69. *The set of sections of a fibre bundle is defined by*

$$\Gamma(B, M) := \{s \in \mathrm{Hom}_{\mathbf{Mf}}(B, M) \mid \pi s = \mathrm{id}_B\}$$



we now describe sections in terms of the trivializations

consider section $s \in \Gamma(B, M)$

- get family (s_α) with $s_\alpha := \mathrm{pr}_F \psi_\alpha f : U_\alpha \to F$

- (s_α) satisfies: for all α, β: $\rho_{\alpha,\beta}(u)(f_\beta(u)) = f_\beta(u)$ for all u in $U_{\alpha,\beta}$

– we say that (s_α) is compatible

Lemma 2.70. *There is a bijection between the sets:*

1. $\Gamma(B, M)$
2. *compatible familes* (s_α)

Proof. $s \in \Gamma(B, M)$ given:

- get compatible family (s_α) by

– $s_\alpha := \mathrm{pr}_F \psi_\alpha s$

compatible family (s_α) given

- define $s \in \Gamma(B, M)$ by

– $b \mapsto [b, s_\alpha(b)]_\alpha$ for any α with $b \in U_\alpha$

– check using compatibility relation: does not depend on choice of α

– check: s is smooth

check: these constructions are inverse to each other

□

Example 2.71. $\mathrm{pr} : M \times \mathbb{R} \to \mathbb{R}$

$\Gamma(M, M \times \mathbb{R}) \cong C^\infty(M)$

$s \mapsto (m \mapsto \mathrm{pr}_\mathbb{R} s(m))$

$f \mapsto (m \mapsto (m, f(m))$ □

Example 2.72. - associated to cocycle $(\Lambda^n T(\phi' \phi^{-1})^{-1,*})\phi$:

$\Omega^n(M) := \Gamma(M, \Lambda^n T^* M)$

- n-forms on M

have map $d : C^\infty(M) \to \Omega^1(M)$

- describe locally:

- $f \mapsto (df_\phi)$

- $df_\phi := d(f\phi^{-1})\phi : U \to \mathbb{R}^{n,*}$

– check:

$$df_{\phi'} = d(f\phi'^{,-1})\phi' = d(f\phi^{-1}\phi\phi'^{,-1})\phi' = d(f\phi^{-1})\phi \circ T(\phi\phi'^{,-1})\phi' = T(\phi'\phi^{-1})^{*,-1}\phi(d(f\phi^{-1})\phi) = T(\phi'\phi^{-1})^{*,-1}df_{\phi}$$

□

2.4.4 Vector bundles and dual bundles

in case the typical fibre of a bundle has an additional structure which is preserved by the values of cocycle the total space of the bundle has a corresponding structure

a vector bundle is a fibre bundle with a vector bundle structure on fibres

V - vector space

Definition 2.73. *A vector bundle with typical V over B is a fibre bundle $\pi : E \to B$ with typical fibre V together with vector space structures on the fibres E_b such that there exists a cover of B by local trivializations (ψ_α) which are fibrewise vector space isomorphisms. Vector bundle morphisms are bundle morphisms which are fibrewise linear.*

the associated cocyle to such a trivialization $\rho_{\alpha,\beta}$ takes values in $GL(V)$ - the linear automorphisms of V

vice versa:

- assume that cocycle has values in $GL(V)$

- define linear structure on E_b as follows:

– chose α with $b \in U_\alpha$

– define structures by $[u, v]_\alpha + \lambda[u, v']_\alpha := [u, v + \lambda v']_\alpha$

– this is well-defined since cocyle is linear

– by construction: $E \to B$ is a vector bundle

$E \to B$ - a vector bundle

-$\Gamma(B, E)$ becomes $C^\infty(B)$-module

– s, s' two sections

— define: $(s + s')(b) := s(b) + s'(b)$

— define: $fs(b) := f(b)s(b)$

– show that the operations produce again smooth sections:

– calculate for local sections: $s + fs'$ is represented by $(s_\alpha + fs'_\alpha)_\alpha$ - has smooth members

$\pi : E \to B$ - vector bundle, $e \in E$, $b := \pi(e)$

Lemma 2.74. *1. There exists a section s in $\Gamma(B, E)$ with $s(b) = e$*

2. If $s \in \Gamma(B, E)$ satisfies $s(b) = 0$, then there exists a finite family of sections (t_i) in $\Gamma(B, E)$ and a finite family (f_i) in $C^\infty(B)$ such that $f_i(b) = 0$ for all i and $s = \sum_i f_i t_i$

the point in 1. is: the section exists globally!

Proof. 1.:

choose local trivialization $\psi : \pi^{-1}(U) \to U \times V$

- $(b, v) := \psi(e)$

- choose $\chi \in C_c^\infty(U)$ with $\chi(b) = 1$

- define $s \in \Gamma(B, M)$ by: $b \mapsto \begin{cases} \psi^{-1}(b, \chi(b)v) & b \in U \\ 0 & else \end{cases}$

2.:

- (v_i) basis of V

- (v^i) dual basis of V^*

- $u \mapsto s^i(u) := v^i(\mathrm{pr}_V \psi(\chi(u)s(u)) : U \to \mathbb{R}$

– ith component of s in trivialization

– vanishes at b and is compactly supported on U

- Taylor

– there is decomposition $s^i = \sum_{j=1}^n f_j^i g^{i,j} =$ with $f_j^i \in C_c^\infty(U)$ and $f_j^i(b) = 0$ $(n = \dim_b B)$

- define $t^{i,j} : U \to E$ by: $t^{i,j}(u) := \psi^{-1}(u, \chi(u)g^{i,j}(u)v_i)$

–extend by zero to all of B

- have $s = (1-\chi^2)s + \sum_{i,j} f^i_j t^{i,j}$

□

dual bundle of a vector bundle $\pi : E \to B$:

- define set $E^* := \bigsqcup_{b\in B} E^*_b$
- have projection $\pi^* : E^* \to B$
- $\psi : \pi^{-1}(U) \to U \times V$
- $\psi^* : \pi^{*,-1}(U) \to U \times V^*$
– $\psi^*(e^*) := (\pi^*(e^*), (v \mapsto e^*(\psi^{-1}(u,v))))$
– if $(\rho_{\alpha,\beta})$ - $GL(V)$-valued cocycle for E, then $(\rho^{*,-1}_{\alpha,\beta})$ is $GL(V^*)$-valued cocycle for E^*
- get topology and smooth structure on E^* such that $\pi^* : E^* \to B$ is vector bundle

Definition 2.75. $\pi^* : E^* \to$ *is called the dual bundle of* $\pi : E \to B$.

this works for other functors of tensor algebra as well

- e.g. $V \mapsto S^2(V^*)$
– yields bundle of symmetric bilinear forms $E^2(E^*) \to B$

have pairing $\langle -, - \rangle : \Gamma(B,E) \times_{C^\infty(B)} \Gamma(B,E^*) \to C^\infty(B)$

- $s \otimes \kappa \mapsto \kappa(b)(s(b))$
- check smoothness

Proposition 2.76. *The pairing induces an isomorphism of $C^\infty(B)$-modules*

$$\Gamma(B,E^*) \cong \mathrm{Hom}_{C^\infty(B)}(\Gamma(B,E), C^\infty(B)) \ .$$

Proof. κ in $\Gamma(B,E^*)$

- get $\hat\kappa \in \mathrm{Hom}_{C^\infty(B)}(\Gamma(B,E), C^\infty(B))$ by: $\hat\kappa(s)(b) := \kappa(b)(s(b))$
- $\hat\kappa(fs)(b) = \kappa(b)(f(b)s(b)) = f(b)\hat\kappa(s)(b)$ shows $C^\infty(B)$-linearity

$\hat\kappa$ in $\mathrm{Hom}_{C^\infty(B)}(\Gamma(B,E), C^\infty(B))$

- define κ in $\Gamma(B, E^*)$ as follows:

– $b \in B$

– define $\kappa(b) : E_b \to \mathbb{R}$ such that:

– $\kappa(b)(e) = \hat{\kappa}(s)(b)$, s any section of E with $s(b) = e$

— well-defined: s' second section

— $s - s' = \sum_i f_i t_i$for sections t_i with $f_i(b) = 0$

— $\hat{\kappa}(s')(b) - \hat{\kappa}(s)(b) = \sum_i f_i(b)\kappa(t_i) = 0$

check smoothness of κ

check that these constructions are inverse to each other

check $C^\infty(B)$-linearity of isomorphism

□

$s \in \Gamma(M, E^*)$

- define $\tilde{s} : E \to \mathbb{R}$ by $\tilde{s}(e) := s(\pi(e))(e)$

- is fibrewise linear

– $C^\infty_{f-lin}(E, \mathbb{R}) \subseteq C^\infty(E, \mathbb{R})$ functions which are fibrewise linear

Lemma 2.77. *We have a bijection $s \mapsto \tilde{s}$ between $\Gamma(M, E^*)$ and $C^\infty_{f-lin}(E, \mathbb{R})$.*

Proof. $\tilde{s} \in C^\infty_{f-lin}(E, \mathbb{R})$

- define $s(b)$ such that $s(b)(e) = \tilde{s}(e)$ for all $e \in E_b$. □

Example 2.78. *T^*M is the dual bundle of TX*

- $\Omega^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathcal{X}(M), C^\infty(M))$

2.4.5 Principal bundles

G - a Lie group

$\pi : M \to B$

a fibrewise right action of G on M is a right action $M \times G \to M$ such that

$$\begin{array}{ccc} M \times G & \xrightarrow{(m,g)\mapsto mg} & M \\ & \underset{\pi \mathrm{pr}_M}{\searrow} & \downarrow \pi \\ & & B \end{array}$$

commutes

Definition 2.79. *A G-principal bundle over B is a fibre bundle $\pi : M \to B$ with typical fibre G together with a fibre-wise right G-action on M such that there exists a cover of B by local trivializations (ψ_α) with $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ which is G-equivariant. Principal bundle morphisms are bundle morphisms which are G-equivariant.*

- the associated cocyle has values in right-G-equivariant maps $G \to G$

- a right G-equivariant map $\rho : G \to G$ is given by left-multiplication with $\rho(e)$

- hence the coycle $\rho_{\alpha,\beta}$ has values in G (which acts on G by left multiplication)

vice versa:

- given a G-valued cocycle the associated fibre bunde is a G-principal bundle

– we define the G-action by $[u, g]_\alpha h := [u, gh]_\alpha$.

assume that $M \to B$ is a G-principal bundle

- assume that there exists a section $s \in \Gamma(B, M)$

- then we define smooth map $B \times G \to M$, $(b, g) \mapsto s(b)g$

– is a bijection

– inverse is smooth (check in trivializations)

– $s_\alpha : U_\alpha \to G$

– $(u, g) \mapsto s_\alpha(u)g$

– inverse $(u, h) \mapsto (u, s_\alpha(u)^{-1}h)$

Corollary 2.80. *There is a bijection between $\Gamma(B, M)$ and G-equivariant bundle isomor-*

phisms

$$\begin{array}{ccc} B \times G & \xrightarrow{\cong} & M \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

Corollary 2.81. *A G-principal bundle is trivial if and only if it has a section.*

Example 2.82. The map $S^1 \to S^1$ given by $z \mapsto z^n$ is a C_n-principal bundle. It is not trivial. □

2.4.6 Frame bundles and associated vector bundles

$\pi : E \to B$ - a vector bundle with typical fibre V

- get associated frame bundle $\mathrm{Fr}(E) \to B$

– a frame of E_b is an isomorphism $s : V \to E$

– the underlying set of $\mathrm{Fr}(E)$ is the set of frames of the fibres of E

– the projection $p : \mathrm{Fr}(E) \to B$ sends the frames of the fibre E_b to b

- the group $GL(V)$ acts from the right on $\mathrm{Fr}(E)$ by precomposition: $(s, g) \mapsto s \circ g$

- in order to define manifold structure find local trivializations and observe that cocycle is smooth

– choose $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ local trivialization for E

- get $\Psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times GL(V)$ by $\Psi_\alpha(s) = (p(s), \psi_\alpha(p(s), s(-))$

- reproduces $GL(V)$-valued cocycle $\rho_{\alpha,\beta}$ of E now considered with values in $\mathrm{Aut}_{\mathbf{Mf}}(GL(V))$

– this cocycle is smooth (since $GL(V)$ is Lie group)

- get associated $GL(V)$-principal bundle which will be denoted by $\mathrm{Fr}(E) \to B$

$M \to B$ - G-principal bundle

- $\kappa : G \to GL(V)$ homomorphism of Lie groups

- G-valued cocycle $\rho_{\alpha,\beta}$ for $M \to B$ gives $GL(V)$-valued cocycle $\kappa(\rho_{\alpha,\beta})$

- get associated vector bundle: notation $M \times_{G,\kappa} V \to B$

– have map $M \times V \to M \times_{G,\kappa} V$ given by

$([u,g]_\alpha, v) \mapsto [u, \kappa(g)v]_\alpha$

- this is well-defined and smooth

- induces the equivalence relation such that $(m, \kappa(g)v) \sim (mg, v)$ for all g in G on $M \times V$

– Actually: $M \times_{G,\kappa} V$ is the quotient of $M \times V$ by this equivalence relation

- write $[m, v]$ for the image of (m, v)

have G-action on $C^\infty(M, V)$ by

$(gf)(m) := \kappa(g) f(mg^{-1})$

- can talk about fixed points $C^\infty(M, V)^G$

Lemma 2.83. $\Gamma(B, M \times_{G,\kappa} V) \cong C^\infty(M, V)^G$

Proof. want that $s(\pi(m)) = [m, f(m)]$ for all m in M

given $s \in \Gamma(B, M \times_{G,\kappa} V)$

- define $f : M \to V$ as follows:

– let $m \in M$, then $s(\pi(m)) = [m, v]$

— this is the unique representative of $s(\pi(m))$ with first entry m

— set $f(m) := v$

— check: $f(mg) = \kappa(g)^{-1} v$

— check smoothness: $f \circ \psi_\alpha^{-1}(u, g) = \kappa(g)^{-1} s_\alpha(u)$

given $f \in C^\infty(M, V)^G$

- define $s \in \Gamma(B, M \times_{G,\kappa} V)$ by $s(b) = [m, f(m)]$ for any $m \in M_b$

– check: well-defined

– check smooth

check: these construction are mutually inverse

□

Example 2.84. $E \to B$ - vector bundle with fibre V

- $\mathrm{Fr}(E) \to B$

- $\kappa = \mathrm{id}_{GL(V)}$

then $\mathrm{Fr}(E) \times_{GL(V),\mathrm{id}_{GL(V)}} V \cong E$

- map $[s, v] \mapsto s(v)$

□

$E \to B$ - vector bundle with typical fibre V

$\kappa : G \to GL(V)$ - homomorphism

Definition 2.85. *A reduction of the structure group of E to G is a pair $M \to B$ of a G-principal bundle and an isomorphism of vector bundles $M \times_G V \stackrel{\cong}{\to} E$.*

Example 2.86. A reduction of the structure group to the trivial group is the same as a trivialization

$V = V_0 \oplus V_1$

- $GL(V_0) \times GL(V_1) \subseteq GL(V)$

a reduction of the structure group to $GL(V_0) \times GL(V_1)$ is equivalent to an decomposition $E_0 \oplus E_1 \cong E$

- $GL(V)^+ = \{A \in GL(V) \mid \det(A) > 0\}$

a reduction of the structure group to $GL(V)^+$ is the same as the choice of an orientation

if V has a scalar product - get $O(V) \subseteq GL(V)$

a reduction of the structure group to $O(V)$ is the same as the choice of an metric on E

□

2.4.7 Pull-back

$f : B' \to B$ - map of manifolds

- get $h^* : C^\infty(B) \to C^\infty(B')$ - pull-back of functions $h^* f := f \circ h$.

extend this to fibre bundles $M \to B$

- $s(h(b'))$ is in $M_{h(b')}$
- want a new bundle over B' with fibre $M_{h(b')}$ over b'

$\pi : M \to B$ - fibre bundle with typical fibre F

- $f : B' \to B$ morphism
- consider pull-back in sets

$$\begin{array}{ccc} M' & \xrightarrow{H} & M \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{h} & B \end{array}$$

- (U, ψ) - local trivialization of π
- induces

$$\psi' : \pi'^{,-1}(h^{-1}(U)) \to U' \times F \ , \quad m' \mapsto (\pi'(m), \mathrm{pr}_F \psi(H(m)))$$

- (U', ψ') local trivialization of π'
- cocycle: (ρ'_{ψ_1,ψ_0}) (indexed by the local trivializations of π)

– $\rho'_{\psi_1,\psi_0}(u') = \rho_{\psi_1,\psi_0}(h(u))$

Definition 2.87. *$\pi' : M' \to B'$ is called the pull-back of $\pi : M \to B$ along h.*

often write $M' := h^* M$

- the pull-back of a vector bundle is again a vector bundle
- the pull-back of a principal bundle is again a principal bundle

Lemma 2.88. *The square*

$$\begin{array}{ccc} M' & \xrightarrow{H} & M \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{h} & B \end{array}$$

is a cartesian square in **Mf***.*

Proof. Exercise: □

pull-back of sections:

- $h^* : \Gamma(B, M) \to \Gamma(B', h^*M)$

-. $s \mapsto (b' \mapsto h^*s = (b', s(h(b'))) \in M'$

Example 2.89. $f : M \to M'$ - morphism of manifolds

- interpret $TF : TM' \to TM$ as:

$Df : TM' \to f^*TM$ by universal property of pull-back □

Example 2.90. pull-back of forms:

$f : M' \to M$

- $f^* : \Omega^1(M) \to \Omega^1(M')$

- $f^*T^*M \overset{Df^*}{\to} T^*M'$

- $f^* : \Omega^1(M) \to \Gamma(M', f^*T^*M) \overset{Df^*}{\to} \Gamma(M', T^*M') = \Omega^1(M')$

commutes:

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{f^*} & C^\infty(M') \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ \Omega^1(M) & \xrightarrow{f^*} & \Omega^1(M') \end{array}$$

exercise:

□

Example 2.91. M, N - manifolds

- $E \to M$, $F \to N$ - vector bundles

$\mathrm{pr}_M : M \times N \to M$, $\mathrm{pr}_N : M \times N \to N$ projections

- write $E \boxplus F := \mathrm{pr}_M^* E \oplus \mathrm{pr}_N^* F \to M \times B$

Example 2.92. have isomorphism $T(M \times N) \to TM \boxplus TN$

- given by $D\mathrm{pr}_M \oplus D\mathrm{pr}_N$

□

□

2.5 Vector fields

2.5.1 The commutator

Definition 2.93. $\mathcal{X}(M) := \Gamma(M, TM)$ *is called the space of vector fields on* M

is $C^\infty(M)$ module

define action $\Gamma(M, TM) \times C^\infty(M) \to C^\infty(M)$

- $(X, f) \mapsto (m \mapsto X(m)(f))$

some formulas:

- have rule $(gX)(f) = gX(f)$
- Leibnitzrule: $X(gf) = X(f)g + fX(g)$
- could say: X is in $\mathrm{Der}(C^\infty(M), \mathrm{id}_{C^\infty(M)})$
- $X(f)(m) = df(m)(X(m))$

Lemma 2.94. *For* X, Y *in* $\mathcal{X}(M)$ *there exists a uniquely determined* Z *in* $\mathcal{X}(M)$ *such that* $Z(f) = X(Y(f)) - Y(X(f))$ *for all* f *in* $C^\infty(M)$

Proof. observe: $f \mapsto X(Y(f)) - Y(X(f))(m)$ is a derivation

$$
\begin{aligned}
X(Y(fg)) - Y(X(fg)) &= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
&= X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g)) \\
&\quad -Y(X(f))g - X(f)Y(g) - Y(f)X(g) - fY(X(g)) \\
&= (X(Y(f)) - Y(X(f)))g + f(X(Y(g)) - Y(X(g)))
\end{aligned}
$$

evaluate at m

- define value $Z(m)$ as this derivation

– Z satisfies the formula

– must check smoothness: Exercise! (already done) □

local formula:

- write $[X, Y] := Z$

- local formula on chart on U

- $[X, Y]_{|U} = [X^i \partial_i, Y^j \partial_j] = (X^j \partial_j Y^i - Y^j \partial_j X^i)\partial_i$

Lemma 2.95. *$\mathcal{X}(M)$ with $[-, -]$ forms a Lie algebra*

note: $[X, fY] = f[X, Y] + X(f)Y$

- $[-, -]$ is not $C^\infty(M)$ - bilinear

$h : M \to M'$ diffeomorphism

- $X \in \mathcal{X}(M)$

define h_*X such that $h^*(h_*Xf) = X(h^*f)$ for all f in $C^\infty(M)$

– get $h_*X(m') := Th(h^{-1}(m'))X(h^{-1}(m'))$

Lemma 2.96. $h_*[X, Y] = [h_*X, h_*Y]$

Proof. check chain rule: $h^*(h_*[X, Y])(f) = [X, Y](h^*f)$

$h^*[h_*X, h_*Y](f) = h^*h_*X(h_*Y(f)) - h^*h_*Y(h_*X(f)) = h^*Xh^*(h_*Y(f)) - Yh^*(h_*X(f)) = [X, Y](h^*f)$ □

Example 2.97. $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$

$X \boxplus Y := D\mathrm{pr}_M \mathrm{pr}_M^* X \oplus D\mathrm{pr}_N \mathrm{pr}_N^* Y \in \mathcal{X}(M \times N)$

$[X_0, X_1] \boxplus [Y_0, Y_1] = [X_0 \boxplus Y_0, X_1 \boxplus Y_1]$

$\square$

the following explains meaning of commutator:

$I \subseteq \mathbb{R}$ open, $0 \in I$

- consider map $\Phi : I \times M \to M$

– write $\Phi(t, m) = \Phi_t(m)$ (family of endomorphisms of M smoothly parametrized by I)

– assume $\Phi_0 = \mathrm{id}_M$

- get vector field $X := \Phi'$ (derivative by time at 0)

– $X(m) := T\Phi(0, m)(\partial_t)$

– $X(m) := (\partial_t)_{|t=0}\Phi_t(m)$

– Y in $\mathcal{X}(M)$

– define $\Phi_{t,*}Y \in \mathcal{X}(M)$ by

– consider $\Phi_{t,*}Y(m) := T\Phi_t(\Phi_t(m))^{-1}(Y(\Phi_t(m))))$

— note that for every $m \in M$ the inverse $T\Phi_t(m)^{-1}$ exists for small $|t|$ since $d\Phi_0(m) = \mathrm{id}_{T_mM}$

Lemma 2.98. $(\partial_t)_{t=0}\Phi_{t,*}Y(m) = [X, Y](m)$

Proof. calculate in chart

- use Taylor expansion and only keep constant and linear terms in t

$\Phi_t(m) = m + tX(m) + O(t^2)$

$T\Phi_t(\Phi(m)) = T(m + tX(m)) + O(t^2) = 1 + tTX(m) + O(t^2)$

$T\Phi_t(\Phi(m))^{-1} = 1 - tTX(m) + O(t^2)$

$$
\begin{aligned}
T\Phi_t^{-1}(\Phi_t(m))(Y(\Phi_t(x))) &= (1 - tTX(m))Y(m + tX(m) + O(t^2)) + O(t^2) \\
&= Y(m) - tTX(m)(Y(m)) + tTY(m)(X(m)) + O(t^2) \\
&= Y(m) + t[X,Y](m) + O(t^2)
\end{aligned}
$$

□

2.5.2 Integral curves

$X \in \mathcal{X}(M)$ given

- consider intervals $I \subseteq \mathbb{R}$

- for curve $\gamma : I \to M$ set: $\gamma'(t) := T\gamma(t)(\partial_t) \in T_{\gamma(t)}M$

Definition 2.99. *A curve $\gamma : I \to M$ is an integral curve of X if $\gamma'(t) = X(\gamma(t))$ for all $t \in I$.*

fix $m \in M$, $t_0 \in \mathbb{R}$

Proposition 2.100. *There exists a unique maximal integral curve $\gamma : I \to M$ of X with $\gamma(t_0) = m$*

Proof. local existence and uniqueness:

- in chart at m: apply Picard- Lindeloef

- get interval I such that there is a unique integral curve $\gamma : I \to M$ with $\gamma(t_0) = m$

unique continuation:

- $\gamma_0, \gamma_1 : I \to \mathbb{R}$ two integral curves

- $\gamma_0(t_0) = \gamma_1(t_0)$

– then $\gamma_0 = \gamma_1$

— $J := \{\gamma_0 = \gamma_1\}$

— show by contradiction that $J = I$

— J is closed in I and contains t_0

— assume: $J \neq I$

— assume: $\sup J < \sup I$

—- case: $\inf J > \inf I$ similar

—- $t_1 := \sup J$

—- $\gamma_0(t_1) = \gamma_1(t_1)$ (since J is closed)

—- then also $[t_1, t_1 + \epsilon) \in J$ for some small $\epsilon > 0$ by local uniqueness

– contradiction!

apply Zorn to find maximal integral curves

□

if $\gamma : I \to M$ is maximal

- if $\sup I \neq \infty$ then $\lim_{t \uparrow \sup I} \gamma(t)$ does not exist

- if $\inf I \neq -\infty$ then $\lim_{t \downarrow \inf I} \gamma(t)$ does not exist

consider open subset U such that $\{0\} \times M \subseteq U \subseteq \mathbb{R} \times M$

- $\Phi : U \to M$ some map

- write $\Phi(t, m) := \Phi_t(m)$

Definition 2.101. Φ *is called a flow of* X *if*

1. $\Phi_0 = \mathrm{id}_M$
2. *For every* m *in* M *the curve* $t \mapsto \Phi_t(m)$ *is an integral curve of* X.

Proposition 2.102. *There exists a unique maximal flow of* X.

Proof. - $\Phi_{|U \cap \mathbb{R} \times \{m\}}$ is the maximal integral curve of X with $\gamma(0) = m$

- check smoothness and openness of U

- use smooth dependence of solutions of ODE on initial conditions □

formulas: $\Phi_t \Phi_s = \Phi_{t+s}$ (where defined)

- $\Phi_{-t} = \Phi_t^{-1}$

$\frac{d}{dt}_{|t=0} \Phi_t^* f = X(f)$

$\frac{d}{dt}_{|t=0} \Phi_{t,*}(Y) = [X, Y]$

Example 2.103. Newton Mechanics

M - position space of a mechanical system (encodes positions)

- TM - phase space (encodes position and velocity)

- $X \in \mathcal{X}(TM)$ - encodes law of involution

- integral curve $\gamma : I \to TM$ - time evolution of the system with initial condition $\gamma(0) = Z$

– base point of Z in M is initial condition

– Z itself is initial velocity

modelling circle

- Physical problem: find the correct M and X modelling the reality

- Mathematical problem: find γ

- Physical problem, verify model: compare prediction of the model with some measurement

– correct model if necessary

- Application: make predictions for not yet measured evolutions

Examples:

- mass point in force: $M = \mathbb{R}^3$

- X by Newtons Law

Example:

- rigid body

- $M = \mathbb{R}^3 \times SO(3)$ (center of mass and orientation in space)

- X by Newtons Law

2.5.3 Fundamental vector fields and actions

G - Lie group

- use notation $\mathfrak{g} := T_e G$

consider manifold M with right action $a : M \times G \to M$

- use $T_{(m,g)}(M \times G) \cong T_m M \oplus T_g G$

– $\mathfrak{g} \to T_m M \oplus \mathfrak{g} \stackrel{Ta(m,e)}{\to} T_m M \oplus \mathfrak{g} \stackrel{\mathrm{pr}_{T_m M}}{\to} T_m M$

– for X in $\mathfrak{g}$ set $X^\sharp(m) := Ta(m,e)(X) \in T_m M$

— fundamental vector of the action at m for X

– let m vary

– get fundamental vector field $X^\sharp \in \mathcal{X}(M)$

consider case $G = M$

- for $g \in G$ let L_g, R_g left- and right multiplication by g

– $X^\sharp(h) = TL_g(e)(X)$.

$L_g L_h = L_{gh}$ implies

- $TL_g(h)(X^\sharp(h)) = TL_g(h)TL_h(e)(X) = TL_{gh}(e)(X) = X^\sharp(gh)$

– shorter $L_{g,*}X^\sharp = X^\sharp$

Definition 2.104. *The vector space* ${}^G\mathcal{X}(G) := \{X \in \mathcal{X}(G) \mid (\forall g \in G \mid L_{g,*}X = X)\}$ *is called the space of left invariant vector fields on* G*.*

for X in $\mathfrak{g}$ have $X^\sharp \in {}^G\mathcal{X}(G)$ - left invariant vector field

- any left-invariant vector field is uniquely is determined by value at e

- have isomorphism ${}^G\mathcal{X}(G) \stackrel{\cong}{\to} \mathfrak{g}$ given by $X \mapsto X(e)$

– is inverse to $X \mapsto X^\sharp$

- $L_{h,*}[-,-] = [L_{h,*}, L_{h,*}]$ shows:

– $[-,-]$ restricts to ${}^G\mathcal{X}(G)$

— $\mathfrak{g}$ - becomes sub-Lie algebra of $\mathcal{X}(G)$

- get induced Lie algebra structure on $\mathfrak{g}$

Definition 2.105. $\mathfrak{g}$ *is called the Lie algebra of* G*.*

- $X \mapsto X^\sharp$ is homomorphism of Lie algebras by definition

– $[X, Y]^\sharp = [X^\sharp, Y^\sharp]$

Example 2.106. V - vector space

- $GL(V) \subseteq \mathrm{End}(V)$ open

- $T_e GL(V) = \mathrm{End}(V)$

- $X^\sharp(g) = TL_g(e)(X) = gX$

- $[X, Y] = X(gY) - Y(gX) = XY - YX$

□

consider general action of G on M

Lemma 2.107. *The map* $\mathfrak{g} \to \mathcal{X}(M)$*,* $X \mapsto X^\sharp$*, is a homomorphism of Lie algebras.*

Proof. consider map $f : M \times G \to M \times G$, $(m, g) \mapsto (mg, g)$

- is diffeomorphism, inverse $(m, g) \mapsto (mg^{-1}, g)$

- $f_*(0 \oplus X) = \mathrm{pr}_M^* X^\sharp \oplus \mathrm{pr}_G^* X$

– omit to write pr

- $[(0 \oplus X), (0 \oplus Y)] = 0 \oplus [X, Y]$

- $[(X^\sharp \oplus X), (X^\sharp \oplus X)] = f_*[(0 \oplus X), (0 \oplus Y)] = f_*(0 \oplus [X, Y]) = [X, Y]^\sharp \oplus [X, Y]$

- read of $[X^\sharp, Y^\sharp] = [X, Y]^\sharp$

□

$\phi : G \to H$ - homomorphism of Lie groups

$d\phi(e) : \mathfrak{g} \to \mathfrak{h}$

Lemma 2.108. *$d\phi(e)$ is homomorphism of Lie algebras.*

Proof. get action of G on H by $(h, g) \mapsto h\phi(g)$

- for X in $\mathfrak{h}$

– $X_H^\sharp$ - fundamental vector field of G-action on H

– is in ${}^H\mathcal{X}(H)$

– $X_H^\sharp(e) = d\phi(e)(X)$

$$d\phi(e)([X, Y]) = [X_H^\sharp, Y_H^\sharp](e) = [d\phi(e)(X), d\phi(e)(Y)]$$

□

$L_g R_h = R_h L_g$ implies

- $R_{g,*}$ preserves ${}^G\mathcal{X}(G)$

- get (anti)action $\mathrm{Ad} : G \to GL(\mathfrak{g})$ by automorphisms of Lie algebras

- $\mathrm{ad} := d\mathrm{Ad}(e) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ (anti)homomorphism of Lie algebras

Lemma 2.109. $\mathrm{ad}(X)(Y) = -[X, Y]$.

Proof. Exercise? □

$X \in \mathfrak{g}$

- $X^\sharp \in {}^G\mathcal{X}(G)$

Lemma 2.110. *The maximal integral curves of X have domain $\mathbb{R}$*

Proof. $\gamma : I \to G$ integral curve of $X^\sharp$ with $\gamma(t_0) = e$

- then $g\gamma$ is integral curve of $X^\sharp$ with $\gamma(t_0) = g$

– $(g\gamma)' = dL_g(\gamma(t))(X^\sharp(\gamma(t))) = X^\sharp(g\gamma(t))$

$\gamma : I \to G$ maximal integral curve

- assume: $t_0 := \sup I < \infty$

- then

$$\gamma(t) := \begin{cases} \gamma(t) & t \in I \\ \gamma(t_0)\gamma(t-t_0) & t \in I - t_0 \end{cases}$$ is extension of integral curve to $I \cup (t_0 + I)$

- contradiction to maximality

□

$\Phi : \mathfrak{g} \times \mathbb{R} \times G \to G, \qquad (X, t, g) = \Phi_t^X(g)$

- flow of $X^\sharp$ starting at m at time t

Definition 2.111. *We define the exponential map* $\exp : \mathfrak{g} \to G$, $\exp(X) := \Phi_1^X(e)$.

Example 2.112. for $GL(V)$

- $\Phi_t^X(g) = ge^{tX}$
- $\exp(X) = e^X$ - usual matrix exponential □

Example 2.113. consider G-action on M

- $X \in \mathfrak{g}$
- $X_M^\sharp$ - fundamental vector field
- $\gamma(t) := m \exp(tX)$ is an integral curve of $X_M^\sharp$, hence defined on all of $\mathbb{R}$

– calculate derivative at t_0

–$(\partial_s)_{s=t} m \exp(sX) = (\partial_s)_{s=0} m \exp(tX) \exp(sX) = X_M^\sharp(\gamma(t))$ □

3 Connections

3.1 Linear connection on vector bundles bundles

3.1.1 Existence and classification

recall:

have differential $d : C^\infty(M) \to \Omega^1(M)$

- consider this as map $\mathcal{X}(M) \times C^\infty(M) \ni (X, f) \mapsto X(f) := df(X)$
- generalizes to V-valued functions $h \in C^\infty(M, V)$:

– write $(X,h) \mapsto \nabla^{\text{triv}}_X h = X(h)$

— componentwise application of X

— uniquely characterized by

— $v^*(\nabla^{\text{triv}}_X h) = X(v^* h)$ for every $v^* \in V^*$

formulas:

$$\nabla^{\text{triv}}_{X+X'} h = \nabla^{\text{triv}}_X h + \nabla^{\text{triv}}_{X'} h \ , \quad \nabla^{\text{triv}}_{fX} h = f \nabla_X h$$

– $C^\infty(M)$ -linear in the first argument

$$\nabla^{\text{triv}}_X (h+h') = \nabla^{\text{triv}}_X h + \nabla^{\text{triv}}_X h' \ , \quad \nabla^{\text{triv}}_X (hf) = f \nabla^{\text{triv}}_X h + X(f) h$$

– $\mathbb{C}$-linear and Leibnitz rule in the second argument

$E \to B$ - vector bundle

- want to consider $\nabla : \mathcal{X}(M) \times \Gamma(B,E) \to \Gamma(B,E)$ with these properties:

Definition 3.1. *A linear connection on E is a map $\nabla : \mathcal{X}(B) \times \Gamma(B,E) \to \Gamma(B,E)$ (written as $\nabla(X,s) = \nabla_X s$) which is $C^\infty(B)$-linear in the first argument, $\mathbb{C}$-linear in the second and satisfies the Leibnitzrule $\nabla_X(fs) = f\nabla_X s + X(f)s$.*

Example 3.2. E is trivial

- can choose trivialization $\psi : E \to B \times V$

– get identification $\Gamma(B,E) \cong C^\infty(B,V)$

– $s \mapsto h_s : b \mapsto \text{pr}_V \psi(s(b))$

– $h \mapsto s_h : b \mapsto \psi^{-1}(b, h(b))$

define connection ∇ on E such that $h_{\nabla_X s} = \nabla^{\text{triv}}_X h_s$

- ∇ depends on choice of trivialization

- ψ' second trivialization, get ∇', $s \mapsto h'_s$ and $h \mapsto s'_h$

- $\psi' \psi^{-1}(u,v) = (u, \rho(u)(v))$ transition function

– $\rho : B \to GL(V) \subseteq \mathrm{End}(V)$

– $h'_s = \rho \cdot h_s$

have $C^\infty(B)$-module isomorphism

$$\Gamma(B, T^*M \otimes \mathrm{End}(E)) \cong \mathrm{Hom}_{C^\infty(B)}(\mathcal{X}(B) \otimes_{C^\infty(B)} \Gamma(B,E), \Gamma(B,E))$$

sends ω to map $X \otimes s \mapsto (b \mapsto \omega(b)(X(b)) \cdot s(b))$

write $\omega(X) \cdot s := \omega(X, s)$

– define $\omega \in \Gamma(B, T^*M \otimes \mathrm{End}(E))$ such that $h_{\omega(X)\cdot s} = \rho^{-1} d\rho(X) \cdot h_s$

– $h'_{\nabla'_X s} = \nabla^{\mathrm{triv}}_X h'_s = \nabla^{\mathrm{triv}}_X (\rho h_s) = \rho(\nabla^{\mathrm{triv}}_X h_s + \rho^{-1} d\rho(X) h_s) = \rho h_{\nabla_X s + \omega(X) s} = h'_{\nabla_X s + \omega(X) s}$
read of: $\nabla' = \nabla + \omega$

□

b in B

$X, X' \in C^\infty(B)$, $s, s' \in \Gamma(B, E)$

- $\nabla_X s(b)$ is locally determined at b

Lemma 3.3. *If $X(b) = X'(b)$ and there exists a neighbourhood U of b such that $s_{|U} = s'_{|U}$, then $(\nabla_X s)(b) = (\nabla_{X'} s')(b)$.*

Proof. Assume that $f, f' \in C^\infty(B)$ and $f(b) = 0$, $f' \equiv 0$ near B (in particular $f'(b)$ but also all derivatives vanish)

- $(\nabla_{fX} s)(b) = f(b)(\nabla_{fX} s)(b) = 0$

- $(\nabla_X (f's))(b) = f'(b)(\nabla_X s)(b) + X(f')(b)s(b) = 0$

under the assumption can write $X - X' = fY$ and $s - s' = f't$ for such a function

□

for $X \in T_b B$ define: $\nabla_X s := \nabla_{\tilde{X}} s(b)$ for any $\tilde{X} \in \mathcal{X}(B)$ with $\tilde{X}(b) = X$

Lemma 3.4. *Linear connections exist and form an affine space over* $\Gamma(B, T^*B \otimes \mathrm{End}(E))$.

Proof. (U_α, ψ_α) covering of B by local trivializations

- locally finite

- get connection ∇^α in U_α (e.g. the trivial one)

- choose partition of unity (χ_α) subordinate to covering

- define $\nabla = \sum_\alpha \chi_\alpha \nabla^\alpha$

– interpretation:

– $\nabla_X s(b) = \sum_\alpha \chi_\alpha(b)(\nabla^\alpha_X s)(b)$

– if $b \in U_\alpha$, then $(\nabla^\alpha_X s)(b)$ is well-defined by Lemma 3.3

check:

∇ is linear connection:

Leibniz:

$$\begin{aligned}
\nabla_X(fs)(b) &= \sum_\alpha \chi_\alpha(b)(\nabla^\alpha_X fs)(b) \\
&= f(b)\sum_\alpha \chi_\alpha(b)(\nabla^\alpha_X s)(b) + X(f)(b)\sum_\alpha \chi_\alpha(b)s(b) \\
&= f\nabla_X(s)(b) + X(f)s(b)
\end{aligned}$$

∇, ∇' two linear connections

- $\omega : \mathcal{X}(M) \times \Gamma(B, E) \to \Gamma(B, E)$

- $(X, s) \mapsto \nabla'_X s - \nabla_X s$

– is $C^\infty(B)$-binlinear

– find unique $\omega \in \Gamma(B, T^*B \otimes \mathrm{End}(E))$ such that $\omega(X) \cdot s = \nabla'_X s - \nabla_X s$

if ∇ is a connection and $\omega \in \Gamma(B, T^*B \otimes \mathrm{End}(E))$, then $\nabla + \omega$ is also a connection

□

consider pull-back situation

$$\begin{array}{ccc} h^*E & \xrightarrow{k} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{h} & B \end{array}$$

∇ - linear connection on E

Lemma 3.5. *There is a unique linear connection $h^*\nabla$ on h^*E such that*

$$k((h^*\nabla_{X'}h^*s)) = \nabla_X s$$

for any $b' \in B'$, $X' \in T_{b'}B'$ and $X := Th(b')(X')$ and $s \in \Gamma(B, E)$.

Proof. ∇' any connection on E'

- write $h^*\nabla = \nabla' + \omega$

– determined ω from condition:

- $k(\omega(b')(X') \cdot (h^*s)(b')) = \nabla_Y s - k(\nabla'_{X'}h^*s)$

– in order to see that ω is wel–defined:

– must show that right-hand side only depends on value of s:

— $b := h(b')$

— assume $s = ft$ with $f(b) = 0$

— $\nabla_Y ft - k(\nabla'_{X'}h^*(ft)) = Y(f)t(b') - k(X(h^*f)h^*t(b')) = (Y(f) - X(h^*f))t(b) = 0$

—- used $k(h^*t(b')) = t(b)$

—- $Y(f) = X(h^*f$ since $Y = Th(b')(X)$

- hence get ω as desired, is uniquely determined

□

3.1.2 Curvature

$E \to B$ vector bundle

∇ - linear connection

- interpret ∇ as map $\Gamma(B,E) \to \Gamma(B, T^*B \otimes E) = \Omega^1(B,E)$

- $s \mapsto (X \mapsto \nabla_X s)$

$s \in \Gamma(B,E)$

Definition 3.6. *s is called parallel of $\nabla s = 0$.*

Example 3.7. consider ∇^{triv} on $C^\infty(B,V)$

$\nabla^{\text{triv}} h = 0$ is equivalent to the assertion that h is constant

fix $b \in B$ and $v \in V$

- there exists $h \in C^\infty(B,V)$ with $h(b) = v$ and $\nabla^{\text{triv}} h = 0$

- take constant function with value h

will see that a similar assertion for general connections on vector bundles is not true

□

in the following $X, Y \in C^\infty(B)$, $s \in \Gamma(B,E)$

Lemma 3.8.

$$(X,Y,s) \mapsto F^\nabla(X,Y) \cdot s := \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s$$

is C^∞-linear in each argument and therefore determines an element $F^\nabla \in \Omega^2(\mathrm{End}(E))$.

Proof.

$$\begin{aligned} \nabla_{fX}(\nabla_Y s) - \nabla_Y(\nabla_{fX} s) - \nabla_{[fX,Y]} s &= f\nabla_X(\nabla_Y s) - f\nabla_Y(\nabla_X s) - f\nabla_{[X,Y]} s - Y(f)\nabla_X s + Y(f)\nabla_X s \\ &= f(\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s) \end{aligned}$$

$$\begin{aligned}
\nabla_X(\nabla_Y fs) - \nabla_Y(\nabla_X fs) - \nabla_{[X,Y]} fs &= \nabla_X(f\nabla_Y s + Y(f)s) - \nabla_Y(f\nabla_X s + X(f)s) \\
&\quad -f\nabla_{[X,Y]}s - [X,Y](f)s \\
&= f\nabla_X(\nabla_Y s) + X(f)\nabla_Y s + Y(f)\nabla_X s + X(Y(f))s \\
&\quad -f\nabla_Y(\nabla_X s) - Y(f)\nabla_X s - X(f)\nabla_Y s - Y(X(f))s \\
&\quad -f\nabla_{[X,Y]}s - [X,Y](f)s \\
&= f(\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s)
\end{aligned}$$

□

Definition 3.9. *F^{∇} is called the curvature of the connection ∇.*

Example 3.10. have $F^{\nabla^{\text{triv}}} = 0$

- this is just the equality

- $X(Y(h)) - Y(X(h)) = [X,Y](h)$ - definition of commutator □

Lemma 3.11. *If $s \in \Gamma(B,E)$ is parallel, then $F^{\nabla} \cdot s = 0$.*

Proof. clear □

Corollary 3.12. *Fix $b \in B$. If for any e in E there exists a parallel section with $s_e(b) = e$, then $F^{\nabla}(b) = 0$.*

Proof. $(F^{\nabla}(X,Y)(b) \cdot e)(b) = (F^{\nabla}(X,Y) \cdot s_b)(b) = 0$

□

$$F^{\nabla+\omega}(X,Y) = F^{\nabla}(X,Y) + \nabla_X\omega(Y) - \nabla_Y\omega(X) - \omega([X,Y]) + [\omega(X),\omega(Y)] \quad (1)$$

- define $\nabla \wedge \omega \in \Omega^2(M, \mathrm{End}(E))$ by

$$\nabla\omega(X,Y)(s) := \nabla_X(\omega(Y)s) - \nabla_Y(\omega(X)s) - \omega([X,Y])s$$

- is $C^\infty(B)$-multilinear and therefore well-defined

$$F^{\nabla+\omega} = F^{\nabla} + \nabla \wedge \omega + [\omega, \omega] \tag{2}$$

Example 3.13. $E = B \times \mathbb{R}$

- identify End($\mathbb{R}$) with trivial bundle with fibre $\mathbb{R}$

- $\nabla = \nabla^{\text{triv}} + \omega$

- $\nabla^{\text{triv}} \wedge \omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = d\omega(X, Y)$

– Cartan formula

– $[\omega(X), \omega(Y)] = 0$

- hence $F^{\nabla^{\text{triv}}+\omega} = d\omega$

curvature can be non-trivial

□

Example 3.14. Physics language

- ∇ - gauge field

- for trivialization of bundle $\nabla = \nabla^{\text{triv}} + \omega$

– ω - gauge potential (depends on the trivialization, nota physical quantity)

– change of trivialization (gauge transformation):

– $\omega' = \omega + \rho^{-1} d\rho$

– $F^{\nabla} = \nabla^{\text{triv}} \wedge \omega + [\omega, \omega]$ - field strength (measurable effect of the field)

choice of bundle depends on what one wants to model

- usually additional structures preserved: complex structures, metrics □

Example 3.15. *if* $\dim(B) \leq 1$, *then curvature always vanishes*

Lemma 3.16. $F^{h^*\nabla} = h^* F^{\nabla}$

Proof. Exercise. □

Example 3.17. $B \times V \to B$ - trivial bundle

- ∇^{triv} - trivial connection

- $h_{\nabla^{\text{triv}}_X s} = X(h_s)$

- $P \in \Gamma(B, \text{End}(E))$

- family of projections

– $\text{tr}P \in C^\infty(M)$

– $\text{tr}P(b) = \dim E_b \in \mathbb{Z}$

– $\text{tr}P = \text{rk}P$ locally constant

– $F := \text{im}(P) = \ker(1-P)$ is subbundle of E

– for $s \in \Gamma(B, F)$ have $\nabla^{\text{triv}}_X s \in \Gamma(B, E)$

– ∇ on F by: $\nabla_X s := P\nabla^{\text{triv}}_X s$

— check Leibnitz, use $Ps = s$

— $\nabla_X(fs) = Pf\nabla^{\text{triv}}_X s + PX(f)s = f\nabla_X s + X(f)s$

∇ is the projection of ∇^{triv} to X

calculate curvature

$P^2 = P$

- $X(P^2) = X(P)P + PX(P) = X(P)$

- $PX(P)P + PX(P) = PX(P)$ hence $PX(P)P = 0$

$$
\begin{aligned}
F^{\nabla}(X,Y)s &= P\nabla^{\text{triv}}_X P\nabla^{\text{triv}}_Y s - P\nabla^{\text{triv}}_Y P\nabla^{\text{triv}}_X s - P\nabla^{\text{triv}}_{[X,Y]} s \\
&= PF^{\nabla^{\text{triv}}} s + PX(P)\nabla^{\text{triv}}_Y s - PY(P)\nabla^{\text{triv}}_X s \\
&= PX(P)(1-P)\nabla^{\text{triv}}_Y s - PY(P)(1-P)\nabla^{\text{triv}}_X s \\
&= PX(P)(1-P)\nabla^{\text{triv}}_Y Ps - PY(P)(1-P)\nabla^{\text{triv}}_X Ps \\
&= PX(P)(1-P)Y(P)Ps - PY(P)(1-P)X(P)Ps
\end{aligned}
$$

$F^{\nabla}(X,Y) = PX(P)(1-P)Y(P)P - PX(P)(1-P)Y(P)P$

Example 3.18. $i : S^2_r \subseteq \mathbb{R}^3$

- sphere of radius r

- $E = r^*T\mathbb{R}^3 \to S^2_r$ - trivial

- $P : E \to TS^2_r$ - orthogonal projection

– get connection ∇ by projecting ∇^{triv}

- $P(\xi)(Z) = Z - r^{-2}\langle \xi, Z\rangle \xi$

choose coordinates near northpole

$\xi(x, y) \mapsto (x, y, \sqrt{r^2 - x^2 - y^2})$

matrix for P

$$P(x,y) = \begin{pmatrix} 1 - r^{-2}x^2 & 1 - r^{-2}yx & r^{-2}x\sqrt{r^2 - x^2 - y^2} \\ 1 - r^{-2}xy & 1 - r^{-2}y^2 & yr^{-2}\sqrt{r^2 - x^2 - y^2} \\ 1 - xr^{-2}\sqrt{r^2 - x^2 - y^2} & 1 - r^{-2}y\sqrt{r^2 - x^2 - y^2} & (x^2 + y^2)r^{-2} \end{pmatrix}$$

$$X(P)(0) = r^{-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad Y(P)(0) = r^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad 1 - P(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(1-P(0))X(P)(0)P(0) = r^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad (1-P(0))Y(P)(0)P(0) = r^{-1}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$F^\nabla(X, Y) = P(0)Y(P)(0)(1 - P(0))X(P)(0)P(0) = r^{-2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \square$$

3.1.3 Parallel transport

$B = I$ - interval, $t_0 \in I$

$E \to B$ - vector bundle, $e_0 \in E_{t_0}$

∇ - connection

Lemma 3.19. *There exists a unique parallel section $s \in \Gamma(I, E)$ such that $s(t_0) = e_0$.*

Proof. - solve ODE $\nabla_{\partial t} s = 0$ with initial condition $s(t_0) = e_0$

local existence:

– analyse locally in trivialization

- $\nabla = \nabla^{\text{triv}} + \omega$

- $\nabla_{\partial_t} = \partial_t + \omega(\partial_t)$

– consider s as V-valued function in t

- $I \ni t \mapsto A(t) := \omega(t)(\partial_t) \in \text{End}(V)$

- solve linear system of ODE with non-constant coefficients

– $\partial_t s = -A(t)s$, $s(t_0) = e_0$

– is solvable and solution exists on I

global uniqueness

- s, s' to solutions on I

- $J = \{s = s'\}$ is non-empty (contains t_0)

- is closed (solutions are continuous)

- from local uniqueness: $J = I$

let $J \subseteq I$ maximal interval on which parallel extension s exists

- argue: $J = I$ using local uniqueness

□

$h : I' \to I$ map

- $s \in \Gamma(I, E)$, $\nabla s = 0$

- then $h^* \nabla h^* s = 0$

observe: let s_{e_0} be the parallel section with $s_{e_0}(t_0) = e_0$

- the map $e_0 \mapsto s_{e_0}$ is linear

$E \to B$ - vector bundle

∇ - connection

- $\gamma : [0,1] \to B$ curve

- get map $E_{\gamma(0)} \to E_{\gamma(1)}$

– get linear map $\|^{\gamma} : E_{\gamma(0)} \ni e \mapsto s_e(1) \in E_{\gamma_1}$

— here s_e parallel section of $\gamma^* E \to [0,1]$ (w.r.t. $\gamma^*\nabla$) with value $s(0) = e$

Definition 3.20. *The map $\|^{\gamma} : E_{\gamma(0)} \to E_{\gamma(1)}$ is called the parallel transport along γ.*

some simple properties of parallel transport:

reparametrization invariant:

- $\phi : [0,1] \to [0,1]$ smooth, endpoint preserving

- $\|^{\gamma} = \|^{\phi^*\gamma}$

every path can be reparametrized such that it is constant near endpoints

- can restrict to path's which are constant near endpoints

- can then concatenate

$$\gamma' \sharp \gamma = \begin{cases} \gamma(2t) & t \leq 1/2 \\ \gamma'(2t-1) & t > 1/2 \end{cases}$$

we have

$\|^{\gamma'\sharp\gamma} = \|^{\gamma'} \circ \|^{\gamma}$

$\|^{\gamma^{-1}} = \|^{\gamma,-1}$

- set $\gamma_\tau(t) = \gamma(t\tau)$ - piece of curve from $\gamma(0)$ to $\gamma(\tau)$

- s any section of E

$\|^{\gamma_\tau^{-1}} s(\gamma(\tau)) \in E_{\gamma(0)}$ - depends on τ

– how?

Lemma 3.21. $\partial_\tau \|^{\gamma_\tau^{-1}} s(\gamma(\tau)) = \|^{\gamma_\tau^{-1}} \nabla_{\gamma'(\tau)} s$

Proof. – is correct if s is parallel along γ (both sides vanish)

– more general section $s = f\sigma$ with σ parallel

$\partial_\tau \|^{\gamma_\tau^{-1}} (f\sigma)(\gamma(\tau)) = f(\gamma(\tau))\ \partial_\tau \|^{\gamma_\tau^{-1}} \sigma(\gamma(\tau)) + \gamma'(\tau)(f)\ \|^{\gamma_\tau^{-1}} \sigma(\gamma(\tau))$

$\|^{\gamma_\tau^{-1}} (\nabla_{\gamma'(\tau)} f\sigma) = f(\gamma(\tau)) \|^{\gamma_\tau^{-1}} \nabla_{\gamma'(\tau)} \sigma + \gamma'(\tau)(f) \|^{\gamma_\tau^{-1}} \sigma(\gamma(\tau))$

– is correct for sections of the form $f\sigma$ with σ parallel along γ

– any section is $\mathbb{R}$-linear combination of such □

from now one:

- consider $U \subseteq \mathbb{R}^n$ - starlike rel 0

- bundle $E \to U$

- $V := E_0$

- connection ∇

- define trivialization $\Psi : E \to U \times V$ by radial parallel transport

– $x \in U$ yields curve $\gamma_x(t) := tx$ from 0 to x

- set $\Psi(e) := (\pi(e), \|^{\gamma_{\pi(e)}, -1}(e))$

Corollary 3.22. *A vector bundle on a starlike domain in $\mathbb{R}^n$ is trivial.*

Proof. one can choose a connection

- then have radial trivialization □

write

- $\nabla = \nabla^{\text{triv}} + \omega$

- ω - $\text{End}(V)$-valued one-form

– investigate Taylor expansion of ω at 0

Lemma 3.23. *We have $\omega(tX)(Y) = \frac{t}{2}F^{\nabla}(0)(X,Y) + O(t^2)$.*

Proof. - s radially parallel

- $\nabla^{\mathrm{triv}} s = 0$ by definition of ∇^{triv}

consider X as constant vector field

- $0 = \nabla_X s(tX) = \omega(tX)(X)s(tX)$ for all radially parallel s

– $\omega(tX)(X) \equiv 0$ (as function of t)

– evaluate at $t = 0$

— $\omega(0)(X) = 0$ for all X

– derive at $t = 0$

– hence $X\omega(X)(0) = 0$

– polarization

X, Y - constant vector fields

– $X\omega(Y) + Y\omega(X) = 0$

– $\frac{1}{2}(X\omega(Y) - Y\omega(X)) = X\omega(Y) = (\partial_t)_{|t=0}\omega(tX)(Y)$

– $\frac{1}{2}(\nabla \wedge \omega)(X,Y) = X\omega(Y)$

— no commutator

– by (2): $\frac{1}{2}(\nabla \wedge \omega)(0)(X,Y) = \frac{1}{2}F^{\nabla}(0)(X,Y)$

– $\omega(tX)(Y) = \frac{t}{2}F^{\nabla}(0)(X,Y) + o(t^2)$

□

interpretation:

consider concatenation of linear paths:

$0 \to tX \to tX + tY \to 0$

- calculate parallel transport up to order t

- $e \to e \to e - \omega(tX)(tY)e \to (e - \omega(tX)(tY)e)$

- alltogether $e \mapsto e - \frac{t^2}{2} F^\nabla(X,Y)s + O(t^3)$

Lemma 3.24. *We have $\nabla = \nabla^{\mathrm{triv}}$ if and only if $F^\nabla = 0$.*

Proof. $\Rightarrow$

- clear

$\Leftarrow$

s - radially parallel section

- $\nabla_Y^{\mathrm{triv}} s = 0$ by definition

- must show that $\nabla_Y s = 0$

– fix vector X in U

— show $\nabla_Y s(X) = 0$

– $\nabla_X s(tX) = 0$ (s radially parallel)

– γ_{tX} curve from 0 to X

– $\partial_t \|^{\gamma_{tX},-1} \nabla_Y s(tX) = \|^{\gamma_{tX},-1} \nabla_X \nabla_Y s(tX) = \|^{\gamma_{tx},-1} F^\nabla(X,Y)s(tX) = 0$

- $\nabla_Y s_e(0) = 0$ (initial condition)

– set $t = 1$

hence $\nabla_Y s(tX) = 0$ for all t □

U - starlike

- $x, y \in U$

- γ curve from x to y

Corollary 3.25. *If $F^\nabla = 0$, then the parallel transport $\|^\gamma : E_x \to E_y$ is independent of γ.*

3.1.4 Tensor algebra with connections, the first Chern class

$E, F \to B$ vector bundles

∇^E, ∇^F connections

Lemma 3.26. *1. There is a unique connection* $\nabla^{E\oplus F}$ *on* $E \oplus F$ *such that*

$$\nabla^{E\oplus F}(s \oplus t) = \nabla^E s \oplus \nabla^F t \ .$$

2. There is a unique connection $\nabla^{E\otimes F}$ *on* $E \otimes F$ *such that*

$$\nabla^{E\otimes F}(s \otimes t) = \nabla^E s \otimes t + s \otimes \nabla^F t \ .$$

3. There is a unique connection $\nabla^{\mathrm{Hom}(E,F)}$ *such that*

$$(\nabla^{\mathrm{Hom}(E,F)}\phi)(s) = \nabla^F(\phi(s)) - \phi(\nabla^E s) \ .$$

Proof. Exercise. Here is a trick for the tensor product:

write $E \otimes F$ as $\mathrm{Hom}(E^*, F)$

□

$E \to B$ - vector bundle

- ∇ - connection

- define $\nabla \wedge - : \Omega^k(B, E) \to \Omega^{k+1}(B, E)$

$$\begin{aligned}\nabla \wedge \omega(X_0, \dots, X_k) \ &:= \ \sum_{i=0}^{k}(-1)^i \nabla_{X_i}\omega(X_0, \dots, \hat{X}_i \dots, X_k) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j \dots, X_k)\end{aligned}$$

Lemma 3.27. $\nabla \wedge \omega$ *is well-defined.*

Proof. must check:

- formula is alternating in (X_i)

- formula ist $C^\infty(B)$-linear in the X_i □

for 1-form:

$$\nabla \wedge \omega(X,Y) = \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega([X,Y])$$

for 2-form

$$\begin{aligned} \nabla \wedge \omega(X,Y,Z) &= \nabla_X \omega(Y,Z) + \nabla_Y \omega(Z,X) + \nabla_Z \omega(X,Y) \\ &\quad + - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y) \end{aligned}$$

for trivial bundle under $\Omega(B, B \times \mathbb{R}) \cong \Omega(B)$ and $\nabla = \nabla^{\text{triv}}$: $\nabla \wedge - = d$ - de Rham differential

calculate:

$\nabla \wedge \nabla(s)(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = F^\nabla s$

Corollary 3.28. $\nabla \wedge - : \Omega(M,E) \to \Omega(M,E)$ *is a differential of a chain complex if and only if* $F^\nabla = 0$

note:

- $\Omega(B,E)$ is $\Omega(B)$ - module

- $\nabla(\omega \wedge s) = d\omega \wedge s + (-1)^{|\omega|} \omega \wedge \nabla^E s$

- $\nabla \wedge \nabla \wedge = F^\nabla \wedge$

$E \to B$ - vector bundle

∇ connection

Lemma 3.29. *(Bianchi identity)*

$$\nabla^{\text{End}(E)} \wedge F^\nabla = 0 \ .$$

Proof. verify locally

- can assume that commutators of X, Y, Z vanish

– take coordinate vector fields

- $F^\nabla(X,Y) = [\nabla_X, \nabla_Y]$

- $\nabla_X^{\mathrm{End}(E)} F^\nabla(Y,Z) = [\nabla_X, [\nabla_Y, \nabla_Z]]$

assertion is now Jacobi identity for endomorphisms of a vector space □

$E \to B$ - vector bundle

- $\mathrm{tr} : \mathrm{End}(E) \to B \times \mathbb{R}$ bundle morphism

- ∇ on E

- ∇^{triv} on $B \times \mathbb{R}$

Lemma 3.30. $\nabla^{\mathrm{Hom}(\mathrm{End}(E), B\times\mathbb{R})}\mathrm{tr} = 0$

Proof. - to show: $X(\mathrm{tr}(\phi)) = \mathrm{tr}(\nabla_X \phi)$

- local trivialization

– sections of E are vector valued functions

– sections of $\mathrm{End}(E)$ are matrix valued functions

- $\nabla^E = d + \omega$

- $\nabla_X^{\mathrm{End}(E)} \phi = X(\phi) + [\omega(X), \phi]$

- $\mathrm{tr}(\nabla_X^{\mathrm{End}(E)} \phi) = \mathrm{tr}(X(\phi)) + \mathrm{tr}([\omega(X), \phi]) = X(\mathrm{tr}(\phi))$

□

$E \to B$ - vector bundle

- ∇ - connection

- $\mathrm{tr}F^\nabla \in \Omega^2(B)$

Lemma 3.31. $d\mathrm{tr}F^\nabla = 0$

Proof. - assume that mutual commutators of X, Y, Z vanish

- Cartan formula

- $d\mathrm{tr}F^\nabla(X,Y,Z) = X(\mathrm{tr}F^\nabla(Y,Z)) - Y(\mathrm{tr}F^\nabla(X,Z)) + Z(\mathrm{tr}F^\nabla(X,Y))$

- get $d\mathrm{tr}F^\nabla(X,Y,Z) = \mathrm{tr}(\nabla_X^{\mathrm{End}(E)} F^\nabla(Y,Z) + \nabla_Y^{\mathrm{End}(E)} F^\nabla(Z,X) + \nabla_Z^{\mathrm{End}(E)} F^\nabla(X,Y)) = 0$ with Bianchi □

dependence on the connection

$\mathrm{tr}F^{\nabla+\omega} = \mathrm{tr}F^{\nabla} + \mathrm{tr}(\nabla \wedge \omega) + \mathrm{tr}[\omega, \omega]$

- $\mathrm{tr}[\omega, \omega] = 0$

- $\mathrm{tr}(\nabla \wedge \omega)(X, Y) = \mathrm{tr}(\nabla_X^{\mathrm{End}(E)}\omega(Y) - \nabla_Y^{\mathrm{End}(E)}\omega(X)) = X\mathrm{tr}(\omega(Y)) - Y\mathrm{tr}(\omega(X)) = (d\mathrm{tr}\omega)(X, Y)$

– Cartan formula

Definition 3.32. *The vector space*

$$H^n_{dR}(B) := \frac{\ker(d : \Omega^n(B) \to \Omega^{n+1}(B))}{\mathrm{im}(d : \Omega^{n-1}(B) \to \Omega^n(B))}$$

is called the nth de Rham cohomology of B.

Corollary 3.33. *The class $c_1(E) := [\mathrm{tr}F^{\nabla}] \in H^2_{dR}(B)$ is independent of the choice of the connection.*

Definition 3.34. *$c_1(E)$ is called the first Chern class of E.*

if E is trivial

- E admits trivial connection ∇^{triv} with zero curvature

- conclude $c_1(E) = 0$

vice versa:

- if $c_1(E) \neq 0$, then E is not trivial.

Note: we will see later that $c_1(E) = 0$ always

3.1.5 Metrics and connections

$E \to B$ - vector bundle

- $h \in \Gamma(B, S^2(E^*))$

– $b \in B$

— $h(b) \in S^2(E_b^*)$ - symmetric bilinear form

Definition 3.35. *h is called a metric on E if $h(b) > 0$ for every b in B.*

Definition 3.36. *The pair (E, h) is called an euclidean vector bundle.*

Example 3.37. $\psi : E \cong B \times V$ - trivialization

- choose metric h^V on V

- get metric on E such that ψ is fibrewise isometry

□

$E \to B$ vector bundle

Lemma 3.38. *There exists a metric on E.*

Proof. cover B by local trivializations (U_α, ψ_α)

- (χ_α) - partition of unity

- get local metrics h^α

- define for $b \in B$ and $e, e' \in E_b$:

$$h(e, e') := \sum_\alpha \chi_\alpha(b) h^\alpha(b)(e, e')$$

- h is a metric on E

□

Lemma 3.39. *Every subbundle $F \subset E$ has a complement.*

Proof. choose metric on E

- $P \in \Gamma(B, \mathrm{End}(E))$

- $P(b)$ - orthogonal projection onto F

- $F^\perp := \ker(1 - P)$

have deomposition $E \cong F \oplus F^\perp$

□

note: $h = h^F \oplus h^{F^\perp}$

$E \to B$ vector bundle

- h^V - metric on V

- h metric on E

– a frame $\phi : V \to E$ is orthogonal if it is an isometry

– get subbundle $O(E, h) \subseteq \mathrm{Fr}(E)$ of orthogonal frames

– is a $O(V, h^V)$ - principal bundle

- have isomorphism $O(E, h) \times_{O(V,h^V)} V \cong E$

– metric provides reduction of structure group to $O(V, h^V)$

vice versa: assume $E \cong P \times_{O(V,h^V)} V$

- get metric h such that $h([p, v], [p, v']) = h^V(v, v')$

∇ - connection

Definition 3.40. *h is compatible with ∇ if $\nabla^{S^2(E^*)}h = 0$.*

also say: ∇ is a metric connection

note: $\nabla_X^{S^2(E^*)}h(s, t) = X(h(s, t)) - h(\nabla_X s, t) - h(s, \nabla_X t)$

- hence compatibility is equivalent to relation

- $dh(s, t) = h(\nabla s, t) + h(s, \nabla t)$

Example 3.41. $E \cong B \times V$

h induced from h^V

- ∇^{triv} is compatible with h □

Example 3.42. $E \to B$ vector bundle

- ∇ connection

- h metric, compatible with ∇

$P \in \Gamma(B, \mathrm{End}(E))$ - family of projections

- $F = \mathrm{im}(P)$

– have restricted metric h^F

- if $P^* = P$, then $P\nabla$ is compatible with h^F

$dh^F(s,t) = h(\nabla s, t) + h(s, \nabla t) = h(\nabla s, Pt) + h(Ps, \nabla t) = h(P\nabla s, t) + h(s, P\nabla t) = h^F(\nabla^F s, t) + h(s, \nabla^F t)$

□

(E, h) euclidean vector bundle

$\gamma : [0,1] \to B$ - a curve

- $\|^\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$

Lemma 3.43. *If ∇ and h are compatible, then $\|^\gamma$ is isometric.*

Proof. s, t - parallel sections along γ

- $e = s(0)$, $e' = s'(0)$

$\partial_t h(s, s') = h(\nabla_{\gamma'(t)} s, s') + h(s, \nabla_{\gamma'(t)} s') = 0$

- $h(e, e') = h(s, s')(0) = h(s, s')(1) = h(\|^\gamma(e), \|^\gamma(e'))$

□

(E, h) euclidean vector bundle

- ∇ - connection

– define new connection characterized by

$$h(\nabla^*_X s, t) = X(h(s,t)) - h(s, \nabla_X t)$$

- $t \mapsto X(h(s,t)) - h(s, \nabla_X t)$ is $C^\infty(B)$-linear

– hence there is a unique section $\nabla^*_X s \in \Gamma(B, E)$ satisfying condition

– check that $(X, s) \mapsto \nabla^*_X s$ is a connection

Definition 3.44. *∇^* is called the adjoint connection.*

∇ and h are compatible if and only if $\nabla = \nabla^*$

$(\nabla^*)^* = \nabla$

- interpret h as isomorphism $h : E \to E^*$

– then $\nabla^* = h^{-1}\nabla^{E^*}h$

define $\omega := \nabla^* - \nabla$

Definition 3.45. *The connection* $\nabla^u := \nabla + \frac{1}{2}\omega$ *is called the orthogonalization of* ∇

- ∇^u is compatible with h

Corollary 3.46. *Every euclidean vector bundle admits a metric connection.*

$\nabla, \nabla + \omega$ are both compatible if and only $\omega(X) = -\omega(X)^*$ for all X

Lemma 3.47. *If* ∇ *is compatible, then* $F^{\nabla}(X,Y) = -F^{\nabla}(X,Y)^*$

Proof. Exercise □

Corollary 3.48. *For any vector bundle* $E \to B$ *we have* $c_1(E) = 0$.

Proof. E has metric

- can choose metric connection

- $F^{\nabla}(X,Y)$ is antisymmetric

- $\mathrm{tr}F^{\nabla^u}(X,Y) = 0$

- cohomology class $c_1(E)$ contains 0

□

Remark 3.49. to get non-trivial cohomology classes consider

$s(\nabla)_n := \mathrm{tr}(\underbrace{F^{\nabla} \wedge \dots F^{\nabla}}_{2n}) \in \Omega^{4n}(B)$

- then $ds_n(\nabla) = 0$

- $s_n(E) := [s_n(\nabla)] \in H^{4n}_{dR}(B)$ does not depend on ∇

these classes may indeed be non-trivial □

3.2 Connection of fibre bundles

3.2.1 Horizontal bundles for submersions

$\pi : M \to B$ smooth map

Definition 3.50. *π is called:*

1. *a submersion if $T\pi(m) : T_m M \to T_{\pi(m)} B$ is surjective for every m in M.*
2. *an immersion if $T\pi(m) : T_m M \to T_{\pi(m)} B$ is injective for every m in M.*

Example 3.51. $\pi : M \to B$ - a locally trivial fibre bundle

- then π is a submersion □

consider submersion $\pi : M \to B$

- $D\pi : TM \to \pi^* TB$ surjective

- $\dim(\ker(D\pi))$ has locally constant rank

- $T^v\pi := \ker D\pi \to M$ is a vector bundle bundle

Definition 3.52. *The subbundle $T^v\pi$ of TM is called the vertical subbundle of π.*

Definition 3.53. *A horizontal bundle for π is a subbundle $T^h M$ of TM such $D\pi_{|T^h M} : T^h M \to \pi^* TB$ is an isomorphism.*

observe: assume that $T^h M$ is horizontal bundle

$T^v\pi \oplus T^h M \to TM$ is bundle isomorphism

- injective: $T^v\pi \cap T^h M = 0$ (since otherwise $D\pi_{|T^h M}$ not injective)

- surjective: both bundles have the same dimension

Lemma 3.54. *Horizontal bundles for $\pi : M \to B$ exist.*

Proof. choose metric on TM

- get notion of orthogonal complement

- take $T^h M := T^v\pi^\perp$ □

Example 3.55. $\pi : E \to B$ vector bundle

- have canonial isomorphism $i : \pi^* E \cong T^v \pi$

– fix base point $e \in E_b$

– fibre of $(\pi^* E)_e$ is canonically isomorphic to E_b

– for $f \in (\pi^* E)_e$ consider curve $t \mapsto e + tf$ in E

– tangent vector $i(e)(f)$ at $t = 0$ is element of TE

– $\pi(e + tf) = b$ for all t implies $T\pi(e)(i(e)(f)) = 0$

— hence $i(e)(f) \in T^v \pi$

check in chart: i is a bundle isomorphism

∇ - connection on E

- will see that it determines a horizontal subbundle $T^{h,\nabla} E$

– $e \in E_b$

– describe $T_e^{h,\nabla} E$

– we can find a section s with $s(b) = e$ and $\nabla s(b) = 0$

— only in the single point b, in general not on a larger subset

— in local trivialization:

—-$\nabla = \nabla^{\text{triv}} + \omega$

—- $\nabla_X s(b) = 0$ means $X(s)(b) + \omega(b)(X)e = 0$

—- $s(b + X) = s(b) - \omega(b)(X)e + O(X^2)$

—- $Ts(b)(X) = -\omega(b)(X)$ (does not depend on choice of s)

—- define $T_e^{h,\nabla} E = Ts(b)(T_b B)$

—- $\pi \circ s = \text{id}$ implies $D\pi(e)_{|T_e^{h,\nabla} E}$ is isomorphism

note: can recover ∇ from $T^{h,\nabla} M$

□

$\pi : M \to B$ submersion

- $T^h M$ given

- can define horizontal lift of vectors and vector fields.

b in B

- $m \in M_b$

- $X \in T_bB$

Definition 3.56. *$X^h \in T_mM$ is called the horizontal lift of X if $T\pi(m)(X^h) = X$ and $X^h \in T^h_mM$.*

- X^h is uniquely determined by X

- $X^h = (T\pi_{|T^h_mM})^{-1}(X)$

consider now vector fields

- $X \in \mathcal{X}(B)$

- define $X^h \in \mathcal{X}(M)$ such that $X^h(m)$ is the horizontal lift of $X(\pi(m))$

Definition 3.57. *X^h is called the horizontal lift of X.*

- get map $\mathcal{X}(B) \to \mathcal{X}(M)$, $X \mapsto X^h$ horizontal lift

- ist $C^\infty(B)$ -linear: $(fX)^h = \pi^*(f)X^h$

consider curve $\gamma : I \to B$

Definition 3.58. *A horizontal lift of γ is a curve $\tilde{\gamma} : I \to M$ with*

1. *$\pi \circ \tilde{\gamma} = \gamma$*
2. *$\gamma'(t)$ is horizontal for every $t \in I$*

consider deviation from being a Lie algebra homomorphism

Lemma 3.59. *The map $\mathcal{X}(B) \times \mathcal{X}(B) \to \Gamma(M, T^v\pi)$*

$$\mathcal{X}(B) \times \mathcal{X}(B) \ni (X, Y) \mapsto T(X, Y) = [X^h, Y^h] - [X, Y]^h$$

takes values in $\Gamma(M, T^v\pi)$ and is $C^\infty(B)$-linear.

Proof. $C^\infty(B)$ -linearity

$$\begin{aligned}
T(fX,Y) &= [(fX)^h, Y^h] - [fX,Y]^h \\
&= [\pi^*(f)X^h, Y^h] - [fX,Y]^h \\
&= \pi^*(f)[X^h,Y^h] - f[X,Y]^h - Y^h(\pi^*(f))X^h + \pi^*(Y(f))X^h \\
&= \pi^*(f)T(X,Y)
\end{aligned}$$

used: $Y^h(\pi^*(f))(m) = T\pi(m)(Y^h(m))(f) = Y(\pi(m))(f) = \pi^*(Y(f))(m)$

- hence $Y^h(\pi^*(f)) = \pi^*(Y(f))$

verticality:

must show that $D\pi(m)(T(X,Y))(m) = 0$ for all m

- suffices to show that $T(X,Y)(\pi^*(f)) = 0$ for all $f \in C^\infty(B)$

$$\begin{aligned}
T(X,Y)(\pi^*(f)) &= [X^h,Y^h](\pi^*(f)) - [X,Y]^h(\pi^*(f)) \\
&= X^h(Y^h(\pi^*(f)) - Y^h(X^h(\pi^*(f))) - \pi^*([X,Y](f)) \\
&= X^h(\pi^*(Y(f))) - Y^h(\pi^*(X(f))) - \pi^*([X,Y](f)) \\
&= \pi^*(X(Y(f))) - \pi^*(Y(X(f))) - \pi^*([X,Y](f)) \\
&= 0
\end{aligned}$$

□

Definition 3.60. *T is called the curvature of $T^h\pi$*

thus $T \in \Gamma(M, \Lambda^2 T^h M \otimes T^v\pi)$

Example 3.61. Example: $M = B \times F$

- $T^hM = \mathrm{pr}^*TB \subseteq TB \boxplus TF \cong M$

- $T = 0$ □

$m \in M_b$, $X, Y \in T_bB$

- then $T(m)(X,Y) \in T^v_m(X,Y)$ is defined

Definition 3.62. *T is called the curvature of the horizontal subbundle T^hM.*

Example 3.63. $\pi : E \to B$ vector bundle

- ∇ - connection

- $T^{h,\nabla}M$ - associated horizontal subbundle

Lemma 3.64. *For $e \in E_b$ and $X, Y \in T_bB$ we have $T(X,Y)(e) = -i(e)(F^{\nabla}(b)(X,Y)(e))$*

Proof. - have explicit formula for horizontal lift in coordinates:

- notation for coordinates:

- for E: (b, v),

– $b \in \mathbb{R}^n$ base coordinate ,

– $v \in V$ - fibre coordinate

- for TE: (b, v, β, ξ),

– $b, \beta \in \mathbb{R}^n$,

– $v, \xi \in V$

$\pi(b, v) := b$

- $T\pi(b, v)(\beta, \xi) = (b, \beta)$

- $(b, \beta) \in T_nB$

- vertical vectors: $(b, v, 0, \xi) \in T^v_{(b,v)}E$

– $\nabla = \nabla^{\text{triv}} + \omega$

- horizontal lift of (b, β) at (b, v): $(b, \beta)^h = (b, v, \beta, -\omega(b)(\beta)(v))$

- for coordinate field: $b \mapsto (b, \beta)$ (consider β as constant function in b)

– horizontal lift: $(b, v) \mapsto (b, v, \beta, -\omega(b)(\beta)(v))$

rite in the target $[b, v, 0, ...]$

$$
\begin{aligned}
T(b,v)((b,\beta),(b,\beta')) &= [(b,v)\mapsto(b,v,\beta,-\omega(b)(\beta)(v)),(b,v)\mapsto(b,v,\beta',-\omega(b)(\beta')(v))] \\
&= -\beta(\omega(-)(\beta')(v)) + \beta'(\omega(-)(\beta)(v)) + \\
&\quad \omega(b)(\beta')(\omega(b)(\beta)(v)) - \omega(b)(\beta)(\omega(b)(\beta')(v)) \\
&= (\nabla\wedge\omega)(b)(\beta',\beta)(v) + [\omega(b)(\beta'),\omega(b)(\beta)](v) \\
&= -F^{\nabla}(b)((b,\beta),(b,\beta'))(v)
\end{aligned}
$$

□

□

consider pull-back situation

$$
\begin{array}{ccc}
M' & \xrightarrow{k} & M \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
B' & \xrightarrow{h} & B
\end{array}
$$

connection $T^h\pi$ induces connection $T^h\pi'$ by pull-back

$dk : TM' \to k^*TM \cong T^vM \oplus T^hM$

- restricts to isomorphism $dk_{|T^v\pi'} : T^v\pi' \to T^v\pi$
- T^hM' characterized by: $T^h_{m'}M' = (Dk(m'))^{-1}(T^h_{k(m')}M')$
- then $dk = dk_{|T^v\pi'} \oplus dk_{T^h_{m'}M'} : T^v\pi' \oplus T^hM' \to T^v\pi \oplus T^hM$
- write $T^hM' = h^*T^hM$

obervation:

Corollary 3.65. *If γ' is horizontal curve in M', then $k \circ \gamma'$ is horizontal in M*

Definition 3.66. *A morphism $\pi : M \to B$ between manifold (topological spaces) is called proper if for every compact $K \subseteq B$ the preimage $\pi^{-1}(K)$ is compact.*

Example 3.67. $\pi : M \to B$ a fibre bundle with compact fibre F

- then π is proper

$\pi : (0, \infty) \to \mathbb{R}$ is not proper

- $\pi^{-1}([-1, 1]) = (0, 1]$ is not compact

If M is compact, then every map out of M is proper.

$\square$

$\pi : M \to B$ submersion

- $T^h M$ - horizontal bundle

- $\gamma : I \to B$ - curve

- $t_0 \in I$

Proposition 3.68. *If π is proper, then for every $m_0 \in M_{\gamma(t_0)}$ there exists a unique horizontal lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(t_0) = m_0$.*

Proof. assume $B = I \subseteq \mathbb{R}$ - interval

- $\partial_t \in \mathcal{X}(I)$

- $\partial_t^h \in \mathcal{X}(M)$

- $\tilde{\gamma}$ must be integral curve of ∂_t^h

– therefore uniqueness

existence

claim: the integral curve γ^h of ∂_t^h with $\gamma^h(t_0) = m_0$ exists on I

by contradiction

- $J \subseteq I$ max. existence interval of γ^h

- $\pi \circ \gamma^h(t) = t$

- assume $\sup(J) = t < \sup(I)$

- from ODE theory: $\gamma^h(s)$ does not have accumulation point for $s \uparrow t$

- chose $\epsilon > 0$ such that $[t - \epsilon, t] \subseteq I$

- note that for $s \geq t - \epsilon$ we have $\gamma^h(s) \in \pi^{-1}([t - \epsilon, t])$

– $\pi^{-1}([t-\epsilon, t])$ is compact

– hence such accumulation point exists

— contradiction

general base

- pull-back along $\gamma : I \to B$

$$\begin{array}{ccc} M' & \xrightarrow{k} & M \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ I & \xrightarrow{\gamma} & B \end{array}$$

- find horizontal lift $\tilde{\gamma}' : I \to M'$

- then $\tilde{\gamma} = k \circ \tilde{\gamma}'$

□

Example 3.69. properness is necessary:

here is a counterexample

– $(0, \infty) \to \mathbb{R}$

– $t_0 = 1$

— $\gamma^h(t) := t$ exists only on $(0, \infty)$ (and not on $\mathbb{R}$) □

consider parallel transport

$\pi : M \to B$ - submersion

$T^h M$ given

- $\gamma : [0, 1] \to B$ - a curve

- pull-back

$$\begin{array}{ccc} \gamma^* M & \xrightarrow{k} & M \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ I & \xrightarrow{\gamma} & B \end{array}$$

- get induced $\gamma^* T^h M$

- $m_0 \in M_{\gamma(0)}$

assume that π is proper (or γ^h exists for other reasons)

- can define horizontal lift of γ with start in m_0

- take $k \circ \gamma^h$

– denote now also as γ^h

– define $\|^{\gamma}(m_0) := \gamma^h(1)$

Definition 3.70. *The map $\|^{\gamma} : M_{\gamma(0)} \to M_{\gamma(1)}$ is called the parallel transport along γ with respect to $T^h M$.*

here is a list of (essentially obvious) properties

- $\|^{\gamma} : M_{\gamma(0)} \to M_{\gamma(1)}$ is diffeomorphism

- is reparametrization invariant

- $\|^{\gamma' \sharp \gamma} = \|^{\gamma'} \circ \|^{\gamma}$

- $\|^{\gamma^{-1}} = \|^{\gamma, -1}$

- if $T = 0$, then $\|^{\gamma}$ is deformation invariant in γ

Lemma 3.71. *A proper submersion $M \to I$ is a trivial bundle.*

Proof. use parallel transport

fix $t_0 \in I$

for $t \in i$ define $\gamma_t(u) := (1 - u)t_0 + ut$

- curver from t to t_0

define

$\Psi : M \times I \times M_{t_0}$

- $\Psi(m) := \|^{\gamma_{\pi(m)}}(m)$

□

Lemma 3.72 (Ehresmann Theorem)**.** *A proper submersion is a locally trivial fibre bundle.*

Proof. - choose connection

- b in B

- choose chart at B with range a starlike domain in $\mathbb{R}^n$

- use radial parallel transport to trivialize

- $M \to B \times M_b$

- $M \ni m \mapsto (\pi(m), \|^{\gamma_{\pi(m)},-1}(m)) \in B \times M_b$

– here γ_x is curve $t \mapsto tx$ from 0 to x

□

3.2.2 Connections on principal bundle

G - Lie group

$\pi : P \to B$ - a G-principal bundle

- have right G-action $g \mapsto R_g$

- can ask that horizontal bundles are G-invariant.

Definition 3.73. *A principal bundle connection on $\pi : P \to B$ is a G-invariant horizontal bundle.*

$\mathfrak{g}$ - Lie algebra of G

- $X \in \mathfrak{g}$ - $X^\sharp \in \mathcal{X}(P)$ fundamental vector field of action

– $X^\sharp(p) = (\partial_{t)_{|t=0}} R_{\exp(tX)}(p)$

- in trivialization $P = B \times G$

- interpret X in ${}^G\mathcal{X}(G)$

– have $X^\sharp(b, g) = 0 \oplus X(g) \in T_bB \oplus T_gG \cong T_{(b,g)}(B \times G)$

– the values of $X^\sharp(p)$ for all $X \in \mathfrak{g}$ generates $T^v\pi$

- G acts on itself by conjugation: $(g, h) \mapsto \alpha_g(h) := g^{-1}hg$

- action fixes e

- G acts on $T_eG = \mathfrak{g}$ by Lie algebra homomorphism $\mathrm{Ad}(g) := T\alpha_g(e) \in \mathrm{End}(\mathfrak{g})$

- by definition: $(\partial_t)_{|t=0} g^{-1}\exp(tX)g = \mathrm{Ad}(g^{-1})(X)$

$$\begin{aligned} TR_g(p)(X^\sharp(p)) &= TR_g(\partial_t)_{|t=0} R_{\exp(tX)}(p) \\ &= (\partial_t)_{|t=0} R_g R_{\exp(tX)}(p) \\ &= (\partial_t)_{|t=0} R_{g^{-1}\exp(tX)g}(pg) \\ &= (\mathrm{Ad}(g^{-1}(X))^\sharp(pg) \end{aligned}$$

write $\mathfrak{g}$ instead of $P \times \mathfrak{g}$

define form $\omega : \Omega^1(M, \mathfrak{g})$ by the following conditions:

- $T^h P = \ker(\omega)$

- $\omega(p)(X^\sharp(p)) = X$ for all $X \in \mathfrak{g}$

– this determines $\omega(p)$ since $T_p P \cong T_p^h P \oplus T_p^v \pi$ and $X \mapsto X^\sharp(p)$, $\mathfrak{g} \to T_p^v \pi$ is isomorphism

- G-invariance of $T^h P$ implies G-invariance of ω

Lemma 3.74. *For every g in G we have* $R_g^* \omega = \mathrm{Ad}(g)\omega$

Proof. $\mathrm{Ad}(g) \in \mathrm{End}(\mathfrak{g})$ is applied to the values

for horizontal vectors: $H \in T_p^h P$

$(R_g^*\omega)(p)(H) = \omega(pg)(TR_g(X)) = 0$ since $TR_g(X) \in T_{pg}^h P$ by invariance of $T^h P$

for vertical vectors:

$$\begin{aligned} (R_g^*\omega)(p)(X^\sharp(p)) &= \omega(pg)(TR_g(p)X^\sharp(p)) \\ &= \omega(pg)((\mathrm{Ad}(g^{-1})(X))^\sharp(pg)) = \mathrm{Ad}(g^{-1})(X) \\ &= \mathrm{Ad}(g^{-1})(\omega(p)(X^\sharp(p))) \end{aligned}$$

□

Definition 3.75. *A form* $\omega \in \Omega^1(P, \mathfrak{g})$ *with*

1. *$\omega(p)(X^\sharp(p)) = X$ for all $X \in \mathfrak{g}$ and $p \in P$*

2. *$R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for all g in G*

is called a connection 1-form.

Connection one-form provide an alternative description of principal bundle connections

- T^hP determines ω
- ω determines T^hP by $T^hP = \ker(\omega)$

Maurer-Cartan form

$\theta \in \Omega^1(G, \mathfrak{g})$

- is the unique principal bundle connection 1-form on $G \to *$
- θ is determined by: for X left invariant: $\theta(X) = X(e)$
- $\theta(g) = dL_{g^{-1}}(g)$
- write often as $g^{-1}dg$

leads to

$d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg = [g^{-1}dg, g^{-1}dg]$

structure equation:

$d\theta = [\theta, \theta]$

$P \to B$ - G - principal bundle

$p \in P$ induces map $i_p : G \to P$, $i_p(g) := pg$

Corollary 3.76. *$\omega \in \Omega^1(P, \mathfrak{g})$ is a connection 1-form if and only if $i_p^*\omega = \theta$ for every p in P.*

we say that ω is fibrewise Mauerer-Cartan

P - G-principal bundle

write $\mathrm{Ad}(P) := P \times_G \mathfrak{g}$ for associated vector bundle

Lemma 3.77. *Principal bundle connections exists and from an affine space over* $\Omega^1(B, \mathrm{Ad}(P))$

Proof. $P = B \times G$ trivial

- $\mathrm{pr}_G^*\theta$ is connection 1-form

$\pi : P \to B$ general

- choose local trivializations (U_α, Ψ_α)

- get principal bundle connections $\omega_\alpha \in \Omega^1(\pi^{-1}(U_\alpha), \mathfrak{g})$

– pull-back of Maurer-Cartan form

- choose partition of unity (χ_α)

- $\omega(p) := \sum_\alpha \chi(\pi(p))\omega_\alpha(p)$

– check that it is fibrewise Maurer-Cartan

ω, ω' - two connection 1-forms

- $\delta := \omega' - \omega \in \Omega^1(P, \mathfrak{g})$

- $\delta_{|T^v\pi} = 0$

- define $\bar{\delta}(b) \in T_b^*B \otimes \mathrm{Ad}(P)$

- $\bar{\delta}(b)(X) = [p, \delta(p)(\tilde{X})]$ for any $p \in P$ and lift $\tilde{X}$ in T_pP

– indepence of lifts: two lift differ by vertical vectors

– independence of p:

– $[pg, \delta(pg)(TR_g(\tilde{X}))] = [pg, \mathrm{Ad}(g^{-1})(\delta(p)(X))] = [p, \delta(p)(X)]$

- get $\bar{\delta} \in \Omega^1(B, \mathrm{Ad}(P))$

– vice versa: $\bar{\delta}$ given

if ω is connection 1-form and $\bar{\delta} \in \Omega^1(B, \mathrm{Ad}(P))$

- define $\delta(p)(\tilde{X}) := Z \in \mathfrak{g}$ such that $[p, Z] = \bar{\delta}(\pi(p))(T\pi(X))$

check: $\omega' := \omega + \delta$ is connection 1-form

□

note: if G is not compact then $\pi : P \to B$ is not proper

- so the general result about existence horizontal lifts of curves do not apply

- but such lifts exist

Lemma 3.78. *Horizontal lifts of curves with respect to a principal bundle connection exist.*

Proof. $\pi : P \to I$ - G-principal bundle

- $T^h P$ - principal bundle connection

- $\gamma : J \to I$ max. horizontal lift

- assume $\sup(J) = t_1 < \sup(I)$

choose any point $p \in P_t$

- there is horizontal curve $\sigma : (t - \epsilon, t + \epsilon) \to P$ with $\sigma(t) = p$

- for any g in G: σg is also horizontal

- there is g in G such that $\gamma(t - \epsilon/2) = \sigma(t - \epsilon/2)g$

- can prolong γ up to $t + \epsilon$ with $s \mapsto \sigma(s)g$

– contradiction to maximality of J

□

consider curvature

$T \in \Gamma(P, \Lambda^2 \pi^* T^* B \otimes T^v P)$

- want to express this in terms of ω

set

$\Omega := d\omega + [\omega, \omega] \in \Omega^2(P, \mathfrak{g})$

- $\Omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) + \omega([X, Y]) - [\omega(X), \omega(Y)]$

Lemma 3.79.

1. *$R_g^* \Omega = \mathrm{Ad}(g^{-1})\Omega$*
2. *If X is vertical, then $\Omega(X, Y) = 0$*
3. *$\omega(p)(T(p)(X, Y)) = -\Omega(p)(X^h, Y^h)$ for $X, Y \in T_{\pi(p)} B$*

Proof. use

- $\mathrm{Ad}(g)$ is Lie algebra auto of $\mathfrak{g}$
- $R_g^* d = dR_g^*$

$$\begin{aligned}
R_g^*\Omega &= R_g^*(d\omega + [\omega,\omega]) \\
&= dR_g^*\omega + [R_g^*\omega, R_g^*\omega]) \\
&= d\mathrm{Ad}(g^{-1})\omega + [\mathrm{Ad}(g^{-1})\omega, \mathrm{Ad}(g^{-1})\omega]) \\
&= \mathrm{Ad}(g^{-1})d\omega + \mathrm{Ad}(g^{-1})[\omega,\omega] \\
&= \mathrm{Ad}(g^{-1})\Omega
\end{aligned}$$

X in $\mathfrak{g}$

$\omega(X^\sharp) = X$ - constant function with value X

- $X^\sharp(f) = (\partial_t)_{|t=0} R^*_{\exp(tX)} f$
- $[X^\sharp, Y] = (\partial_t)_{|t=0} DR^{-1}_{\exp(tX)}(R^*_{\exp(tX)}(Y))$
- $R_g^*(\omega(Y)) = R_g^*(\omega)(DR_g^{-1}(R_g^*(Y)))$
- $(\partial_t)_{|t=0}\mathrm{Ad}(\exp(tX)(Y')) = -[X, X']$

$$\begin{aligned}
\Omega(X^\sharp, Y) &= X^\sharp(\omega(Y)) - Y(\omega(X^\sharp)) - \omega([X^\sharp, Y]) + [\omega(X^\sharp), \omega(Y)] \\
&= X^\sharp(\omega(Y)) - Y(X) + \omega([X^\sharp, Y]) + [X, \omega(Y)] \\
&= (\partial_t)_{|t=0} R^*_{\exp(tX)}(\omega(Y)) - \omega((\partial_t)_{|t=0} DR^{-1}_{\exp(tX)}(R^*_{\exp(tX)}(Y))) + [X, \omega(Y)] \\
&= (\partial_t)_{|t=0}\mathrm{Ad}(\exp(tX))\omega(Y) + \omega((\partial_t)_{|t=0} DR^{-1}_{\exp(tX)}(R^*_{\exp(tX)}(Y))) \\
&\quad -\omega((\partial_t)_{|t=0} DR^{-1}_{\exp(tX)}(R^*_{\exp(tX)}(Y))) + [X, \omega(Y)] \\
&= -[X, \omega(Y)] + [X, \omega(Y)] \\
&= 0
\end{aligned}$$

use that ω vanishes on horizontal vectors:

- $\Omega(X^h, Y^h) = d\omega(X^h, Y^h) = -\omega([\tilde{X}, \tilde{Y}])$
- $\omega(T(X, Y)) = \omega([X^h, Y^h])$

$\square$

$\rho : G \to GL(V)$ any representation

- write also $\rho : \mathfrak{g} \to \mathrm{End}(V)$ for derivative at e (Lie algebra homomorphism)

-$P(V) := P \times_G V$ associated bundle

- define $\Omega^n(P,V)^{h,G}$ (horizontal and G-invariant sections) as the subspace of $\Omega^n(P,V)$ of sections with:

1. $\alpha(X_1,\ldots,X_n) = 0$ if X_1 is vertical
2. $R_g^*\alpha = \rho(g^{-1})\alpha$

Lemma 3.80. *We have a bijection between*

$$\Omega^n(P,V)^{h,G} \stackrel{\cong}{\to} \Omega^n(B,P(V)) \ , \quad \omega \mapsto \bar{\omega}$$

such that

$$\bar{\alpha}(b)(X_1,\ldots,X_n) = [p, \alpha(p)(\tilde{X}_1,\ldots,\tilde{X}_n)]$$

for any $p \in P_b$ and lifts $\tilde{X}_i$ of X_i

Proof. well defined:

- independent of choice of lifts:

– two lifts differ by vertical vector

– α vanishes on vertical vectors

- independent on p

– $p' = pG$

– can take lifts $R_{g,*}\tilde{X}_i$

– $\alpha(pg)(R_{g,*}\tilde{X}_1,\ldots,R_{g,*}\tilde{X}_n) = \rho(g^{-1})\alpha(p)(\tilde{X}_1,\ldots,\tilde{X}_n)$

– $[pg, \rho(g^{-1})v] = [p,v]$

inverse map:

$\alpha(p)(\tilde{X}_1, \ldots, \tilde{X}_n) = Z$ where

- $\bar{\alpha}(X_1, \ldots, X_n) = [p, Z]$

- $X_i = T\pi_*(\tilde{X}_i)$

□

$R^\omega \in \Omega^2(B, \mathrm{Ad}(P))$ correspond to Ω.

Definition 3.81. $R^\omega \in \Omega^2(B, \mathrm{Ad}(P))$ *is called the curvature of the principal bundle connection* ω

note: $R^{\omega+\delta} = R^\omega + \nabla \wedge \delta + [\delta, \delta]$

3.2.3 Associated vector bundles

$\rho : G \to \mathrm{End}(V)$ representation

- $\rho(P) := P \times_G V$ - associated vector bundle

- apply ρ to the cocycle for P

identify section spaces $\Gamma(B, \rho(P)) \cong \Omega^0(B, \rho(P)) \cong C^\infty(P, V)^G$

- $s \mapsto \tilde{s}$

– recall $\tilde{s} : P \to V$, $R_G^* \tilde{s} = \rho(g^{-1})\tilde{s}$

– get s back: $s(b) = [p, \tilde{s}(p)]$

$T^h P$ - principal bundle connection

- define linear connection such that for X in $\mathcal{X}(B)$

$$\widetilde{\nabla_X s} = X^h(\tilde{s})$$

checks

1. $X^h(\tilde{s})$ corresponds to section:

   – use that X^h is invariant

   – X^h commutes with R_g^*

- $R_g^*(X^h(\tilde{s})) = X^h(R_g^*\tilde{s}) = X^h(\rho(g^{-1})(\tilde{s})) = \rho(g^{-1})(X^h(\tilde{s}))$

2. $(X, s) \mapsto \nabla_X s$ is $C^\infty(B)$-linear in X: clear

3. $(X, s) \mapsto \nabla_X s$ satisfies Leibnitz rule: exercise

relation between curvatures:

have bundle morphism $\mathrm{Ad}(P) \to \mathrm{End}(\rho(P))$

- $P(\rho) : [p, X] \mapsto [p, \rho(X)]$

– well defined: $[pg, \mathrm{Ad}(g^{-1})(X)] \mapsto [pg, d\rho(\mathrm{Ad}(g^{-1})(X))] = [pg, \rho(g^{-1})\rho(X)\rho(g^{-1})] = [p, \rho(X)]$

– extends to $P(\rho) : \Omega^2(B, \mathrm{Ad}(P)) \to \Omega^2(B, \mathrm{End}(\rho(P))$

Lemma 3.82. *We have the relation* $F^\nabla = P(\rho)(R^\omega)$

Proof. Exercise! □

$\gamma : [0, 1] \to B$ - curve in B

- $\tilde{\gamma}$ horizontal lift an P

- $t \to [\tilde{\gamma}(t), v]$ is parallel section of $\rho(P)$ along γ

- the parallel transport $\|^\gamma : \rho(P)_{\gamma(0)} \to \rho(P)_{\gamma(1)}$ is given by

- $[\tilde{\gamma}(0), v] \mapsto [\tilde{\gamma}(1), v]$

from vector bundle connection to principal bundle connection on frame bundle

- ∇ linear connection on $E \to B$ given

- p in $\mathrm{Fr}(E)$, $\pi(p) = b$

- can choose local section $f : B \to P$ such that

- $f(b) = p$

- the section $b' \mapsto f(b)(v) \in E$ is parallel in b

- define $T_p^h P := Tf(T_b B)$

- check: this determines a principal bundle connection

- under $\mathrm{id}(\mathrm{Fr}(E)) \cong E$ get back ∇ as associated linear connection

3.2.4 Quotients

M - manifold

G - Lie group

- G acts from the right on M

Definition 3.83. *G acts freely if $mg = m$ for some m in M implies that $g = e$.*

Definition 3.84. *G acts properly if $M \times G \to M \times M$, $(m, g) \mapsto (m, mg)$ is proper.*

- properness is a topological propery

G acts on topological space M

in the following: G is a group acting from the right on a topological space

Lemma 3.85. *The quotient map $\pi : M \to M/G$ is open.*

Proof. the quotient is characterized by universal property

- it follows that topology of M/G is generated by the subsets U with $\pi^{-1}(U)$ open

- this is the maximal topology such that π continuous

consider $W \subseteq M$ open

- want to show that $\pi(W)$ is open

– enough to show that $\pi^{-1}(\pi(W))$ is open

– but $\pi^{-1}(\pi(W)) = \bigcup_{g \in G} Wg$ is open

— this last step uses that we consider quotient by group action and not an arbitrary quotients by some equivalence relation □

Lemma 3.86. *If M is Hausdorff and G acts properly, then M/G is Hausdorff.*

Proof. by contradiction:

consider $\bar{m}, \bar{m}'$ in $\bar{M}$

assume: they are not separated by open sets

- consider preimages m, m'

- for every V, V' separating m, m' in M

- $VG \cap V'G \neq \emptyset$

– equiv: $V \cap V'G \neq \emptyset$

- consider decreasing families for such neighborhoods: (V_i), (V_i')

– get for every i:

– $m_i \in V_i$, $m_i' \in V_i'$, $g_i \in G$ with $m_i' g_i = m_i$

- conclude:

– $m_i \to m$

– $m_i' \to m'$

- conclude: $(m_i', m_i' g_i) \to (m', m)$

– by properness of $M \times G \to M \times M$: (m_i', g_i) has accumulation point (m', g)

– by continuity: $gm' = m$

– this implies: $\bar{m}' = \bar{m}$ - a contradiction ☐

Proposition 3.87. *If G acts freely and properly, then the set M/G has a manifold structure such that $\pi : M \to M/G$ is smooth and a G-principal bundle.*

Proof. set $B := G/M$ as topological quotient

- clarify general topological properties:

– $\pi : M \to B$ is open

– by properness of action: B is Hausdorff

– B is second countable

— $(U_i)_i$ - countable base of topology of M

— $(\pi(U_i G))_i$ is a countable base of topology of B

– B is paracompact

— we will show that B is locally euclidean:

— in particular it is locally compact

— a locally compact second countable Hausdorff space is paracompact

construct vertical bundle:

- $X \in \mathfrak{g}$

– for every m in M:

- $\mathfrak{g} \ni X \mapsto X^\sharp(m)$ is injective

– here is the argument:

— if $X^\sharp(m) = 0$, then (by uniqueness of integral curves) $m \exp(tX) = m$ for all t

— by freeness of action: $\exp(tX) = e$ for all t

— apply $(\partial_t)_{|t=0}$: $X = 0$

- define $T^v\pi \subseteq TM$ to be generated by the values of fundamental vector fields

– has constant rank $\dim(\mathfrak{g})$

– is a subbundle

- $b \in B$

– construct chart of B at b

— choose $m \in M_b$

— choose vector fields $Y_1, \ldots, Y_r$ near m complementary to $T^v\pi$ at m

– there exists nbhd $0 \in U \subseteq \mathbb{R}^r$ such that

– $H(t_1, \ldots, t_r) := \Phi^{Y_r}_{t_r} \circ \cdots \circ \Phi^{Y_1}_{t_1}(m)$ is defined for $(t_1, \ldots, t_r) \in U$

consider G-equivariant map $F : U \times G \to M$ given by $(t, g) \mapsto H(t)g$

claim: $TF(0, e)$ is isomorphism:

- $TF(0,e)(\partial_i) = Y_i(m)$

- $TF(0,e)(X) = X^\sharp(m)$

- one can choose U and $e \in V \subseteq G$ such that $F : U \times V \to M$ is diffeomorphism

- claim: can make U smaller such that $F : U \times G \to M$ is diffeomorphism into image

– differential DF is isomorphism (by G-invariance calculation at m implies same at mg)

– enough to show first: this map is injective

– otherwise: find sequences (x_i), (x_i') in U and (g_i), (g_i') in G such that

— $(x_i, g_i) \neq (x_i', g_i')$ for all i

— $F(x_i, g_i) = F(x_i', g_i')$

— $x_i \to 0$, $x_i \to 0$.

— set $h_i := g_i^{-1} g_i'$

— then by equivariance: $F(x_i, e) = F(x_i', h_i)$

— $H(x_i')h_i = H(x_i) \to m$ converges

— by properness $h_i \to h$ (after going to subsequence)

— get $mh = m$

— by freeness: $h = e$

— but then (x_i, e) and (x_i', h_i) belong to $U \times V$ for large i

— conclude $x_i = x_i'$, $h = e$

— $(x_i, g_i) = (x_i', g_i')$ for large i - contradiction

define chart ϕ of B near $b = [m]$ by:

$\phi([m']) = \mathrm{pr}_1(F^{-1}(m'))$

- is independent of choice of representative of $[m]$

- is continuous: $\phi^{-1}(W) = \mathrm{pr}_1(\pi^{-1}(W))$ is open since π is continuous and pr_1 is open.

- its inverse is $t \mapsto \pi \circ H(t)$ is also continuous transition functions

define ϕ' similarly using F'

- $\phi'(\phi^{-1}(t)) = \mathrm{pr}_1(F'^{-1}(H(t)))$ is smooth

□

Example 3.88. G- Lie group

$P \to B$ - G- principal bundle

- $B \cong P/G$

- $\rho : G \to GL(V)$ - representation
- G acts on $P \times V$ by $(p, v)g \mapsto (pg, \rho(g^{-1})v)$
- $P \times V \to (P \times V)/G = P \times_{G,\rho} V$ is G-principal bundle

□

Corollary 3.89. *If G is compact and acts freely on M, then we have a G-principal bundle $M \to M/G$.*

Corollary 3.90. *If G is a closed subgroup of a Lie group H, then we have a G-principal bundle $H \to H/G$.*

here we use "Cartan's Theorem": A closed subgroup of a Lie group is a submanifold.

Example 3.91. many interesting manifolds arrise as quotients in this way

1. $GL(V)/O(V, h^V)$ - manifold of scalar products on V
2. $SO(n+1)/SO(n) \cong S^n$ - oriented lines in $\mathbb{R}^{n+1}$
3. $U(n+1)/U(n) \times U(1) \cong \mathbb{CP}^n$ - lines in $\mathbb{C}^{n+1}$
4. $O(n+m)/O(n) \times O(m) = \mathrm{Gr}(n, m)$ - n-planes in $\mathbb{R}^{n+m}$
5. $U(n)/\underbrace{U(1) \times \cdots \times U(1)}_{n\times}$ - manifold of decompositions $\mathbb{C}^n = L_1 \oplus \cdots \oplus L_n$ into lines

□

4 Riemannian geometry

4.1 Connections on the tangent bundle

M manifold

- consider connections ∇ on TM

- have torsion tensor

– $T^\nabla \in \Omega^2(M, TM)$: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

- we say that ∇ is torsion-free if $T^\nabla = 0$

- for $\omega \in \Omega^1(M, \mathrm{End}(TM))$

– $T^{\nabla+\omega}(X,Y) = T^\nabla(X,Y) + \omega(X)(Y) - \omega(Y)(X)$

Example 4.1. ∇ - any connection on TM

- $\nabla' := \nabla - \frac{1}{2}T^\nabla$ is torsionfree:

- interpret: $T^\nabla \in \Omega^1(M, \mathrm{End}(TM))$

– $T^\nabla(X)(Y) := T^\nabla(X,Y)$

– $\nabla'_X Y := \nabla_X Y - \frac{1}{2}T^\nabla(X,Y)$ □

Definition 4.2. *A Riemannian metric on M is a metric g on TM. A Riemannian manifold is a pair (M,g)*

Proposition 4.3 (Levi-Civita connection)**.** *On a Riemannian manifold there exists a unique connection which is compatible with the metric and torsion free.*

Proof. uniqueness: ∇, ∇' two such connections

- $\nabla' = \nabla + \omega$

- torsionfreeness of both: $\omega(X)Y - \omega(Y)X = 0$

- compatibility with metric: $g(\omega(X)Y, Z) = -g(Y, \omega(X)Z)$

– will show: these two conditions imply that $\omega = 0$

— calculate for arbitrary X, Y, Z:

$$\begin{aligned} g(\omega(X)Y, Z) &= g(\omega(Y)X, Z) \\ &= -g(X, \omega(Y)Z) \\ &= -g(X, \omega(Z)Y) \\ &= g(\omega(Z)X, Y) \\ &= g(\omega(X)Z, Y) \\ &= -g(Z, \omega(X)Y) \\ &= -g(\omega(X)Y, Z) \end{aligned}$$

— hence $g(\omega(X)Y, Z) = 0$ for all X, Y, Z

– this shows that $\omega = 0$

existence:

want to define $\nabla_X Y$ by :

$$\begin{aligned} 2g(\nabla_X Y, Z) \; := \; & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & -g([X, Z], Y) - g([Y, Z], X) + g([X, Y], Z) \end{aligned}$$

here $X, Y, Z \in \mathcal{X}(M)$

- claim: $\nabla_X Y \in \mathcal{X}(M)$

- must check $C^\infty(M)$-linearity of r.h.s. in Z:

– insert fZ:

— terms which derive f: $X(f)g(Y, Z) + Y(f)g(X, Y) - X(f)g(Z, Y) - Y(f)g(X, Z) = 0$

– must check $C^\infty(M)$-linearity of r.h.s. in X:

— insert fX:

— terms which derive f: $Y(f)g(X, Z) - Z(f)g(X, Y) + Z(f)g(X, Y) - Y(f)g(X, Z) = 0$

– must check Leibnitzrule of r.h.s. in Y:

— insert fY:

— terms which derive f: $X(f)g(Y,Z) - Z(f)g(X,Y) + Z(f)g(X,Y) + X(f)g(Y,Z) = 2X(f)g(Y,Z)$

— this the expected term

have now well-defined connection ∇

compatible with metric:

- use vector fields with vanishing commutator

- $2g(\nabla_X Y, Z) + 2g(\nabla_X Z, Y) = 2Xg(Y,Z)$ ok

torsion free :

- use vector fields with vanishing commutator

$2g(\nabla_X Y, Z) - 2g(\nabla_Y X, Z) = 0$ ok

□

Definition 4.4. *The connection described in Prop. 4.3 is called the Levi-Civita connection.*

Example 4.5. (M,g) Riemannian

- ∇^M - Levi-Civita connection

- $i : N \subseteq M$ submanifold

- $g^N := Di^* g$ is Riemannian metric

- $P : i^*TM \to TN$ orthogonal projection

Lemma 4.6. *$P\nabla^M$ is Levi-Civita connection on N.*

Proof. P is orthogonal

- $P\nabla^M$ is compatible with metric

– locally near N have product structure: $\mathbb{R}^n \times \mathbb{R}^{m-n}$ such that N corresponds to $\mathbb{R}^n \times \{0\}$

– $X, Y \in \mathcal{X}(N)$

– can extend to $\tilde{X}, \tilde{Y}$ in M (constant in $\mathbb{R}^{m-n}$-direction)

– then $[\tilde{X}, \tilde{Y}]$ has values in TN

$$\begin{aligned} T^{P\nabla^M}(X,Y) &= P\nabla_{\tilde{X}}\tilde{Y} - P\nabla_{\tilde{Y}}\tilde{X} - [X,Y] \\ &= P(\nabla_{\tilde{X}}\tilde{Y} - P\nabla_{\tilde{Y}}\tilde{X} - [\tilde{X},\tilde{Y}] \\ &= PT^{\nabla}(\tilde{X},\tilde{Y}) \\ &= 0 \end{aligned}$$

□

Example 4.7. $(\mathbb{R}^m, g_{eu})$ is Riemannian manifold

- g_{eu}. - canonical metric

- ∇^{triv} is Levi-Civita connection

$N \subseteq \mathbb{R}^m$ submanifold

- $i : N \to \mathbb{R}^m$. - inclusion

- $Di : TN \to i^*T\mathbb{R}^m$

- $i^*g_{eu} =: g$ is induced Riemannian metric

- $P\nabla^{\text{triv}}$ is Levi-Civita connection

– is the tangential component of the derivative

historically important observation:

- a priori: the connection $P\nabla^{\text{triv}}$ depends on the embedding

- Levi-Civita: (1917 for surfaces) $P\nabla^{\text{triv}}$ only depends on induced metric, but not on embedding

– we already know this

- later generalized by Weyl

□

notation for curvature $R := F^{\nabla} \in \Omega^2(M, \text{End}(TM))$

- note $R(X,Y)$ is antisymmetric since ∇ is compatible with metric

4.2 The Riemannian distance

(M,g) Riemannian

- $\gamma : [0,1] \to M$ path

– $\gamma' : [0,1] \to TM$ speed

Definition 4.8. *The length of γ is defined by*

$$\ell(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt \ .$$

properties of the length:

Lemma 4.9.

1. *$\ell(\gamma)$ is reparametrization invariant.*
2. *$\ell(\gamma^{\mathrm{op}}) = \ell(\gamma)$*
3. *$\ell(\gamma_0 \sharp \gamma_1) = \ell(\gamma_0) + \ell(\gamma_1)$*

Proof. Exercise: □

assume: M is path-connected

- write $\gamma : m \to m'$ for path from m to m'

Definition 4.10. *We define $d : M \times M \to [0,\infty)$ by*

$$d(m,m') := \inf_{\gamma : m \to m'} \ell(\gamma) \ .$$

Lemma 4.11. *d is a metric on M which defines the topology.*

Proof.

$d(m,m) = 0$

- use constant path

$d(m,m') = d(m',m)$

- use $\ell(\gamma^{\mathrm{op}}) = \ell(\gamma)$

$d(m, m') \leq d(m, m'') + d(m'', m')$

- if $\gamma_0 : m \to m''$ and $\gamma_1 : m'' \to m$, then $\gamma_1 \sharp \gamma_0 : m \to m''$

– $\ell(\gamma_1 \sharp \gamma_0) = \ell(\gamma_0) + \ell(\gamma_1)$

– but we have more path's from m to m' to approximate $d(m, m')$ which do not go over m''

consider chart $\phi : U \to \mathbb{R}^n$, $\phi(m) = 0$

- have Euclidean metric d_{eu} on U (induced via ϕ)

– Claim: There exists a constants $c, C > 0$ such that $c d_{eu}(m, m') \leq d(m, m') \leq C d_{eu}(m, m')$.

— this implies assertion about topology

— both metrics define the neighborhood filter at m

define $\|X\|^2$ using g_{eu}

- by continuity and local compactness after making U smaller:

– there exists $C, c > 0$ such that: $c^2\|X\|^2 \leq g(x)(X, X) \leq C^2\|X\|^2$ for all X

$x \in U$

assume that $B_{d_{eu}}(0, \|x\|) \subseteq U$

- upper estimate:

- take linear curve $\gamma(t) := tx$ from 0 to x

$d(0, x) \leq \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt \leq \int_0^1 \sqrt{g(tx)(x, x)}dt \leq \int_0^1 C\|x\|dt = C\|x\|$

lower estimate

- $\gamma : 0 \to x$ in U any curve

– first inequality below:

— straight curves are shortest in euclidean space

— mean value theorem

– $c\|x\| \leq c\int_0^1 \|\gamma'(t)\|dt \leq \int_0^1 \sqrt{g(\gamma'(t),\gamma'(t))}dt = \ell(\gamma)$

- every curve which leaves U is even longer

– minimize over all γ: $c\|x\| \leq d(0,x)$

this also shows that $d(m,m') = 0$ implies $m = m'$

□

Question:

- can the distance be realized by a curve?

- how can one characterize such a curve?

4.3 Geodesics

(M,g) - Riemannian

$\gamma : [0,1] \to M$

Definition 4.12. *The energy of γ is defined by*

$$E(\gamma) := \int_0^1 g(\gamma'(t),\gamma'(t))dt \ .$$

no square root

Cauchy-Schwarz: $\ell(\gamma) \leq \sqrt{E(\gamma)}$

- equality if $g(\gamma'(t),\gamma'(t)) = \text{const}$

– in this case $g(\gamma'(t),\gamma'(t)) = \ell(\gamma)^2$

a family of curves with fixed ends is a smooth map $\gamma : I \times [0,1] \to M$ such that $\gamma(u,0)$ and $\gamma(u,1)$ are constant

- here $I \subseteq \mathbb{R}$

- write $\gamma(u,t) := \gamma_u(t)$

Definition 4.13. *γ is critical for E if for every family of curves with fixed ends $(\gamma_u)_{u\in I}$ with $\gamma = \gamma_0$*

$$(\partial_u)_{|u=0}E(\gamma_u) = 0 \ .$$

∇ - Levi-Civita

Proposition 4.14. *γ is critical for E if and only if*

$$\nabla_{\partial_t}\gamma'(t) = 0 \ .$$

Proof. write $\partial_u\gamma = \gamma^\sharp$

use that ∇ is compatible with metric and torsion free

$$\begin{aligned}
(\partial_u)_{|u=0}E(\gamma_u) &= \int_0^1 (\partial_u)_{|u=0} g(\gamma_u'(t),\gamma_u'(t))dt \\
&= 2\int_0^1 g(\nabla_{\partial_u}\gamma_u'(t),\gamma'(t))_{|u=0}dt \\
&\overset{T^\nabla=0}{=} 2\int_0^1 g(\nabla_{\partial_t}\gamma^\sharp(t),\gamma'(t))dt \\
&= \int_0^1 \partial_t g(\gamma^\sharp(t),\gamma'(t))dt - \int_0^1 g(\gamma^\sharp(t),\nabla_{\partial_t}\gamma'(t))dt \\
&= g(\gamma^\sharp,\gamma')|_0^1 - \int_0^1 g(\gamma^\sharp(t),\nabla_{\partial_t}\gamma'(t))dt \\
&= -\int_0^1 g(\gamma^\sharp(t),\nabla_{\partial_t}\gamma'(t))dt
\end{aligned}$$

- can arrange (γ_u) such that $\gamma^\sharp$ is arbitrary vector field along γ

– in chart $\gamma_u = \gamma + u\gamma^\sharp$

– globally glue using partition of unity

- conclude $\nabla_{\partial_t}\gamma'(t) = 0$ as necessary and sufficient condition

□

Definition 4.15. *A curve γ in M satisfying $\nabla_{\partial_t}\gamma' = 0$ is called a geodesic.*

- in ccordinates

– $\nabla = \nabla^{\text{triv}} + \omega$

– $\nabla_{\partial_t} = \partial_t + \omega(\gamma(t))(\gamma'(t))$

– $\nabla_{\partial_t}\gamma'$ is equation: $\partial_t\gamma' + \omega(\gamma(t))(\gamma'(t))(\gamma'(t)) = 0$

– is second order ODE

– in ccordinates:

— set $\Gamma^i_{j,k}\partial_i = \omega(\partial_j)(\partial_k)$

– ODE: $\gamma''^{,i} = -\Gamma^i_{j,k}\gamma^j\gamma^k$

corresponds to vector field $S \in \Gamma(TM, T(TM))$

- S is called the geodesic spray

– in coordinates

– x of M

– (x, ξ) of TM

– $S(x, \xi) = (\xi, -\omega(x)(\xi)(\xi))$

– solution of geodesic equation uniquely determined by $\gamma'(0) \in TM$

Lemma 4.16. *A geodesic has constant (absolute) speed*

Proof.

γ - a geodesic

- $\partial_t g(\gamma', \gamma') = 2g(\nabla_{\partial_t}\gamma', \gamma') = 0$

□

- for every X in TM there exists maximal interval $[0, a(X))$ such that the geodesic with initial condition X exists

– scale invariance

— if $\gamma : I \to M$ is geodesic, then $\gamma(st) : s^{-1}I \to M$ is also one

— for $a < a(X)$

— then $t \to \gamma(at) : [0, 1] \to M$ exists with $\gamma'(0) = aX$

Corollary 4.17. *There exists a maximal neighbourhood U of the zero section of TM such that for every $X \in U$ there exists a geodesic $\gamma^X : [0,1] \to M$ with $\gamma^{X,\prime}(0) = X$. This geodesic is unique*

Definition 4.18. *The map $\exp : U \to M$, $X \mapsto \gamma^X(1)$ is called the exponential map.*

for m in M write $\exp_m : (U \cap T_m M) \to M$ for the restriction

Lemma 4.19. $\exp_m$ *is diffeomorphism near* 0

Proof. - $X \in T_m M$

- interpret X in $T_0(T_m M)$

– $T \exp_m(X) = (\partial_t)_{|t=0} \exp_m(tX) = X$

- $D \exp_m(0) = \mathrm{id}_{T_m M}$

- in particular: is invertible

□

- $\exp_m$ is called exponential chart/coordinates

- $t \mapsto \exp_m(tX)$ is geodesic with $\gamma'(0) = X$

Example 4.20. $(\mathbb{R}^n, g_{eu})$

- Levi-Civita connection is ∇^{triv}

- x in $\mathbb{R}^n$

- X in $T_x\mathbb{R}^n \cong \mathbb{R}^n$

- geodesic with initial condition (x, X) is $\gamma(t) := x + tX$

– indeed: $\gamma'(t) \equiv X$

– $\nabla^{\mathrm{triv}}_{\partial_t}(\gamma'(t)) = 0$

Exponential map: $\exp(x)(X) = x + X$ □

Example 4.21. $S^2 \subseteq \mathbb{R}^3$

- induced metric:

- claim: big circles are geodesics

consider w.l.o.g. $S^2 \cap \{z=0\}$ parametrized as $\gamma(t) = (\cos(t), \sin(t), 0)$

- $\gamma'(t) = (-\sin(t), \cos(t), 0)$

- $\nabla_{\partial_t}\gamma'(t) = P\nabla^{\mathrm{triv},\mathbb{R}^3}_{\partial_t}\gamma'(t) = P(-\cos(t), -\sin(t), 0) = 0$

– vector points perpendicular to sphere

consider circle of latitude

- $\sigma(t) := (\sqrt{1-h^2}\cos(t), \sqrt{1-h^2}\sin(t), h)$

- $\sigma'(t) = (-\sqrt{1-h^2}\sin(t), \sqrt{1-h^2}\cos(t), 0)$

- $\nabla^{\mathrm{triv}}_{\partial_t}\sigma'(t) = (-\sqrt{1-h^2}\cos(t), -\sqrt{1-h^2}\sin(t), 0)$

- $P\nabla^{\mathrm{triv}}_{\partial_t}\sigma'(t) \neq 0$ (h-component is missing) -

– σ is not a geodesic

□

4.4 Families of geodesics and Jacobi fields

want to understand $T\exp_m$

- $(X_u)_u$ - family of vectors in T_mM

- $(t \to \exp_m(tX_u))$ - family of geodesics

- want to understand vector field $(\partial_u)_{|u=0}\exp_m(tX_u)$ as function of t

$(\gamma_u)_u$ - family of curves

- smooth map $I \times J \to M$, I, J intervals

Definition 4.22. *$(\gamma_u)_u$ is a family of geodesics if γ_u is a geodesic for every u in I.*

notation:

- γ' - derivative by t

– $\gamma^\sharp$ - derivative by u

- interpret formulas on pull-back of TM to $I \times J$

$$
\begin{aligned}
\nabla_{\partial_t}\nabla_{\partial_t}\gamma^\sharp &\overset{T^\nabla=0}{=} \nabla_{\partial_t}\nabla_{\partial_u}\gamma' \\
&\overset{R}{=} \nabla_{\partial_u}\nabla_{\partial_t}\gamma' + R(\gamma',\gamma^\sharp)\gamma' \\
&\overset{\nabla_{\partial_t}\gamma'=0}{=} R(\gamma',\gamma^\sharp)\gamma'
\end{aligned}
$$

$\gamma : I \to M$ - geodesic

Definition 4.23. *A section $J \in \Gamma(I, \gamma^*TM)$ is called a Jacobi field if it satisfies the ODE*

$$\nabla_{\partial_t}\nabla_{\partial_t}J - R(\gamma', J)\gamma' = 0 \ .$$

- second order linear ODE

- space of Jacobi field is $2n$-dimensional with $n = \dim(M)$

- fix $t_0 \in I$

- Jacobi field Y is uniquely determined by $J(t_0)$ and $(\nabla_{\partial_t}J)(t_0)$

Example 4.24. Jacobi fields in $\mathbb{R}^n$

- $\gamma(t) = tX$

- fix Y, Z in $\mathbb{R}^n$

- then $J(t) = Y + tZ$ is Jacobi field

- in fact $tX + u(Y + tZ) = t(X + Z) + uY$ is family of geodesics

- alternatively: check ODE

□

Lemma 4.25. *$T\exp_m(X) : T_mM \to T_{\exp_m(X)}M$ is the linear map which sends Y in T_mM to the value of the Jacobi field J at $t = 1$ along $t \mapsto \exp_m(tX)$ with initial values $J(0) = 0$ and $\nabla_{\partial_t}J(0) = Y$.*

Proof. consider $J := t \mapsto T\exp_m(tX)(Y) = (\partial_u)_{|u=0}\exp_m(t(X + uY))$

- is Jacobi field J with

- $J(0) = 0$ (set $t = 0$ and differentiate by u)

- $\nabla_{\partial_t} J(0) = (\nabla_{\partial_t})_{|t=0}(tT\exp_m(tX)(Y)) = Y$

evaluate map at 1 □

Definition 4.26. *(M, g) has negative/positive curvature if $\pm g(R(X, Y)Y, X) < 0$ for all m in M and lin. independent $X, Y \in T_m M$.*

Proposition 4.27. *If (M, g) has non-positive curvature, then $T\exp_m(X)$ is an isomorphism for every X in the domain of definition.*

Proof. suiffices to show injective

- by contradiction:

– assume:

— $\exp_m(X)$ define

– $T\exp_m(X)(Y) = 0$, but $Y \neq 0$

$\gamma(t) := \exp_m(tX)$ geodesic

- there exists Jacobi field J with

– $J(0) = 0$

– $\nabla_{\partial_t} J(0) = Y$

– $J(1) = 0$

calculate

- scalar multiply ODE for J with J

$$\begin{aligned} 0 &= g(\nabla_{\partial_t}\nabla_{\partial_t} J, J) - g(R(\gamma', J)\gamma', J) \\ &= \partial_t g(\nabla_{\partial_t} J, J) - g(\nabla_{\partial_t} J, \nabla_{\partial_t} J) - g(R(\gamma', J)\gamma', J) \end{aligned}$$

integrate from 0 to 1

- $0 = g(\nabla_t J, J)|_0^1 - \int_0^1 g(\nabla_{\partial_t} J, \nabla_{\partial_t} J)dt - \int_0^1 g(R(\gamma', J)\gamma', J)dt$

- use:

– $\int_0^1 g(\nabla_{\partial_t} J, \nabla_{\partial_t} J)dt > 0$ (since $\nabla_{\partial_t} J(0) \neq 0$)

– use $J(0) = 0,\ J(1) = 0$

– get $\int_0^1 g(R(\gamma', J)J, \gamma')dt > 0$

- contradicts non-positive curvature □

Corollary 4.28. *Assume that (M, g) has non-positive curvature. If $U \subseteq T_m M$ is in the domain of definition and $(\exp_m)_{|U}$ is injective, then it is a diffeomorphism into its image.*

Example 4.29. $\mathbb{R}^n$ is flat

- curvature is non-positive

- $\exp(0)(X) = X$

- is diffeomorphism

$T^n := \mathbb{R}^n/\mathbb{Z}^n$

- $\pi : \mathbb{R}^n \to T^n$ projection $\pi(x) = [x]$

- $T_{[x]}R^n \cong T_x\mathbb{R}^n$ via $T\pi(x)$

- $\exp_{[x]}(d\pi(x)(X)) = \pi(\exp_x(T\pi(x)^{-1}(X)) = \pi(x + T\pi(x)^{-1}(X))$

- $T\exp_{[x]} = T\pi(x) \circ T\exp(x) \circ T\pi(x)^{-1}$ is isomorphism for all x

- $\exp_{[x]}$ is not injective

□

Example 4.30. S^2 in $\mathbb{R}^3$

$N = (0, 0, 1)$ - northpole

- $\exp_m(\pi X) = S = (0, 0, -1)$ for every unit vector X in $T_N S^2$

- $T\exp_m(\pi X) = 0$, in particular not injective

- but S^2 has positive curvature - hence not contradiction □

4.5 Gauss lemma

geodesic balls

- T_mM has metric $g(m)$

– write $\|-\|$ for length

– use this metric to define ball $B(0,r) := \{X \in T_mM \mid \|X\| < r\}$

- assume: $r > 0$ such that $\exp_m$ is defined and diffeomorphism on $B(0,r)$ in T_mM

γ - geodesic

- J Jacobi field along γ

Lemma 4.31. *We have* $g(J(t),\gamma'(t)) = tg(\nabla_{\partial_t}J(0),\gamma'(0)) + g(J(0),\gamma'(0))$.

Proof. - scalar product of ODE by γ':

– use $g(R(\gamma',J)\gamma',\gamma') = 0$ by antisymmetry

– get $g(\nabla_{\partial_t}\nabla_{\partial_t}J,\gamma') = 0$

– $0 = \partial_t g(\nabla_{\partial_t}J,\gamma') - \partial_t g(\nabla_{\partial_t}J,\nabla_{\partial_t}\gamma') = \partial_t g(\nabla_{\partial_t}J,\gamma')$

hence $g(\nabla_{\partial_t}J,\gamma')$ is constant in t

- again: $g(\nabla_{\partial_t}J,\gamma') = \partial_t g(J,\gamma')$

- hence $g(J(t),\gamma'(t)) = tg(\nabla_{\partial_t}J(0),\gamma'(0)) + g(J(0),\gamma'(0))$ □

Corollary 4.32. *For every X in $B(0,r)$ and $Y \in T_mM$ we have*

$$g(T\exp_m(X)(Y), T\exp_m(X)(X)) = g(Y,X)\ .$$

Proof. geodesic $t \mapsto \exp(m)(tX)$

- apply Lemma to Jacobi field with $J(0) = 0$, $\nabla_{\partial_t}J(0) = Y$

- evaluate at $t = 1$ □

$T\exp_m$ preserves scalar products with radial vectors

assume: $r > 0$ such that $\exp_m$ is defined and diffeomorphism on $B(0,r)$ in T_mM

Proposition 4.33.

1. *For every $s \in (0,r)$ the subset $\exp_m(S(0,s))$ is the metric distance s-sphere at m*

2. $\exp_m(B(0,r))$ *is the metric ball at* m *of radius* r *in* M*.*

3. *For* X *in* $B(0,r)$ *the curve* $t \mapsto \exp_m(tX)$ *realizes the distance between* m *and* $\exp_m(X)$*.*

4. *If* $\sigma : [0,T]$ *is any curve from* 0 *to* $\exp_m(X)$ *with* $\ell(\sigma) = \|X\|$*, then* $\sigma(t) = \exp(f(t)X)$ *for* $f : [0,T] \to [0,1]$ *monotoneous.*

Proof. $1 \Rightarrow 2$ is clear

show 2

- if $\|X\| < s$, then $d(m, \exp_m(X)) \leq \|X\| < s$

- hence $\exp_m(X) \notin \exp_m(S(0,s))$

- take $s < s' < r$

- assume that $m' \in M \setminus \exp_m(\bar{B}(0,s'))$

Lemma 4.34. *We have* $d(m,m') \geq s'$*.*

- hence $d(m,m') = s$ implies $m \in \exp_m(S(0,s))$

Proof. γ - curve from m to m'

- a maximal such that $\gamma([0,a]) = \{m\}$

– last time that γ meets m

- b minimal such that $\gamma(v) \in \exp_m(S(0,s'))$

– first time of exit the s'-Ball

- $\sigma := \exp_m^{-1}(\gamma_{|(a,b]})$

– a curve from 0 to the s'-sphere in T_mM (0 excluded)

- write $g(m)$ as $\langle -, - \rangle$ (scalar product on T_mM)

- express $\sigma(t)$ in polar coordinates (for $t \in (a,b]$)

- $\sigma(t) = \rho(t)\xi(t)$, $\xi(t)$ unit vector, $\rho(t) := \|\sigma(t)\|$

– $\xi(t)$ is well-defined since $\sigma(t) \neq 0$ since $t > a$

- $\sigma' = \rho'\xi + \rho\xi'$

- define vector field $Z(X) = X/\|X\|$ on $T_mM \setminus \{0\}$

– is radial unit-norm

– $\xi(t) = Z(\sigma(t))$

$\langle Z(\sigma(t)), \sigma'(t)\rangle = \langle \xi(t), \rho'(t)\xi(t) + \rho(t)\xi'(t)\rangle = \rho'(t)\langle \xi(t), \xi(t)\rangle = \rho'(t)$

here we use: $0 = \partial_t\langle \xi(t), \xi(t)\rangle = 2\langle \xi(t), \xi'(t)\rangle$

- $\tilde{Z}$ - image under $\exp_m(B(0, r))$

- also unit-norm, since $T\exp_m$ preserves length of radial fields

- by Gauss Lemma and since $\tilde{Z}(\gamma(t))$ is radial at $\gamma(t)$:

– $g(\tilde{Z}(\gamma(t)), \gamma'(t)) = \langle Z(\sigma(t)), \sigma'(t)\rangle = \rho'(t)$

— use that $\tilde{Z}$ has unit-norm for second inequality (Cauchy-Schwarz)

$$\begin{aligned}
\ell(\gamma) &\geq \ell(\gamma_{|(a,b]}) \\
&= \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt \\
&\geq \int_a^b g(\tilde{Z}(\gamma'(t)), \gamma'(t)) dt \\
&= \int_a^b \rho'(t) dt \\
&= \rho(b) - 0 \\
&= s'
\end{aligned} \tag{3}$$

- γ was aritrary

- $d(m, m') \geq s'$

□

- see that $\exp_m(S(0,s))$ is s-distance sphere in M at m.

3:

clear: $\ell(t \mapsto \exp_m(tX)) = \|X\|$

– constant speed $\|X\|$

– $d(m, \exp_m(X)) = \|X\|$ by 1. since $X \in S(0,X)$

4:

$\gamma : m \to \exp_m(X)$ with length $\|X\|$

- $0 \leq a$ - last time with $\gamma(a) = 0$

- write $\gamma(t) = \exp_m(\rho(t)\xi(t))$

- Cauchy-Schwarz

$$\begin{aligned} \|X\| &= \ell(\gamma) \\ &\geq \int_a^T \sqrt{g(\sigma'(t), \sigma'(t))}dt \\ &\geq \int_0^T g(\tilde{Z}(\sigma'(t)), \sigma'(t))dt \\ &= \int_0^T \rho'(t)dt \\ &= \|X\| \end{aligned}$$

conclude: second inequality is equality

– $\sqrt{g(\sigma'(t), \sigma'(t))} = g(\tilde{Z}(\sigma'(t)), \sigma'(t))$ for all t

— hence by converse of Cauchy-Schwarz in equality case:

— conclude $\sigma'(t) \sim \tilde{Z}(\sigma(t))$, i.e. σ' points in positive radial direction

— solve $f'(t)\tilde{Z}(\sigma(t))\|X\| = \sigma'(t)$ for f

— f is monotoneous

– with initial condition $f(T) = 1$

- then $\exp_m(f(t)Y) = \sigma(t)$ for $t \in (a, T]$

– since $\exp_m(f(T)X) = \exp_m(X) = \sigma(T)$

– $\partial_t \exp_m(f(t)X) = f'(t)\|X\|\tilde{Z}(\sigma(t)) = \sigma'(t)$

conclude further: σ is constant for $t \leq a$ (otherwise this piece contributes to length)

- set $f(t) = 0$ for $t \in [0, a]$

□

$m \in M$

Lemma 4.35. *There exists an open neighbourhood $m \in W \subseteq M$ and $r > 0$ such that $(\exp_{m'})_{|B(0,r)}$ is a diffeomorphism for all $m' \in W$*

Proof. $U \subseteq TM$ open domain of exp

consider map $f : U \to M \times M$

- $U \ni X \mapsto (\pi(X), \exp_{\pi(X)}(X))$

- $0 \to T_m M \to T_{0_m}(TM) \to T_m M \to 0$ exact

– first map vertical embedding i

– second map $T\pi(m)$

- choose split $s : T_m M \to T_{0_m}(M)$

- $df(0_m)(s(Y) + i(X)) = (Y, X + A(Y))$

- A - some linear map

– $df(0_m)$ is upper triangular, hence invertible

– f is diffeomorphism on neighbourhood $U' \subseteq U$ of 0_m

- choose r and $m \in W$ such that

– r-ball-bundle over W is in U' □

m, m' in M

$\gamma : m \to m'$ curve

on $[0, T]$

Lemma 4.36. *If $\ell(\gamma) = d(m, m')$, then at every $t \in (0, T)$ there exists $\epsilon > 0$ such that $0 < t - \epsilon$ and $t + \epsilon < T$ and $\gamma(t + s) = \exp_{\gamma(t)}(f(s)X)$ for some vector X in $T_{\gamma(t)}M$ for all $s \in (-\epsilon, \epsilon)$.*

Proof. for any $0 \leq a < b \leq T$

$\gamma_{|[a,b]}$ realizes distance between $\gamma(a)$ and $\gamma(b)$

- otherwise could shorten path from m to m'

fix t

- can find $r > 0$ and $s > 0$ such that $(\exp_{m'})_{|B(0,s)}$ is diffeomorphism for all m' in $B(0, r)$
- take ϵ so small that

– $0 < t - \epsilon < t + \epsilon < T$

– $d(\gamma(t - \epsilon), \gamma(t + \epsilon)) < s$

– conclude: $\gamma_{|(t-\epsilon,t+\epsilon)}$ is reparametrized geodesic

– X is tangent at of this geodesic when it hits $\gamma(t)$

□

Corollary 4.37. *If γ is a constant speed curve which realizes the distance between its endpoints, then it is a geodesic.*

4.6 Completeness

(M, g) - Riemannian manifold assume: connected

- have metric d
- (M, d) is metric space

– have notion of completeness

Definition 4.38. *M is metrically complete if (M, d) is a complete metric space*

Definition 4.39. *M is metrically proper if (M, d) is a proper metric space*

Example 4.40. M compact - then metrically complete

$(\mathbb{R}^n, d)$ is complete

$\square$

Definition 4.41. *(M, g) is called geodesically complete at m if the exponential map $\exp_m$ is defined on all of $T_m M$. It is geodesically complete if it is geodesically complete at all points.*

- geodesically complete means: for every X in TM the geodesic with initial condition X exists on all of $\mathbb{R}$

Theorem 4.42 (Hopf-Rinow)**.** *Assume that M is connected. The following assertions are equivalent.*

1. *(M, g) is geodesically complete.*
2. *(M, g) is geodesically complete at a point m.*
3. *The balls $\bar{B}(m, r)$ are compact for all $r > 0$.*
4. *(M, g) is metrically proper.*
5. *(M, d) is metrically complete.*

In this case the distance between every two points in M can be realized by a curve (which can be taken as a geodesic).

Proof. proof shema:

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

and $2 \Rightarrow$ realization of distance (is used for $2 \Rightarrow 3$)

$1 \Rightarrow 2$

trivial

$3 \Rightarrow 4$:

- consider $\bar{B}(m', r')$

- it is contained in $\bar{B}(m, r' + d(m, m'))$

- closed subset of compact, hence itself compact

$4 \Rightarrow 5$:

$(m_i)_{i \in \mathbb{N}}$ - Cauchy sequence

- $\sup_i d(m_i, m) < \infty$

- sequence is contained in compact $\bar{B}(m, r)$ for r sufficiently large

- Cauchy sequence has accumulation point

$5 \Rightarrow 1$:

- by contradiction

- (M, g) not geodesically complete

– take X in TM such that maximal geodesic γ with inital X defined on $[0, T]$

— $\gamma'([0, T])$ is not relative compact by ODE-theory

– but $g(\gamma'(t), \gamma'(t)) = g(X, X)$ for all t

– for any sequence $0 \leq t_n \uparrow T$

— $(\gamma(t_n))$ is Cauchy sequence in M

— use: $d(\gamma(t_n), \gamma(t_m)) \leq |t_n - t_m|$

– has limit in M by metric completeness

— conclude: $\gamma'([0, T])$ is relatively compact

— contradiction

must show

$2 \Rightarrow 3$:

Lemma 4.43. *If (M, m) is geodesically complete at m, then every two points can be connected by a distance-realizing geodesic.*

Proof. choose $r > 0$ such that $(\exp_m)_{|B(0,2r)}$ is diffeomorphism

m' in M

if $d(m,m') < r$: write $m' = \exp_m(X)$

- $t \mapsto \exp_m(tX)$ is geodesic $m \to m'$ which realizes distance

assume now $d(m,m') \geq r$

- choose sequence $(\gamma_k)_{k\in\mathbb{N}}$ of curves $\gamma_k : m \to m'$ with: $\ell(\gamma_k) \to d(m,m')$
- define $t_k \in (0,1)$ first time with $d(m,\gamma_k(t_k)) = r$
- by compactness of $S(m,r)$: take subsequence - can assume $\gamma_k(t_k) \to q$ in $S(m,r)$

- $d(m,m') \leq d(m,\gamma_k(t_k)) + d(\gamma_k(t_k),m') \leq \ell(\gamma_k)$

– $k \to 0$ gives

– $d(m,m') = d(m,q) + d(q,m')$

- chose unique unit vector $X \in T_mM$ such that $q = \exp_m(rX)$
- consider curve $\gamma : [0,d(m,m')] \to M$, $\gamma(t) := \exp(tX)$

– it exists by assumption of geodesic completeness at m

- define subset $I \subseteq [0,d(m,m')]$

$$I := \{t \in [0,d(m,m')] \mid d(m,\gamma(t)) = t \;\&\; d(m,\gamma(t)) + d(\gamma(t),m') = d(m,m')\}$$

- know $r \in I$
- claim: $\sup I = d(m,m')$

assume claim:

- $d(m,\gamma(d(m,m'))) = d(m,m')$
- $d(m,\gamma(d(m,m'))) + d(\gamma(d(m,m')),m') = d(m,m')$, hence $d(\gamma(d(m,m')),m') = 0$

– hence $\gamma(d(m,m')) = m'$

– $\ell(\gamma) = d(m,m')$

— hence γ realizes distance between m and m'

proof of claim:

- by contradiction:

– $t := \sup I < d(m, m')$

— know: $r \leq t$

– $p := \gamma(t)$

– consider $s > 0$ such that $t + 2s < d(m, m')$ and $(\exp_p)_{|B(0,2s)}$ is diffeomorphism

– find x (as above) in $S(p, s)$ such that $d(p, x) + d(x, m') = d(p, m')$

– let $Y \in T_pM$ be unit vector such that $\exp_p(sY) = x$

$$\begin{aligned} d(m, x) &\leq d(m, p) + d(p, x) \\ &= d(m, p) + d(p, m') - d(x, m') \\ &= d(m, m') - d(p, m') + d(p, m') - d(x, m') \\ &= d(m, m') - d(x, m') \\ &\leq d(m, x) \end{aligned}$$

hence $d(m, x) = d(m, p) + d(p, x) = t + s$

set $\sigma(t) = \exp_p(tY)$

- $\ell(\gamma_{|[0,t]}) = d(m, p)$

- $\ell(\sigma_{|[0,s]}) = s$

– $\theta := \gamma_{|[0,t]} \sharp \sigma_{|[0,s]}$ realizes distance between m and x

– this implies that $Y = \gamma'(t)$ by Lemma 4.36

— hence $x = \gamma(t + s)$

– $t + s \in I$ contradiction

$\square$

$2 \Rightarrow 3$:

m in M

- $r > 0$

- must show: $\bar{B}(m,r)$ is compact

$(m_k)_{k\in\mathbb{N}}$ sequence in $\bar{B}(m,r)$

- $\gamma_k : m \to m_k$ geodesic on $[0,1]$, distance realizing

set $X_k := \gamma_k'(0)$

- $\exp_m(X_k) = m_k$
- $\|X_k\| \leq r$ for all k

- assume after passing to subsequence: $X_k \to X$ by compactness of $\bar{B}(0,r)$

– $\|X\| \leq r$

– then $\exp_m(X) = m' \in \bar{B}(m,r)$

– $m_k = \exp_m(X_k) \to \exp_m(X) = m'$

thus $(m_k)_k$ has converging subsequence

□

4.7 Properties of the Riemannian curvature

(M,g) - Riemannian manifold

- ∇ - Levi-Civita connection
- $R \in \Gamma(M, \Lambda^2 T^*M \otimes \mathrm{End}(TM)^a)$ curvature

– recall: $R(X,Y)(Z) <:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Remark 4.44. in some books R is defined with the opposite sign □

define $R \in \Gamma(M, \Lambda^2 T^*M \otimes \Lambda^2 T^*M)$

$$R(X,Y,Z,W) := g(R(X,Y)Z, W)$$

Lemma 4.45 (First Bianchi identity)**.** $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Proof. use torsion freeness

- extend X, Y, Z to local fields, vanishing commutator,

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y \\
&= \nabla_X\nabla_Z Y - \nabla_Y\nabla_Z X + \nabla_Y\nabla_Z X - \nabla_Z\nabla_X Y + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y \\
&= 0
\end{aligned}$$

□

Lemma 4.46 (Second Bianchi identity)**.** $\nabla \wedge R = 0$

Proof. special case of Bianchy for linear connections □

for fields X, Y, Z with mutually vanishing commutator 2. Bianchi means:

- $\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0$

Lemma 4.47. *$R(X,Y,Z,W) = R(Z,W,X,Y)$.*

Proof. antisymmetrie in X, Y + first Bianchy

$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(X,Z,Y,W) + R(Z,Y,X,W)$

antisymmetrie in Z, W + first Bianchy

$R(X,Y,Z,W) = -R(X,Y,W,Z) = R(Y,W,X,Z) + R(W,X,Y,Z)$

add

$2R(X,Y,Z,W) = R(X,Z,Y,W) + R(Z,Y,X,W) + R(Y,W,X,Z) + R(W,X,Y,Z)$

also

$2R(Z,W,X,Y) = R(Z,X,W,Y) + R(X,W,Z,Y) + R(W,Y,Z,X) + R(Y,Z,W,X)$

compare term by term + use antisymmetries

□

hence $R \in \Gamma(M, S^2(\Lambda^2 T^*M))$

consider linear map $R(X,-)Y : TM \to TM$

Definition 4.48. *The Ricci curvature is defined by* $\mathrm{Ric}(X,Y) = -\mathrm{Tr}(R(X,-)Y)$.

Lemma 4.49. *We have* $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$

Proof. (e_i) - ONB

$\mathrm{Ric}(X,Y) = -\sum_i R(X,e_i,Y,e_i)$

- symmetry now obvious ☐

Definition 4.50. *The scalar curvature of M is defined by* $S = \sum_i \mathrm{Ric}(e_i,e_j)$.

Example 4.51. Einstein equation

Definition 4.52. *g satisfies the Einstein equation if* $\mathrm{Ric} = \lambda g$ *for some* $\lambda \in C^\infty(M)$.

Lemma 4.53 (Schur)**.** *If $n \geq 3$ and g satisfies the Einstein equation, then λ is constant.*

Proof. calculate at point

use fields whose derivative vanish in this point

- then commutators also vanish (torsion freeness)

- use second Bianchy

$$\begin{aligned}
U\mathrm{Ric}(X,Y) &= \sum_i g(\nabla_U R(X,e_i)e_i,Y) \\
&= -\sum_i g(\nabla_X R(e_i,U)e_i,Y) - g(\nabla_{e_i} R(U,X)e_i,Y) \\
&= -\sum_i Xg(R(e_i,U)e_i,Y) - e_i g(R(U,X)e_i,Y) \\
&= X\mathrm{Ric}(U,Y) + e_i g(R(U,X)Y,e_i)
\end{aligned}$$

set $X = Y = e_j$ and sum

$$\begin{aligned}
US &= e_j\mathrm{Ric}(U,e_j) + e_i\mathrm{Ric}(U,e_i) \\
&= 2e_j\mathrm{Ric}(U,e_j)
\end{aligned}$$

insert equation $\mathrm{Ric} = \lambda g$ and get:

- $U(\lambda)n = 2e_j(\lambda)g(U, e_j) = 2U(\lambda)$

– $(n-2)U(\lambda) = 0$

— use $n \neq 2$

– conclude: $U(\lambda) = 0$

□

Definition 4.54. *A metric satisfying* $\text{Ric} = \lambda g$ *is called an Einstein metric.*

is a second order non-linear PDE for g

- $\lambda = \frac{S}{n}$

- field equation of general relativity

Given M: does M admit an Einstein metric?

not much known in general, many examples

□

Example 4.55. if (M, g) is Einstein, then $S = n\lambda$ is constant

famous question:

Given M: does M admits a metric with $S > 0$

much is known

□

$H \subseteq T_m M$ 2-plane

choose $X, Y \in H$ orthonormal

Definition 4.56. *The sectional curvature of* M *in direction* H *is defined by*

$$K(H) := R(X, Y, Y, X) \ .$$

independent of choice of X, Y, depends only on H

- second choice

- $X' = aX + bY$

- $Y' = -bX + aY$

- with $a^2 + b^2 = 1$

$$\begin{aligned}
R(X', Y', Y', X') &= R(aX + bY, -bX + aY, -bX + aY, aX + bY) \\
&= a^2 R(X, Y, -bX + aY, aX + bY) - b^2 R(Y, X, -bX + aY, aX + bY) \\
&= R(X, Y, -bX + aY, aX + bY) \\
&= R(X, Y, Y, X)
\end{aligned}$$

consider V- an euclidean vector space

$R \in V^{*,\otimes 4}$

algebraic symmetries of the curvature tensor

1. $R(X, Y, Z, W) = -R(Y, X, Z, W)$
2. $R(X, Y, Z, W) = -R(Z, W, X, Y)$
3. $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$

note that then also $R(X, Y, Z, W) = -R(X, Y, W, Z)$

for $X, Y \in V$ define $K(X, Y) := R(X, Y, Y, X)$

- this is quadratic in X and Y

Lemma 4.57. *The K determines R. If $R, R' \in V^{*,\otimes 4}$ satisfy the algebraic curvature identities and $K(X, Y) = K'(X, Y)$ for all $X, Y \in V$, then $R = R'$.*

Proof. polarize in X

$R(X + Z, Y, X + T, Y) = R(X, Y, X, Y) + R(T, Y, T, Y) + 2R(X, Y, Z, Y)$

- use symmetry for last term

same with R'

- get $R(X, Y, Z, Y) = R'(X, Y, Z, Y)$

polarise in Y

$R(X,Y+W,Z,Y+W) = R(X,Y,Z,Y)+R(X,W,Z,W)+R(X,Y,Z,W)+R(X,W,Z,Y)$

- no symmetry anymore

get

$R(X,Y,Z,W) + R(X,W,Z,Y) = R'(X,Y,Z,W) + R'(X,W,Z,Y)$

or

$R(X,Y,Z,W) - R'(X,Y,Z,W) = R'(X,W,Z,Y) - R(X,W,Z,Y)$

or

$R(X,Y,Z,W) - R'(X,Y,Z,W) = R(Y,Z,X,W) - R'(Y,Z,X,W)$

$R(X,Y,Z,W) - R'(X,Y,Z,W)$ is invariant under cyclic permutations of X,Y,Z

use first Bianchi $3(R(X,Y,Z,W) - R'(X,Y,Z,W)) = 0$

□

Lemma 4.58. *Assume that $R \in V^{*,\otimes 4}$ satisfies the algebraic curvature identities. If $K(X,Y) = k\|X\|^2\|Y\|^2$ for all X,Y with $X \perp Y$, then*

$$R(X,Y,Z,W) = k\left(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\right) \ .$$

Proof. RHS satisfies with $Y=Z$ and $X=W$

$k\left(\langle Y,Y\rangle\langle X,X\rangle - \langle X,Y\rangle\langle Y,X\rangle\right) = k\|X\|^2\|Y\|^2$

also satisfies curvature identities:

- antisymmetry in X,Y: inspection

- symmetry for exchange $(X,Y) \leftrightarrow (Z,W)$: inspection

- antisymmetry in X,Y: inspection

- first Bianchy

$$
\begin{aligned}
& \langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle \\
& +\langle Z,X\rangle\langle Y,W\rangle - \langle Y,X\rangle\langle Z,W\rangle \\
& +\langle X,Y\rangle\langle Z,W\rangle - \langle Z,Y\rangle\langle X,W\rangle \\
= \;& 0
\end{aligned}
$$

apply Lemma 4.57 □

Remark 4.59. assume $R(X,Y,Z,W) = k\left(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\right)$

$\mathrm{Ric}(X,W) = k(\sum_i \left(\langle E_i,E_i\rangle\langle X,W\rangle - \langle X,E_i\rangle\langle E_i,W\rangle\right) = k(n-1)\langle X,W\rangle$

$R = kn(n-1)$ □

Definition 4.60. *We say that the sectional curvature of* (M,g) *is constant at* m *if* $H \mapsto K(m)(H)$ *is constant.*

Corollary 4.61. *If the sectional curvature of* M *is constant at each point* m *in* M*, then*

$$R(X,Y,Z,W) = \frac{S}{n(n-1)}\left(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\right)$$

for some constant S *(equal to the scalar curvature).*

Proof. at every point m:

apply Lemma 4.58

- $R(m)(X,Y,Z,W) = k(m)\left(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\right)$

- $\mathrm{Ric}(m)(X,W) = k(m)(\sum_i \left(\langle E_i,E_i\rangle\langle X,W\rangle - \langle X,E_i\rangle\langle E_i,W\rangle\right) = k(m)(n-1)\langle X,W\rangle$

- hence (M,g) is Einstein and k is locally constant by Lemma 4.53

$S = kn(n-1)$ (S - scalar curvature)

this gives formula

□

Example 4.62. 1. $(\mathbb{R}^n, g_{eu})$ has constant sectional curvature 0.

2. (S^n, g_{S^n}) (unit sphere in $\mathbb{R}^{n+1}$) has constant sectional curvature 1.

3. $H := \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > 0\}$ with metric: $y^{-2} g_{eu}$ (the hyperbolic space, upper half-space model) has constant sectional curvature -1.

the calculations for the last two examples can be done directly, but are lengthy

- easier by using some theory □

4.8 Isometries and second fundamental form

(M, g), (M', g') - Riemannian manifolds

$f : M \to M'$

Definition 4.63. *f is isometric of $f^*g' = g$.*

an isometric map is an immersion

Remark 4.64. (M', g') - Riemannian manifold

$f : M \to M'$ - immersion

- define $g := f^*g'$

– this is a Riemannian metric on M

– $f : (M, g) \to (M', g')$ is isometric □

- $Df : TM \to f^*TM'$

- $f^*TM' \cong TM \oplus TM^\perp$

– first summand identified via Df

– $P : f^*TM' \to TM$ orthogonal projection

have already seen:

- can express Levi-Civita connection of M in terms of that of M'

Lemma 4.65. $\nabla = Pf^*\nabla'$

- ∇ is tangential component of $f^*\nabla'$

what about the normal component

- define: $N := (1-P) : f^*TM' \to TM^\perp$ - projection on normal direction
- consider $X, Y \in \mathcal{X}(M)$
- $N{\nabla'}_X Y \in \Gamma(M, TM^\perp)$

Proposition 4.66. *The map $I : \mathcal{X}(M) \times \mathcal{X}(M) \to \Gamma(M, TM^\perp)$ given by $(X,Y) \mapsto I(X,Y) := -N{\nabla'}_X Y$ is C^∞-linear and symmetric.*

Proof. - calculate at $m \in M$

- extend here X, Y to vector fields in an open nbhd of $f(m)$

$N{\nabla'}_{fX} Y = fN{\nabla'}_X Y$

$N{\nabla'}_X (fY) = fN{\nabla'}_X Y + X(f)NY = fN{\nabla'}_X Y$ since $NY = 0$

for symmetry: $N{\nabla'}_X Y - N{\nabla'}_Y X = N[X,Y] = 0$

□

hence get $I \in \Gamma(M, S^2TM^* \otimes TM^\perp)$

Definition 4.67. *I is called the second fundamental form of f.*

Example 4.68. $f : \mathbb{R}^1 \to \mathbb{R}^2$ canonical embedding

- get $I = 0$ □

Example 4.69. $f : S^2 \to \mathbb{R}^3$

- ξ - out-pointing normal vector vector field
- trivializes $(TS^2)^\perp$
- calculate $\langle I(X,Y), \xi\rangle$
- because of rot. invariance suffices to calculate it at northpole
- $\langle I(X,Y), \xi\rangle = -\langle {\nabla'}_X Y, \xi\rangle$

– coordinates: (x, y) - projection to (x, y) -plane

– $r := \sqrt{x^2 + y^2}$

– $\xi(x, y) = (x, y, \sqrt{1 - r^2})$

– extend Y to tangential field by $Y - \langle Y, \xi \rangle \xi$

— check: is $\perp \xi$

– $\langle \nabla'_X (Y - \langle Y, \xi \rangle \xi), \xi \rangle = -X \langle Y, \xi \rangle = -\langle Y, \nabla'_X \xi \rangle$

— use here that $\nabla_X \xi \perp \xi$ since ξ is unit vector field

- $(\nabla'_X \xi)(0, 0) = (X, 0)$

- hence $I(X, Y) = \langle Y, X \rangle$

same calculation also shows for $S^n \subseteq \mathbb{R}^{n+1}$

- the second fundamental form satisfies $\langle I(-, -), \xi \rangle = g_{S^n}$

□

(M, g), (M', g') - Riemannian manifolds

- $f : M \to M'$ isometry

- consider geodesic γ in M

- Question: Is $f \circ \gamma$ geodesic in M'?

– $\nabla'_{\partial_t} \gamma' = \nabla_{\partial_t} \gamma' - I(\gamma', \gamma')$

Corollary 4.70. *$f \circ \gamma$ is a geodesic if and only of $I(\gamma', \gamma') \equiv 0$*

Definition 4.71. *f is called totally geodesic if $I = 0$.*

Corollary 4.72. *The following are equivalent:*

1. *If f is totally geodesic.*
2. *then f sends all geodesics in M to geodesics in M'.*

Example 4.73. $\mathbb{R}^n \subseteq \mathbb{R}^{n+m}$ is totally geodesic

$S^n \subseteq \mathbb{R}^{n+1}$ is not totally geodesic □

Gauss equation expresses curvature of M in terms of curvature of M'

$f : M \to M'$ isometric

- will write X for $Tf(m)(X)$ and $X \in T_m M$

I - second fundamental form

Theorem 4.74. *For $X, Y, Z, W \in T_m M$ we have*

$$R(X, Y, Z, W) - R'(X, Y, Z, W) = g'(I(Y, Z), I(X, W)) - g'(I(X, Z), I(Y, W))$$

Proof. $\nabla_X \nabla_Y Z = \nabla'_X \nabla_Y Z + I(X, \nabla_Y Z) = \nabla'_X \nabla'_Y Z + \nabla'_X I(Y, Z) + I(X, \nabla_Y Z)$

$g'(\nabla'_X I(Y, Z), W) = -g'(I(Y, Z), \nabla'_X W) = g'(I(Y, Z), I(X, W))$

- calculate with commuting vector fields which are parallel at the given point m

– $I(X, \nabla_Y Z)(m) = 0$

$g(R(X, Y)Z, W) = g(R'(X, Y)Z, W) + g'(I(Y, Z), I(X, W)) - g'(I(X, Z), I(Y, W))$

□

Example 4.75. calculation of curvature of S^n

- have seen $I = g\xi$ for unit outward normal field ξ

- $R' = 0$

get:

- $R(X, Y, Z, W) = \langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle$

– S^n has constant sectional curvature 1

$\mathrm{Ric} = (n - 1)g$

- S^n is Einstein with $\lambda = n - 1$

$R = n(n - 1)$ - constant positive scalar curvature

□

4.9 Conformal change of the metric

(M, g) - Riemannian manifold

$f \in C^\infty(M)$

- $e^f g$ - new metric

Definition 4.76. *We call $g' := e^f g$ the conformal change of g by e^f.*

Question: how does the Levi-Civita connection and the curvature change

prep:

- vector space V

- $(e_i)_i$ - base of V

- $(e^i)_i$ - dual base of V^*

- consider $V^* \otimes \mathrm{End}(V) \cong V^* \otimes V^* \otimes V$

- $\phi \in V^*$

- can consider:

– $\phi \otimes 1 := \phi \otimes \mathrm{id}_V = \phi \otimes e^i \otimes e_i$

— $\phi(X)(Y) = \phi(X)Y$

– $\phi_\sharp := e^i \otimes \phi \otimes e_i$

– $\phi_\sharp(X)(Y) = \phi(Y)X$

– $\phi_\sharp^* := e^i \otimes \langle e_i, e_k\rangle e^k \otimes \langle \phi, e^j\rangle e_j = e^i \otimes e^i \otimes \phi(e_j)e_j$

- use symbol a for antisymmetrization (without 1/2) in X, Y and in the endormorphism part

– $a(U(X,Y)) := U(X,Y) - U(Y,X) - U(X,Y)^* + U(Y,X)^*$

for $h \in C^\infty(M)$

- $dh \in \Omega^1(M)$

Definition 4.77. *We define the gradient* $\mathrm{grad}(h) \in \mathcal{X}(M)$ *of h by*

$$g(\mathrm{grad}(h), -) = dh \ .$$

locally in ONB $(e_i)_i$:

- $\mathrm{grad}(h) = dh(e_i)e_i$

locally in coordinates:

- $\mathrm{grad}(h) = g^{ij}\partial_j h \partial_j$
– g^{ij} is inverse to $g_{ij} = g(\partial_i, \partial_j)$

Lemma 4.78. *We have*

$$\nabla' = \nabla + \frac{1}{2}(df \otimes 1 + df_\sharp - df_\sharp^*)$$

and

$$R'(X,Y) = R(X,Y) + a(\frac{1}{2}\nabla_X df \otimes Y - \frac{1}{8}\|df\|^2(Y^* \otimes X) + \frac{1}{4}df \otimes Y(f)X)\ .$$

Proof. recall formula for Levi-Civita connection

$$\begin{aligned} 2g(\nabla_X Y, Z) \quad := \quad & Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ & -g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z) \end{aligned}$$

replace g by $e^f g$ get ∇'

$2g(\nabla'_X Y, Z) = 2g(\nabla_X Y, Z) + X(f)g(Y,Z) + Y(f)g(X,Z) - Z(f)g(X,Y)$

$2(\nabla'_X Y - \nabla_X Y) = X(f)Y + Y(f)X - g(X,Y)\mathrm{grad}(f)$

$\nabla'_X - \nabla_X = \omega$

- with $2\omega = df \otimes 1 + df_\sharp - df_\sharp^*$

calculate R':

$R^{'} = R + \nabla \wedge \omega + [\omega, \omega]$

calculate with fields with vanishing commutator

$(\nabla \wedge \omega)(X,Y) = \nabla_X \omega(Y) - \nabla_Y \omega(X)$

$(\nabla \wedge (df \otimes 1))(X,Y) = \nabla_X df(Y)1 - \nabla_Y df(X)1 = X(Y(f)) - Y(X(f)) = 0$

- use $\nabla 1 = 0$ and $[X,Y] = 0$

$$
\begin{aligned}
(\nabla \wedge df_\sharp)(X,Y) &= \nabla_X(df \otimes Y) - (X \leftrightarrow Y) \\
&= \nabla_X df \otimes Y + df \otimes \nabla_X Y - (X \leftrightarrow Y) \\
&= \nabla_X df \otimes Y - (X \leftrightarrow Y)
\end{aligned}
$$

- use torsion-free

$$
\begin{aligned}
(\nabla \wedge df_\sharp^*)(X,Y) &= \nabla_X(Y^* \otimes \mathrm{grad}(f)) - (X \leftrightarrow Y) \\
&= \nabla_X Y^* \otimes \mathrm{grad}(f)) + Y^* \otimes \nabla_X \mathrm{grad}(f) - (X \leftrightarrow Y) \\
&= Y^* \otimes \nabla_X \mathrm{grad}(f) - (X \leftrightarrow Y) \\
&= (\nabla \wedge df_\sharp)(X,Y)^*
\end{aligned}
$$

$2(\nabla \wedge \omega)(X,Y) = a(\nabla_X df \otimes Y)$

$$
\begin{aligned}
4[\omega(X),\omega(Y)] &= ((df \otimes X) \circ (df \otimes Y) + (X^* \otimes \mathrm{grad}(f)) \circ (Y^* \otimes \mathrm{grad}(f)) - (df \otimes X) \circ (Y^* \otimes \mathrm{grad}(f)) \\
&\quad -(X^* \otimes \mathrm{grad}(f)) \circ (df \otimes Y) - (X \leftrightarrows Y) \\
&= Y(f) df \otimes X + X(f) Y^* \otimes \mathrm{grad}(f) - \|df\|^2 Y^* \otimes X - \langle X,Y\rangle df \otimes \mathrm{grad}(f) \\
&\quad -(X \leftrightarrows Y) \\
&= a(df \otimes Y(f)X - \frac{1}{2}\|df\| Y^* \otimes X)
\end{aligned}
$$

thus

$$
R'(X,Y) = R(X,Y) + a(\frac{1}{2}\nabla_X df \otimes Y - \frac{1}{8}\|df\|^2 Y^* \otimes X + \frac{1}{4} df \otimes Y(f)X)
$$

- a means antisymmetrization (without 1/2) in X, Y and in the endormorphism part

- factor 1/8 instead of 1/4 correct! □

Example 4.79. $f = constant$

$\nabla' = \nabla$

$R' = R$ for curvature tensor

but $R'(X, Y, Z, W) = e^f R(X, Y, Z, W)$

- $\mathrm{Ric}' = e^{-f}\mathrm{Ric}$
- $S' = e^{-2f} S$
- $K = e^{-f} K$

e.g. sphere S_r^{n-1} of radius r is isometric to conformal change of unit sphere $g' = r^2 g$

- sectional curvature of S_r is r^{-2}

□

Example 4.80. the upper half plane

- $H := \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > 0\}$
- metric: $y^{-2} g_{eu}$

Definition 4.81. *$(H, y^{-2} g_{eu})$ is called the hyperbolic space.*

Lemma 4.82. *The hyperbolic space is complete and has constant sectional curvature -1.*

Proof. - $y^{-2} = e^f$

- $f = -2\log(y)$
- $df = -2y^{-1} dy$
- $\frac{1}{2}(\nabla_X df \otimes Y) = y^{-2} X^n dy \otimes Y$
- $\frac{1}{8}\|df\|^2 (Y^* \otimes X) = 2^{-1} y^{-2} Y^* \otimes X$
- $\frac{1}{4}(df \otimes Y(f)X) = Y^n y^{-2} dy \otimes X$

$y^4 R'(X, Y, Z, W) = X^n Z^n \langle Y, W\rangle - 2^{-1}\langle Y, Z\rangle\langle X, W\rangle + Y^n Z^n \langle X, W\rangle + (anti - symm)$

- sum of first and third term is symmetric in X, Y
- get

$y^4 R'(X, Y, Z, W) = -(\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle)$

– $R'(X,Y,Z,W) = -(g'(Y,Z)g'(X,W) - g'(X,Z)g'(Y,W))$

– constant sectional curvature $K = -1$

show completeness:

$\mathbb{R}^+ \times \mathbb{R}^{n-1}$ acts by isometry: $(\lambda, z)(x,y) = (\lambda x + z, \lambda y)$

- this action is transitive

- the existence time for the unit speed geodesics on H has a uniform lower bound given by the existence time at some base point

- H is geodesically complete

□

□

4.10 Lie groups

G - a Lie group

- $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ - adjoint representation

- consider Ad-invariant invariant scalar products on $\mathfrak{g}$

Example 4.83. assume: G is compact

- then such a scalar product exists

- dg - normalized invariant volume

- fix any scalar product $\tilde{B}$ on $\mathfrak{g}$

- define $B(X,Y) := \int_G \tilde{B}(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y))dg$

– B is Ad-invariant scalar product

Lemma 4.84. *If $\mathfrak{g}$ is simple, then B is unique up to normalization.*

Proof. - B' second Ad-invariant scalar product

- $B'(X,Y) = B(AX,Y)$ for some symmetric $A \in \mathrm{End}(\mathfrak{g})$

- Ad-invariance of B, B' implies: $\mathrm{Ad}(g)A\mathrm{Ad}(g^{-1}) = A$ for all $g \in G$

– differentiate: $[\mathrm{ad}X, A] = 0$

- if A is not $\lambda 1$, then it has at least two eigenvalues

- λ - eigenvalue

– $\mathfrak{g}(\lambda) \subseteq \mathfrak{g}$ proper eigensubspace

– is an ideal in $\mathfrak{g}$

— $X \in \mathfrak{g}(\lambda)$

— $A([Y, X]) = A(\mathrm{ad}(Y)(X)) = \mathrm{ad}(Y)(A(X)) = \lambda \mathrm{ad}(Y)(X) = \lambda[Y, X]$

– existence of proper ideal is contradiction to simpleness of $\mathfrak{g}$

□

call G simple if $\mathfrak{g}$ is simple

- G compact, simple

- Killingform $-B_G$ is invariant and positive definite

- hence any invariant scalar product is multiple of $-B_G$

□

back to general situation

- for any scalar product B on $\mathfrak{g}$

– define Riemannian metric g_B in G by left-invariant extension of B

– $g_B(h) := TL^*_{h^{-1}}B$

— for left invariant fields $X, Y \in {}^G\mathcal{X}(G)$

— $g_B(X, Y) = B(X(e), Y(e))$

Corollary 4.85. (G, g_B) *is complete.*

Proof. G acts transitively isometrically by isometries on (G, g_B)

□

if we assume that B is Ad-invariant, then can understand Riemannian geometry of (G, g_B) in a simple manner

Lemma 4.86. *The following are equivalent:*

1. *The Riemannian metric g on G is left-and right invariant.*
2. *$B = g(e)$ is* Ad*-invariant.*

Proof. Exercise! □

Lemma 4.87. *If B is* Ad*-invariant, then the Levi-Civita connection on (G, g_B) is determined by $\nabla_X Y = \frac{1}{2}[X, Y]$ for $X, Y \in {}^G\mathcal{X}(G)$.*

Proof. show first: there is a unique connection ∇ on TG such that $\nabla_X Y = \frac{1}{2}[X, Y]$ for $X, Y \in {}^G\mathcal{X}(G)$

- have trivialization $\Phi : TG \cong G \times \mathfrak{g}$

– $X \in T_g G \mapsto (g, TL_{g^{-1}}(g)(X))$

– this determines trivial connection ∇^{triv}

– $X \in {}^G\mathcal{X}(G)$ goes to constant function with value $X(e)$

– this trivial connection satisfies for $\nabla_X Y = 0$ for $X, Y \in {}^G\mathcal{X}(G)$

– consider $\omega \in \Omega^1(G, TG)$ defined by:

– $\omega(X)(Y) = \frac{1}{2} TL_g(e)([TL_{g^{-1}}(g)(X), TL_{g^{-1}}(g)(Y)])$

— i.e. for $X, Y \in {}^G\mathcal{X}(G)$: $\omega(X)(Y) = \frac{1}{2}[X, Y]$

— then $\nabla := \nabla^{\text{triv}} + \omega$ is a connection

—- ∇ satisfies the condition

—- uniqueness is clear since ω is determined by condition

∇ is Levi-Civita:

- calculate with $X, Y, Z \in {}^G\mathcal{X}(G)$

– torsion-free:

– $\nabla_X Y - \nabla_Y X = \frac{1}{2}[X, Y] - \frac{1}{2}[Y, X] = [X, Y]$

– compatible with metric:

— $Xg(Y,Z) = 0$

— $g_B(\nabla_X Y, Z) + g_B(Y, \nabla_X Z) = \frac{1}{2}B([X(e), Y(e)], Z(e)) + \frac{1}{2}B(Y(e), [X(e), Z(e)]) = 0$

– it is here where we use invariance of B

□

$X \in \mathfrak{g}$

- interpret $X \in {}^G\mathcal{X}(G)$

- get integral curve curve $t \mapsto \gamma(t) := \exp(tX)$ in G

– $\gamma(0) = e$

- $\gamma'(t) = X(\gamma(t))$

– $\gamma(t) := \exp((t+s)X) = \exp(tX)\exp(sX)$ (one-parameter subgroup

Lemma 4.88. *Assume that (G, g_B) is defined with invariant B. The curve γ is a geodesic*

Proof. $\gamma'(t) = X(\gamma(t))$

- $\nabla_{\partial_t}\gamma'(t) = \nabla_{\gamma'(t)}X = \nabla_{X(\gamma(t))}X = [X,X](\gamma(t)) = 0$ □

conclude: $\exp = \exp_e$

- exp: exponential map of G in the sense of Lie groups

- $\exp_e$: exponential map of G in the sense of Riemannian geometry

all geodesics are of the form

$t \mapsto g\exp(tX)$ for some g in G and X in $\mathfrak{g}$

Corollary 4.89. *A Lie subgroup H of G is a totally geodesic submanifold.*

curvature:

$R(X,Y)Z = \frac{1}{2}([X,[Y,Z]] - [Y,[X,Z]] - [[X,Y],Z]) = [[X,Y],Z]$

by Jacobi

$\mathrm{Ric}(X,W) = \sum_i g_B([[X,e_i],e_i],W) = -\sum_i g_B([X,e_i],[W,e_i]) = \sum_i g([W,[X,e_i],e_i) = K(W,X)$

– K is the Killing form

Corollary 4.90. *If we choose B proportional to the Killing form, then (G, g_B) is Einstein.*

Remark 4.91. one could ask more generally: for which scalar products B on $\mathfrak{g}$ is (G, g_B) Einstein

- there are many more examples (quite recent) □

4.11 Energy and more

(M, g) - Riemannian

- recall definitions of energy and length of a curve $\gamma : [0, a] \to M$

– $E(\gamma) = \int_0^a g(\gamma'(t), \gamma'(t))dt$

– $\ell(\gamma) = \int_0^a \sqrt{g(\gamma'(t), \gamma'(t))}dt$

Cauchy-Schwarz: $\ell(\gamma)^2 \leq aE(\gamma)$ (for any curve)

$\gamma : m \to m'$

- note: $\ell(\gamma) = d(m, m')$ implies that γ is geodesic

Lemma 4.92. *Assume $\ell(\gamma) = d(m, m')$. Then for any curve $\sigma : m \to m'$ we have $E(\gamma) \leq E(\sigma)$ with equality iff σ is a minimizing geodesic.*

Proof. γ is geodesic

- speed^2 $g(\gamma'(t), \gamma'(t))$ is constant

- speed $d(m, m')/a$

– $E(\gamma) = a \cdot d(m, m')^2/a^2 = \ell(\gamma)^2/a$

- $aE(\gamma) = \ell(\gamma)^2 \leq \ell(\sigma)^2 \leq aE(\sigma)$

- if equality: $\ell(\sigma) = d(m, m')$ and hence σ is minimizing geodesic □

Example 4.93. meridians from north to southpole on S^2 show:

- $E(\gamma) = E(\sigma)$ does not imply $\gamma = \sigma$

□

already know: geodesics are precisely critical curves for E

- $(\gamma_u)_u$ variation of geodesic γ rel endpoints

– $0 = (\partial_u)_{|u=0} E(\gamma_u)$

- we now consider second derivative of $E(\gamma_u)$

– variation field $\gamma_u^\sharp(t) := \partial_u \gamma_u(t)$

— is a section of $\gamma^* TM$

Lemma 4.94.

$$(\partial_u)_{|u=0} E(\gamma_u) = -2 \int_0^a g(\gamma^\sharp, \nabla^2_{\partial_t} \gamma^\sharp + R(\gamma^\sharp, \gamma')\gamma') dt \ .$$

Proof.

$$\begin{aligned} \partial_u E(\gamma_u) &= \int_0^a \partial_u g(\gamma_u', \gamma_u') dt \\ &= 2 \int_0^a g(\nabla_{\partial_u} \gamma_u', \gamma_u') dt \\ &= 2 \int_0^a g(\nabla_{\partial_t} \gamma_u^\sharp, \gamma_u') dt \\ &= -2 \int_0^a g(\gamma_u^\sharp, \nabla_{\partial_t} \gamma_u') dt \end{aligned}$$

- use here ∇ is torsion free for $\nabla_{\partial_u} \gamma_u' = \nabla_{\partial_t} \gamma_u^\sharp$

- $\gamma_u^\sharp(0) = 0$ and $\gamma_u^\sharp(a) = 0$ for partial integration

apply $(\partial_u)_{|u=0}$

$$
\begin{aligned}
(\partial_u^2 E(\gamma_u))_{|u=0} &= -2(\int_0^a g(\nabla_{\partial_u}\gamma_u^\sharp, \nabla_{\partial_t}\gamma_u')dt))_{|u=0} - 2(\int_0^a g(\gamma_u^\sharp, \nabla_{\partial_u}\nabla_{\partial_t}\gamma_u')dt)_{|u=0} \\
&= -2(\int_0^a g(\gamma_u^\sharp, \nabla_{\partial_u}\nabla_{\partial_t}\gamma_u')dt)_{|u=0} \\
&= -2\int_0^a g(\gamma^\sharp, (\nabla_{\partial_u}\nabla_{\partial_t}\gamma_u')_{|u=0})dt
\end{aligned}
$$

- use here γ_0 is geodesic to drop first summand

$$
(\nabla_{\partial_u}\nabla_{\partial_t}\gamma_u')_{|u=0} = \nabla_{\partial_t}(\nabla_{\partial_u}\gamma_u')_{|u=0} + R(\gamma^\sharp, \gamma')\gamma' = \nabla_{\partial_t}^2\gamma^\sharp + R(\gamma^\sharp, \gamma')\gamma'
$$

- drop subscript 0 (for u-variable)

insert this formula - get result □

Remark 4.95. assume $\gamma^\sharp$ is Jacobi field

- then $(\partial_u^2 E(\gamma_u))_{|u=0} = 0$

- Hessian of E has a zero at γ

- the existence of a Jacobi field which vanishes at the endpoints of the geodesic is a strong condition

– the endpoints are called conjugate (will be discussed later) □

lower estimates of symmetric bilinear forms

- V real euclidean vector space

- B - symmetric bilinear form on V

- $c \in \mathbb{R}$

– say: $B \geq c$ if $B(v, v) \geq c$ for every unit vector v in V

- equivalently: write $B(v, w) = \langle Av, w\rangle$ for symmetric endomorphism A

– $B \geq c$ iff all eigenvalues of A are bounded below by c

(M, g) Riemannian manifold

- $\mathrm{Ric}(m)$ is symmetric bilinear form on T_mM

– condition $\mathrm{Ric}(m) \geq c$ makes sense

- say: $\mathrm{Ric} \geq c$ if $\mathrm{Ric}(m) \geq c$ for all m in M

recall definition of diameter of metric space (X, d): $\mathrm{diam}(X) = \sup_{x,x' \in X} d(x, x')$

Theorem 4.96 (Bonnet-Myers). *If* (M, g) *is complete and* $\mathrm{Ric} \geq c > 0$*, then* M *is compact and* $\mathrm{diam}(M) \leq \pi\sqrt{\frac{n-1}{c}}$.

Proof. by contradiction

- assume that there exists m, m' in M with $\ell := d(m, m') > \pi\sqrt{\frac{n-1}{c}}$

– chose minimizing geodesic $\gamma : [0, 1] \to M$ from m to m'

— this is possible by completeness assumption

– γ is also energy minimizing

$(e_i)_{i=1,n}$ parallel ONB γ^*TM

- such that $e_n := \frac{\gamma'}{\ell}$

– $V_j(t) := \sin(\pi t)e_j(t)$ section of γ^*TM

— observe: $V_j(0) = 0$, $V_j(1) = 0$

insert in formula for second variation of energy formula

$$\begin{aligned} E_j'' &:= -2\int_0^1 g(V_j, V_j'' + R(V_j, \gamma')\gamma')dt \\ &= 2\int_0^1 \sin(\pi t)^2(\pi^2 - \ell^2 K(\gamma(t))(e_j(t), e_n(t))dt \end{aligned}$$

sum over $j = 1, \ldots, n-1$

- use

$\sum_j K(\gamma(t))(e_j(t), e_n(t)) = \mathrm{Ric}(e_n(t), e_n(t)) \geq c > \frac{(n-1)\pi^2}{\ell^2}$

$$\sum_{j=1}^{n-1} E_j'' < 2\int_0^1 \sin(\pi t)^2((n-1)\pi^2 - \ell^2 \frac{(n-1)\pi^2}{\ell^2})dt = 0$$

hence $E_j'' < 0$ for at least one j

- can find variation of γ which decreases energy

- contradiction to γ being energy minimizing □

Remark 4.97. the constant in Bonnet-Myers is optimal

- S_r^n has diameter πr

- $\mathrm{Ric} = (n-1)r^{-2}$

□

4.12 Coverings

M - a connected manifold

Definition 4.98. *A covering of M is a fibre bundle $\tilde{M} \to M$ with discrete fibres.*

can characterize coverings by the unique path lifting property

- $\pi : \hat{M} \to M$ a smooth map between manifolds

Lemma 4.99. *The following are equivalent:*

1. *$\pi : \hat{M} \to M$ is a covering.*
2. *π has the unique path lifting property saying: Given any bold diagram*

$$\begin{array}{ccc} \{t_0\} & \xrightarrow{t_0 \mapsto \hat{m}_0} & \hat{M} \\ \downarrow & \overset{\hat{\gamma}^{\hat{m}_0}}{\nearrow} & \downarrow \pi \\ I & \xrightarrow{\gamma} & M \end{array}$$

there exists a unique dotted arrow rendering the diagram commutative

Proof. sketch:

1⇒2:

- $\hat{M} \to M$ has a canonical flat connection $T^h\hat{M} := T\hat{M}$

– (since $T^v\pi = 0$ by discreteness of fibres)

- given bold diagram:

– $\hat{\gamma}^{\hat{m}_0}$ is unique horizontal lift of γ with $\hat{\gamma}(t_0) = \hat{m}_0$

2⇒1:

- $m_0 \in M$

– choose small ball $m_0 \in B \subseteq M$

— for $m \in B$ let $\gamma_m : [0,1] \to B$ be radial curve from m_0 to m

— define $\Phi : B \times \hat{M}_{m_0} \to M$ local trivialization such that $\Phi(b, \hat{m}_0) = \hat{\gamma}_m^{\hat{m}_0}(1)$

□

Definition 4.100. *M is simply connected if every connected covering $\tilde{M} \to M$ is an isomorphism.*

more facts about coverings:

Proposition 4.101. *There exists a connected covering $\tilde{M} \to M$ such that $\tilde{M}$ is simply connected (it is called the universal covering).*

Proof. idea of construction:

- fix point m_0

– a point in $\tilde{M}$ is a pair $(m, [\gamma])$ where $m \in M$, $\gamma : m_0 \to m$ a curve, $[\gamma]$ - homotopy class

— $\tilde{M} \to M$ given by $(m, [\gamma]) \to m$

— define manifold structure such that this is local diffeomorphism

— check unique path lifting:

—- if σ is path in M starting in m

—- unique lift starting in $(m, [\gamma])$ is $t \mapsto (\sigma(t), [\sigma_{\leq t} \sharp \gamma])$

show $\tilde{M}$ is connected

- $(\gamma(t), [\gamma_{\leq t}]$ is path from $(m_0, [\text{const}_{m_0}])$ to $(m, [\gamma])$

check $\tilde{M}$ is simply connected

- $\hat{M} \to \tilde{M}$ covering, connected

– must show that injective:

— assume $\hat{m}_0, \hat{m}_0'$ two points in fibre at $(m_0, [\text{const}_{m_0}])$

— chose path $\hat{\gamma}$ from $\hat{m} \to \hat{m}'$

— $\tilde{\gamma}$ - path in $\tilde{M}$

— is closed loop at $(m_0, [\text{const}_{m_0}])$

— is zero homotopic

—- this implies $\hat{m}_0 = \hat{m}_0'$ (it is at this point where the argument is sketchy since this fact has not been shown above)

∎

Lemma 4.102. *The universal covering has the following universal property: Given bold part of the diagram*

$$\begin{array}{ccc} \{\tilde{m}\} & \xrightarrow{\tilde{m} \to \hat{m}} & \hat{M} \\ \downarrow & \overset{\phi}{\nearrow} & \downarrow \text{covering} \\ \tilde{M} & \longrightarrow & M \end{array}$$

the dotted arrow exists and is unique making the diagram commutative.

Proof. existence:

- $\tilde{m}'$ in $\tilde{M}$

- choose path $\tilde{\sigma} : \tilde{m} \to \tilde{m}'$

- σ - image in M

- $\hat{\sigma}$ - unique lift in $\hat{M}$ starting in $\hat{m}$

- define $\phi(\tilde{m}') = \hat{\sigma}(1)$

- check continuity of ϕ

- uniqueness of ϕ

- $\square$

Corollary 4.103. *The universal covering is uniquely determined up to isomorphism of fibre bundle.*

Definition 4.104. *The group $\pi_1(M)$ of fibrewise diffeomorphisms of $\tilde{M}$ is called the fundamental group of M.*

Lemma 4.105. *$\tilde{M} \to M$ is a $\pi_1(M)$-principal bundle.*

Proof. must show: $\pi_1(M)$ acts simply transitively on fibres

- consider fibre over given point m

- $g \in \pi_1(M)$

- $\tilde{m}', \tilde{m} \in \tilde{M}$ over m

- apply universal property for $\hat{M} = \tilde{M}$

- if $g\tilde{m} = \tilde{m}$, then $g = \mathrm{id}$ by uniqueness clause

- can find g such that $g(\tilde{m}) = \tilde{m}'$ by existence clause

$\square$

(follows easily from universal property)

Remark 4.106. - the usual definition of $\pi_1(M)$ is as the group of homotopy classes of loops $[\sigma]$ in M at some base point m_0 with concatenation

– right-action in the model by $(m, [\gamma])[\sigma] = (m, [\gamma \sharp \sigma])$

$\square$

Corollary 4.107. *If (M, g) is a complete Riemannian manifold with $\mathrm{Ric} \geq c > 0$, then $\pi_1(M)$ is finite.*

Proof. - $\pi : \tilde{M} \to M$ is immersion

- $\tilde{g} := \pi^* g$ satisfies $\tilde{\mathrm{Ric}} \geq c > 0$

- $(\tilde{M}, \tilde{g})$ is also complete

- hence $\tilde{M}$ is compact by Bonnet-Myers

- π has finite fibres

- hence $\pi_1(M)$ is finite $\square$

Example 4.108. choose p, q a primes, different

- let C_p act on $\mathbb{C}^2$ by $[n](z_1, z_2) = (e^{2\pi i \frac{n}{p}} z_1, e^{2\pi i \frac{nq}{p}} z_2)$

- this is isometric

- preserves $S^3 \subseteq \mathbb{C}^2$

- acts freely on S^3

Definition 4.109. *The lense space $L(p, q)$ is the quotient S^3/C_p with respect to this action.*

have covering $S^3 \to L(p, q)$

- can choose metric on $L(p, q)$ such that the covering is isometric

- then $L(p, q)$ has constant sectional curvature 1

- $S^3 \to L(p, q)$ is the universal covering

- $\pi_1(L(p, q)) = C_p$

$\square$

Recall: (M, g)

- if M has $K \leq 0$, then $\exp_m$ is diffeo near every point of T_mM

Lemma 4.110. *If (M, g) is complete and has $K < 0$, then $\exp_m : T_mM \to M$ is a covering.*

Proof. we check unique path lifting property

- equip T_mM with metric $g' := \exp_m^* g$

- radial curves $t \mapsto tX$ are geodesics in this metric

– exist for all times

— (T_mM, g') is complete by Hopf-Rinow

$\gamma : [0, 1] \to M$ path

- $x \in \exp_m^{-1}(\gamma(0))$ start point for lift

– if lift of γ exists, then it is unique (since $\exp_m$ is local diffeo)

– for some $t > 0$ there exists lift $\tilde{\gamma}$ on $[0, t)$ (again by local diffeo)

– let t be maximal with this property

— want to show: $t = 1$

assume $t < 1$

- $t_n \uparrow t$

– $\gamma(t_n) \to \gamma(t)$

– $d(\tilde{\gamma}(0)), \tilde{\gamma}(t_n)) \leq \ell(\tilde{\gamma}_{\leq t_n}) = \ell(\gamma_{\leq t_n})$ is uniformly bounded

— by compactness of balls of $(T_m M, g')$

— get converging subsequence $\tilde{\gamma}(t_n) \to x'$

— consider lift $\tilde{\sigma}$ of γ with $\tilde{\sigma}(t) = x'$ near t

— same limit point as $\tilde{\gamma}$

— $\exp_m$ local diffeo near x'

— $\tilde{\gamma} = \tilde{\sigma}$ for $t' \leq t$

— $\tilde{\sigma}$ extends $\tilde{\gamma}$ to some times larger than t

—- contradiction to maximality of t

□

Corollary 4.111. *If (M, g) is complete with $K \leq 0$, then the universal covering of M is diffeomorphic to $\mathbb{R}^n$.*

Example 4.112. $T^n = \mathbb{R}^n/\mathbb{Z}^n$ (this is the universal covering of the torus)

- has $K = 0$

- $\tilde{T}^n \cong \mathbb{R}^n$ □

Example 4.113. - here many examples of compact quotients of the hyperbolic space

- these are compact Riemannian manifolds with constant negative sectional curvature

□

4.13 Conjugate points

(M, g) - Riemannian manifold

$\gamma : I \to M$ geodesic

$p, q \in I$

Definition 4.114. *The pair of points p, q is called conjugate if there exists a non-zero Jacobi field along γ with $J(p) = 0 = J(q)$.*

Remark 4.115. if p, q is conjugate, and $\gamma(t) = \exp_m((t-p)X)$, then $T\exp_m((q-p)X)$ is not an isomorphism □

Remark 4.116. in the condition for conjugate points can assume that $J \perp \gamma'$

- $n = \dim(M)$

- can decompose space of Jacobi fields into 2-dim subspaces of Jacobifields parallel to γ' and $2n-2$-dim subspace of fields orthogonal to γ'

– this is because of $g(J(t), \gamma'(t)) = g(J(p), \gamma'(p)) + (t-p)g(\nabla_{\partial_t} J(p), \gamma'(p))$

– if $J \simeq \gamma'$ then:

– if $J(p) = 0$, $\nabla_{\partial_t} J(p) \simeq \gamma'(p)$

– $g(J(t), \gamma'(t)) = (t-p)g(\nabla_{\partial_t} J(p), \gamma'(p))$ non-zero linear

– J has no zero other than p

Jacobi fields with two zeros are orthgonoal to γ'

□

consider manifold (M, g), $(\tilde{M}, \tilde{g})$

- $\dim(\tilde{M}) \geq \dim M$

$\gamma : [0, a] \to M$, $\tilde{\gamma} : [0, a] \to \tilde{M}$ geodesics

- $\|\gamma'(t)\| = \|\tilde{\gamma}(t)\|$ - same velocity

J Jacobi along γ, $\tilde{J}$ Jacobi along $\tilde{\gamma}$

write $\nabla_t J = J'$ etc

Theorem 4.117 (Rauch Comparison)**.** *Assume:*

1. $J(0) = 0$, $\tilde{J}(0) = 0$
2. $g(J'(0), \gamma'(0)) = \tilde{g}(\tilde{J}'(0), \tilde{\gamma}'(0))$
3. $\|J'(0)\| = \|\tilde{J}'(0)\|$
4. $\tilde{\gamma}$ *has no conjugate point on* $(0, a]$
5. *for all* $t \in [0, a]$ *and planes* $H \subseteq T_{\gamma(t)}M$ *containing* $\gamma'(t)$ *and* $\tilde{H} \subseteq T_{\tilde{\gamma}(t)}\tilde{M}$ *containing* $\tilde{\gamma}'(t)$ *we have* $K(H) \leq \tilde{K}(\tilde{H})$ *(sectional curvature).*

Then $\|\tilde{J}\| \leq \|J\|$ *with equality at some* t *only if* $\tilde{K}(\tilde{J}(s), \tilde{\gamma}'(s)) = K(J(s), \gamma'(s))$ *for all* $s \in [0, t]$.

Example 4.118. assume: (M, g) has constant section curvature K

- γ geodesic of speed $\|\gamma'(t)\| = v$

- $R(X, Y, Z, W) = K(g(Y, Z)g(X, W) - g(X, Z)g(Y, W))$

– implies with $J \perp \gamma'$

– $R(\gamma', J)\gamma' = -Kv^2 J$

– conclude: $J'' = R(\gamma', J)\gamma' = -Kv^2 J$

– for $K > 0$

– $J(t) = J(0)\cos(\sqrt{K}vt)J(0) + \frac{1}{\sqrt{K}v}\sin(\sqrt{K}vt)J'(0)$

discuss conjugate points:

$J(0) = 0$

$J(q) = 0$, $J'(0) \neq 0$

then $\sin(\sqrt{K}vq) = 0$

- smallest q:

$$q = \frac{2\pi}{v\sqrt{K}}$$

– distance between conjugate points is $\frac{2\pi}{\sqrt{K}}$

□

(M, g) general

Corollary 4.119. *If M has upper sectional curvature bound $k > 0$, then the distance between any two conjugate points on a geodesic with speed v bounded below by $\frac{2\pi}{v\sqrt{k}}$.*

Example 4.120. If M has non-positive curvature than γ has no pairs of conjugate points.

□

the following prepares the proof:

$\gamma : [0, a]$ curve in (M, g)

- $V \in \Gamma(M, \gamma^* TM)$

- $t \in [0, a]$

– define index form by:

$$I_t(V) := \int_0^t \left(\|V'(s)\|^2 + R(\gamma'(s), V(s), \gamma'(s), V(s)) \right) ds$$

$\gamma : [0, a]$ geodesic in (M, g)

- no conjugate points in $(0, a]$

- J - Jacobi along γ, $J \perp \gamma'$

- $V \in \Gamma(M, \gamma^* TM)$, $V \perp \gamma'$

Lemma 4.121. *Jacobi-fields minimize index form for fields $\perp \gamma'$ with given boundary values: If J is a Jacobi field along γ with $J(0) = V(0) = 0$ and $J(t) = V(t)$, then $I_t(J) \leq I_t(V)$ with equality only if $V = J$.*

Proof. choose basis $(J_i)_{i=1,\dots,n-1}$ of Jacobi fields along γ with $J_i(0) = 0$ $J_i \perp \gamma'$

- $J = \sum_i a_i J_i$ for constants $(a_i)_i$

- $V = \sum_i f_i J_i$, $(f_i)_i$ real-valued functions

– note: f_i is smooth at $t = 0$

$$
\begin{aligned}
\|V'\| + R(\gamma', V, \gamma', V) &= g(\sum_i (f_i' J_i + f_i J_i'), \sum_j (f_j' J_j + f_j J_j')) - R(\gamma', \sum_i f_i J_i, \gamma', \sum_j f_j J_j) \\
&= g(\sum_i f_i' J_i, \sum_j f_j' J_j) + g(\sum_i f_i' J_i, \sum_j f_j J_j') + g(\sum_i f_i J_i', \sum_j f_j' J_j) \\
&+ g(\sum_i f_i J_i', \sum_j f_j J_j') + g(\sum_i f_i J_i'', \sum_j f_j J_j)
\end{aligned}
$$

$$
\begin{aligned}
g(\sum_i f_i J_i, \sum_j f_j J_j')' &= g(\sum_i f_i' J_i, \sum_j f_j J_j') + g(\sum_i f_i J_i', \sum_j f_j J_j') + g(\sum_i f_i J_i, \sum_j f_j' J_j') \\
&+(\sum_i f_i J_i, \sum_j f_j J_j'')
\end{aligned}
$$

substract:

$$
\begin{aligned}
&\|V'\| + R(\gamma', V, \gamma', V) - g(\sum_i f_i J_i, \sum_j f_j J_j')' \\
&= g(\sum_i f_i' J_i, \sum_j f_j' J_j) + g(\sum_i f_i J_i', \sum_j f_j' J_j) - g(\sum_i f_i J_i, \sum_j f_j' J_j')
\end{aligned} \tag{4}
$$

will show: the last two terms cancel

- follows from $(g(J_i', J_j) - g(J_i, J_j'))(t) = 0$

- have $(g(J_i', J_j) - g(J_i, J_j'))(0) = 0$

$$
\begin{aligned}
(g(J_i', J_j) - g(J_i, J_j'))' &= (g(J_i'', J_j) + g(J_i', J_j') - g(J_i', J_j') - g(J_i, J_j'') \\
&= R(\gamma', J_i, \gamma', J_j) - R(\gamma', J_j, \gamma', J_i) \\
&= 0
\end{aligned}
$$

- hence $g(\sum_i f_i J_i', \sum_j f_j' J_j) - g(\sum_i f_i J_i, \sum_j f_j' J_j') = 0$

integrate (4) from 0 to t

$I_t(V) = g(V(t), \sum_j f_j J_j'(t)) + \int_0^t \| \sum f_i' J_i \|^2 ds$

$I_t(J) = g(J(t), \sum_j a_j J_j'(t))$

$V(t) = J(t)$ implies $a_i = f_i(t)$

$I_t(V) - I_t(J) = \int_0^t \| \sum f_i' J_i \|^2 d$

this implies both assertions

□

Proof of Rauch. $J = J^\perp \oplus J^\top$

$\tilde{J} = \tilde{J}^\perp \oplus \tilde{J}^\top$

$\|J^\top\| = \|J^\top(0)\| + t\|J^\top(0)'\|$

$\|\tilde{J}^\top\| = \|\tilde{J}^\top(0)\| + t\|\tilde{J}^\top(0)'\|$

hence $\|J^\top\| = \|\tilde{J}^\top\|$

consider now length of orthogonal component

- assume $J \perp \gamma'$ $\tilde{J} \perp \tilde{\gamma}'$

– $J \neq 0$

- set $v := \|J\|$, $\tilde{v} := \|\tilde{J}\|$

– $\tilde{v}$ has no zero on $(0, a]$ (by absense of conjugate points assumption)

l'Hospital

$\lim_{t\to 0} \frac{v(t)}{\tilde{v}(t)} = \lim_{t \to 0} \frac{v''(t)}{\tilde{v}''(t)} = \frac{\|J'(0)\|^2}{\tilde{v}''(t)} = 1$

- use $v''(0) = g(J''(0), J(0)) + 2\|J'(0)\|^2$ and $J'(0) \neq 0$ (since $J \neq 0$)

will show $(\frac{v(t)}{\tilde{v}(t)})' \geq 0$

equivalently: $v'\tilde{v} \geq v\tilde{v}'$

- this implies assertion

fix t

- if $v(t) = 0$, then $v'(t) = 2g(J'(t), J(t)) = 0$

- inequality holds

– similarly if $\tilde{v}(t) = 0$

assume $v(t) \neq 0$, $\tilde{v}(t) \neq 0$

- set $U(s) := \frac{J(s)}{v(t)}$, $\tilde{U}(s) := \frac{\tilde{J}(s)}{\tilde{v}(t)}$

$$\begin{aligned}
\frac{v'(t)}{v(t)} &= \frac{2g(J'(t), J(t))}{v(t)^2} \\
&= 2g(U'(t), U(t)) \\
&= (\|U\|^2)' \\
&= \int_0^t (\|U\|^2)''(s)ds \\
&= 2\int_0^t (\|U'(s)\|^2 + R(\gamma'(s), U(s), \gamma'(s), U(s)))ds \\
&= 2I_t(U)
\end{aligned}$$

analoguous

$\frac{\tilde{v}'(t)}{\tilde{v}(t)} = 2I_t(\tilde{U})$

must show

$I_t(\tilde{U}) \leq I_t(U)$

choose parallel basis $(e_i)_{i=1,\ldots,n}$ of γ^*TM

choose parallel basis $(\tilde{e}_i)_{i=1,\ldots,\tilde{n}}$ of $\tilde{\gamma}^*T\tilde{M}$

such that

- $\gamma'(t) = \|\gamma'\|e_1$, $\tilde{\gamma}'(t) = \|\tilde{\gamma}'\|\tilde{e}_1$

- $e_2(t) = U(t)$, $\tilde{e}_2(t) = \tilde{U}(t)$

this gives isometric and parallel map

- $\phi : \Gamma([0, a], \gamma^*TM) \to \Gamma([0, a], \tilde{\gamma}^*T\tilde{M})$

– $e_i \mapsto \tilde{e}_i$, $i = 1, \ldots, n$

have $I_t(U) \leq I_t(\phi(U))$ (by curvature inequality)

apply Lemma 4.121

$I_t(\tilde{U}) \leq I_t(\phi(U)) \leq I_t(U)$

this gives estimate:

for equality:

$\|\tilde{J}(t)\| = \|J(t)\|$

- then $v'(s)\tilde{v} = v(s)\tilde{v}'(s)$ for all $s \in [0, t]$

- $I_t(\tilde{U}) = I_t(\phi(U))$

- hence $\phi(U)$ is Jacobi field

- compare initial condition and value at t: $\phi(U) = \tilde{U}$

- $\tilde{K}(\tilde{\gamma}'(s), \tilde{J}(s)) = K(\gamma'(s), J(s))$

$\square$